# 15-750:Algorithms in the Real World

**Data Compression**

# Compression in the Real World

Ubiquitous usage. Examples:

- Data storage: file systems, large-scale storage systems (e.g. cloud storage)
- Communication
- Media: Video, audio, images
- Data structures: Graphs, indexes
- Newer: Neural network compression

# Encoding/Decoding

**"Message"** refers to the data to be compressed


The encoder and decoder need to understand common compressed format.

# Lossless vs. Lossy

**Lossless**: Input message = Output message

**Lossy**: Input message ≈ Output message

**Quality of Compression:**

For Lossless?

Runtime vs. Compression (ratio) vs. Generality

For Lossy?

Loss metric (in addition to above)

# How much can we compress?

Q: Can we (lossless) compress any kind of messages?

No!

For lossless compression, assuming all input messages are valid, if one string is compressed, some other must expand.

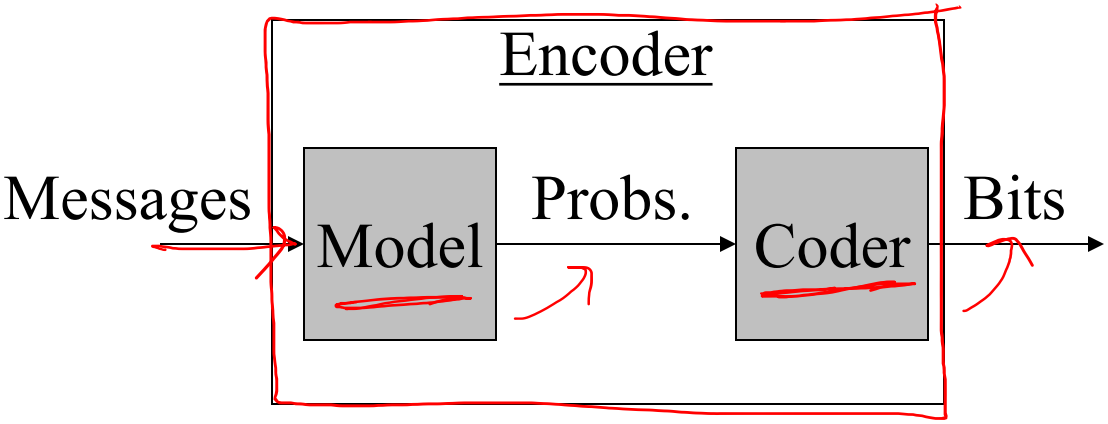
Q: So what we do need in order to be able to compress?

Can compress only if some messages are more likely than other.

That is, there needs to be **bias** in the probability distribution.

# Model vs. Coder

To compress we need a bias on the probability of messages. The **model** determines this bias




Example models:

- Simple: Character counts, repeated strings
- Complex: Models of a human face

# INFORMATION THEORY BASICS

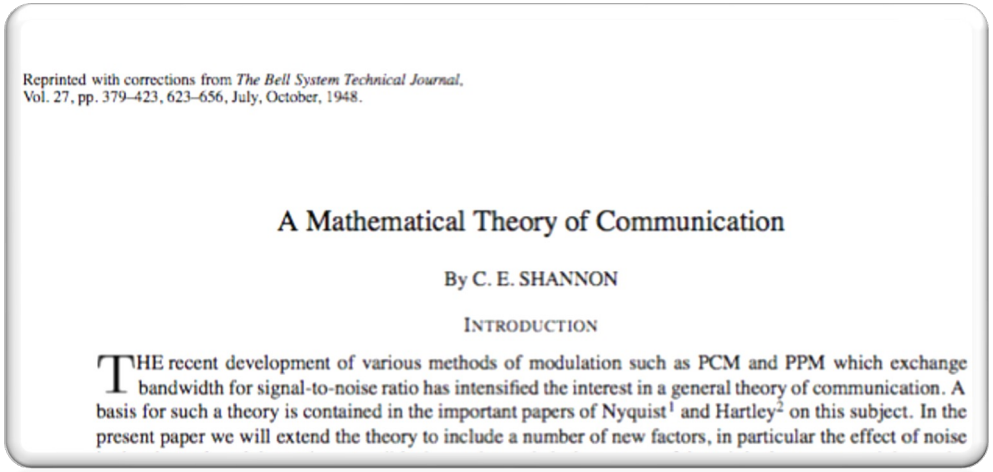
# Information Theory

- Quantifies and investigates "information"
- Fundamental limits on representation and transmission of information
  - What's the minimum number of bits needed to **represent** data? "source coding"
  - What's the minimum number of bits needed to **communicate** data? "channel coding"
  - What's the minimum number of bits needed to **secure** data? cryptography

# Information Theory

Claude E. Shannon

- Landmark 1948 paper: mathematical framework
- Proposed and solved key questions
- Gave birth to information theory



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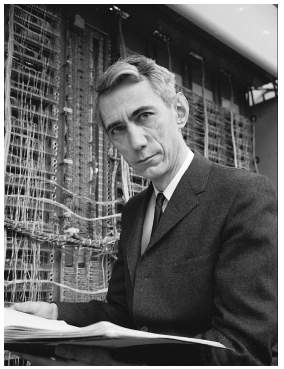
Vol. 27, pp. 379–423, 623–656, July, October, 1948.

A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise



# Information Theory

In the context of compression:

An interface between modeling and coding

**Entropy**

– A measure of information content

Suppose a message can take **n** values from $S = \{s_1, \ldots, s_n\}$ with a probability distribution ***p(s)***.

$p(s_1), p(s_2), \ldots p(s_n)$

One of the n values will be chosen.

"How much choice" is involved? OR

"How much information is needed to convey the value chosen?

# Entropy

Q: Should it depend on the values $\{s_1,\ldots,s_n\}$?

(e.g., American names vs. European names)

No.

Q: Should it depend on p(s)?

Yes.

If P($s_1$)=*1* and rest are all *0*?

No choice. Entropy = 0

**More the bias lower the entropy**

# Entropy

Shannon (1948 paper) lists key properties that an entropy function should satisfy and *shows that "log" is the only function*.

Specifically, $\log\left(\frac{1}{p(s)}\right)$

Intuition for the log function:

- When p(s) is low, entropy should be high
- Suppose two independent messages are being picked then entropy should add up

$s_1$ $s_2$ indep.

$p(s_1)$ $p(s_2)$

# Entropy

For a set of messages $S$ with probability $p(s)$, $s \in S$, the **self information** of $s$ is:

$$i(s) = \log \frac{1}{p(s)} = -\log p(s)$$

Measured in **bits if the log is base 2**.

**Entropy** is the weighted average of self information.

$$H(S) = \sum_{s \in S} p(s) \log \frac{1}{p(s)}$$

# Entropy Example

Binary random variable (i.e., taking two values) with probability p and 1-p

Denoted as $H_2(p)$:

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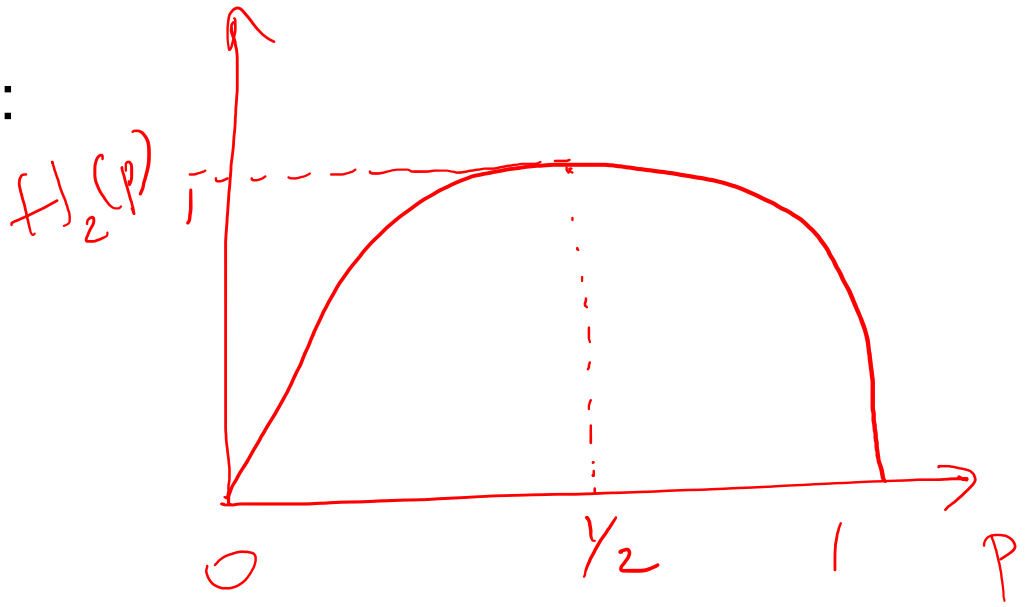
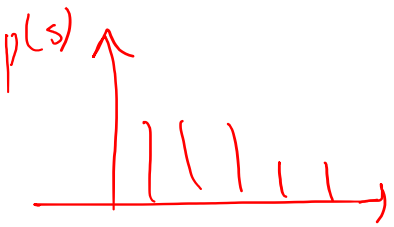



**Highest entropy when equiprobable**

(true for n >2 as well)

# Entropy Example

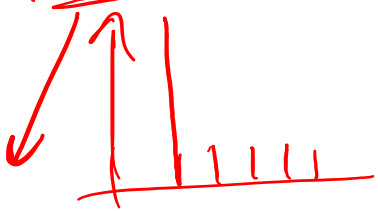



$$p(S) = \{.25, .25, .25, .125, .125\}$$

$$H(S) = 3 \times .25 \log 4 + 2 \times .125 \log 8 = 2.25$$

$$p(S) = \{.5, .125, .125, .125, .125\}$$

$$H(S) = .5 \log 2 + 4 \times .125 \log 8 = 2$$



$$p(S) = \{.75, .0625, .0625, .0625, .0625\}$$

$$H(S) = .75 \log(4/3) + 4 \times .0625 \log 16 = 1.3$$

# Conditional Entropy

Conditional entropy: Information content based on a context

The **conditional probability** $p(s|c)$ is the probability of $s$ in a context $c$.

$\log \frac{1}{p(s|c)}$

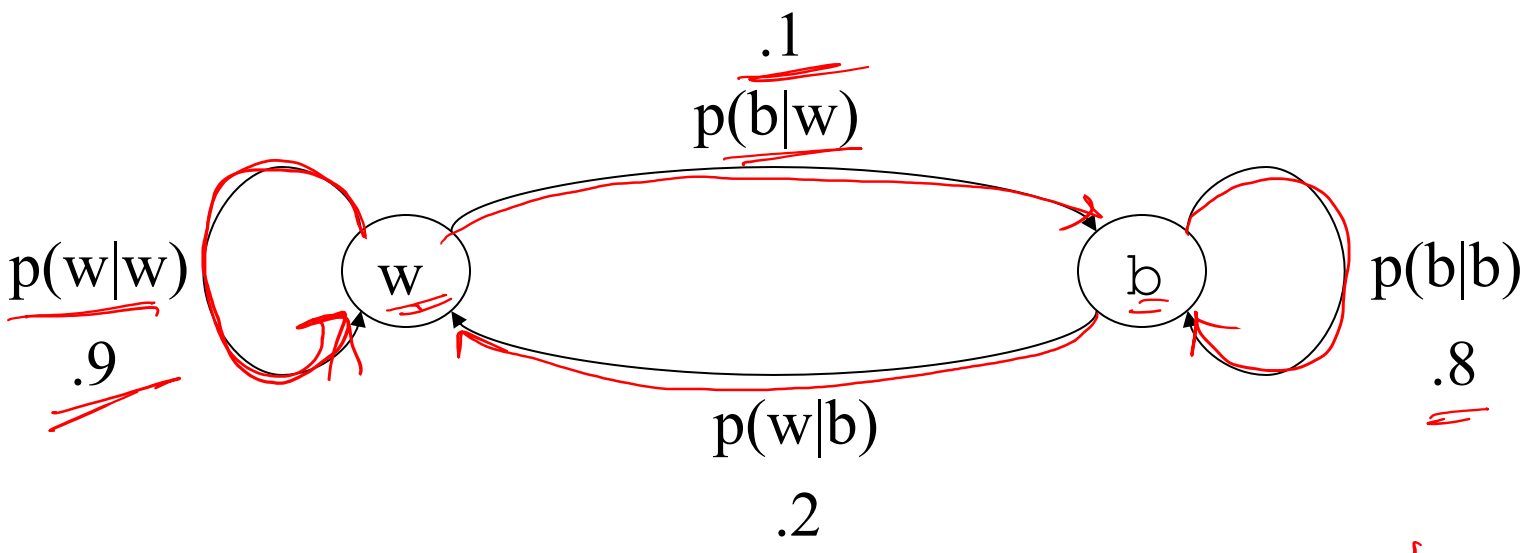
The **conditional entropy** is the weighted average of the conditional self information

$$H(S|C) = \sum_{c \in C} \left( p(c) \sum_{s \in S} p(s|c) \log \frac{1}{p(s|c)} \right)$$

# Types of "sources"

- Sources generate the messages (to be compressed)
- Sources can be modelled in multiple ways
- Independent and identically distributed (i.i.d) source
  - Prob. of each msg is independent of the previous msg
- Markov source
  - message sequence follows a Markov model (specifically Discrete Time Markov Chain, aka DTMC)

# Example of a Markov Chain



# Shannon's experiment

Asked people to predict the next character given the whole previous text. He used these as conditional probabilities to estimate the entropy of the English Language.

The number of guesses required for right answer:

| # of guesses | 1 | 2 | 3 | 4 | 5 | > 5 |
|---|---|---|---|---|---|---|
| Probability | .79 | .08 | .03 | .02 | .02 | .05 |

From the experiment

**H(English) = .6 - 1.3**

In comparison, ASCII uses 7 bits, Unicode and other representations use 8 or even higher

# PROBABILITY CODING

# Terminology

Communication (or a file) is broken up into pieces called **messages**.

Each message come from a **message set** $S = \{s_1, \ldots, s_n\}$ with a **probability distribution** $p(s)$.

**Code $C(s)$:** A mapping from a message set to **codewords**, each of which is a string of bits

**Message sequence:** a sequence of messages

# Variable length codes and Unique Decodability

A **variable length code** assigns a bit string (codeword) of variable length to every message value

e.g. `a = 1, b = 01, c = 101, d = 011`

What if you get the sequence of bits `1011` ?

aba or ca or ad

Is it `aba, ca,` or, `ad`?

A **uniquely decodable code** is a variable length code in which bit strings can always be uniquely decomposed into its codewords.

# Prefix Codes

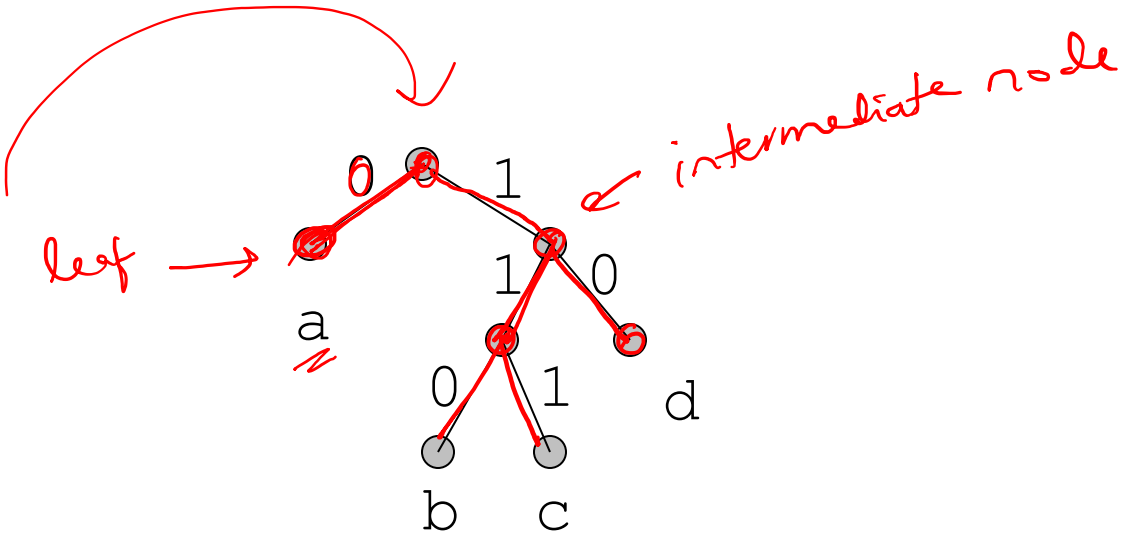
A **prefix code** is a variable length code in which no codeword is a prefix of another word.

e.g., a = 0, b = 110, c = 111, d = 10

All prefix codes are uniquely decodable

# Prefix Codes: as a tree

Prefix codes can be viewed as a binary tree with 0s or 1s on the edges and message values at the leaves:




a = 0, b = 110, c = 111, d = 10

# Average Length

For a code $C$ with associated probabilities $p(c)$ the **average length** is defined as

$$l_a(C) = \sum_{c \in C} p(c) l(c)$$

$l(c)$ = length of the codeword $c$ (a positive integer)

We say that a prefix code $C$ is **optimal** if for all prefix codes $C'$, $l_a(C) \leq l_a(C')$

# Relationship to Entropy

**Theorem (lower bound):** For any probability distribution p(S) with associated uniquely decodable code C,

$$H(S) \leq l_a(C)$$

**Theorem (upper bound):** For any probability distribution p(S) with associated optimal prefix code C,

$$l_a(C) \leq H(S) + 1$$

# Kraft McMillan Inequality

**Theorem (Kraft-McMillan):** For any uniquely decodable code C,

$$\sum_{c \in C} 2^{-l(c)} \leq 1$$

Conversely, for any set of lengths L such that $\sum_{l \in L} 2^{-l} \leq 1$

there is a prefix code C such that

$$l(c_i) = l_i (i = 1,\ldots,|L|)$$

We will use Kraft McMillan for proving the upper bound theorem.

# Proof of the Upper Bound (Part 1)

Assign each message a length: $l(s) = \lceil \log(1/p(s)) \rceil$

We then have

$$\sum_{s \in S} 2^{-l(s)} = \sum_{s \in S} 2^{-\lceil \log(1/p(s)) \rceil} \leq \sum_{s \in S} 2^{-\log(1/p(s))} = \sum_{s \in S} p(s) = 1$$

Then, by the converse part of Kraft-McMillan inequality there is a prefix code with lengths $l(s)$.

# Proof of the Upper Bound (Part 2)

Now we can calculate the average length given l(s)

$$
\begin{aligned}
l_a(S) &= \sum_{s \in S} p(s) l(s) \\
&= \sum_{s \in S} p(s) \cdot \lceil \log(1/p(s)) \rceil \\
&\leq \sum_{s \in S} p(s) \cdot (1 + \log(1/p(s))) \\
&= 1 + \sum_{s \in S} p(s) \log(1/p(s)) \\
&= 1 + H(S)
\end{aligned}
$$

# Another property of optimal codes

**Theorem:** If C is an optimal prefix code for the probabilities $\{p_1, \ldots, p_n\}$, then $p_i > p_j$ implies $l(c_i) \leq l(c_j)$

$s_i \leftarrow c_i$
$s_j \leftarrow c_j$

**Proof:** (by contradiction)

Assume $l(c_i) > l(c_j)$. Consider switching codes $c_i$ and $c_j$.

If $l_a$ is the average length of the original code, the length of the new code is

$$\begin{aligned} l_a' &= l_a + p_j(l(c_i) - l(c_j)) + p_i(l(c_j) - l(c_i)) \\ &= l_a + (p_j - p_i)(l(c_i) - l(c_j)) \\ &< l_a \end{aligned}$$

This is a contradiction since $l_a$ is not optimal