Lecture 22: Approximation (cont.) and Online Decision Making

1. Recap ← Min Makespan
   - New ideas

2. Two new problems (Set Cover / Vertex Cover)
   - New ideas

3. Competitive Analysis model for online algorithms

4. Problems
   - To be continued on Thursday

Turn on recording!
Set Cover:

Input: set system \(S_1, S_2, \ldots, S_m\)

Universe \(U\)

Collection of subsets of \(U\)

Know: union of all sets \(S_1, U \cup S_2, \ldots, U \cup S_m = U\)

Qm. find smallest sub collection of sets such that \(U \cup S_j = U\) for \(j \in C\) and \(|C|\) is minimized

\(C \leq \exists 1, 2, \ldots, m\)
Greedy Algorithm for Set Cover

While I uncovered elements,
pick a set that covers the max # of uncovered elements.

Thm. Suppose that OPTimum solution uses $K$ sets
(a) - then when we've picked $K$ sets cover at least $\frac{(1 - \frac{1}{e})|U|}{63\%}$
(b) - the total # of sets I pick $\leq K(\ln n + 1)$

$T_n = |U|$

Pf (a):
$n_t = \# \text{ of uncovered elements after } g \text{ has picked } t \text{ sets}$

@ time $t$, $E$ set that covers $\geq \frac{n_t}{K}$ uncovered elements

$n_t \leq n_t \left(1 - \frac{1}{K}\right) \Rightarrow n_t \leq \left(1 - \frac{1}{K}\right)^t n$

$\Rightarrow n_K \leq \left(1 + \frac{1}{K}\right)^K n \leq e$
(b) $t = k \ln n \implies n_t \leq (1 - \frac{1}{k})^t n \leq e^{-\frac{t}{k}} n$

$\implies (1 + \ln n) \text{approx}$

Fact 2. This analysis of greedy algo is tight ups to constants.

Fact 3. [Thomem Rejse '98] If $\exists$ a polytime algo that gets $(1 - e^{-\ln n})$ approx to SC $\implies P = NP$. 

Algorithm picks $\Omega(\lg n)$ sets 

OPT picks 2 sets.
Vertex Cover
(Art Gallery Guard problem)

\( G = (V, E) \)

\( C \subseteq V \)

\( s.t. \forall \text{ edge } \{u, v\} \in E, \text{ either } u \text{ or } v \text{ (or both) belong to } C \)

\[ \text{minimize } |C| \]

1. Write problem as Integer LP.
2. "Relax" to LP.
3. "Round" the solution.

\[ \text{LP} \leq \text{ILP} = \text{OPT} \]

\[ \text{OPT} = \text{ILP.} \]

\[ \begin{align*}
\min & \sum_{u \in V} x_u \\
\text{s.t.} & x_u + x_v \geq 1 \\
& \forall \{u, v\} \in E \\
& x_u \in [0, 1] \\
& 0 \leq x_u \leq 1
\end{align*} \]

\"is \ u \ in \ C"
$$\min \sum_{v \in V} x_v$$

subject to \( x_u \geq 1 \quad \forall (u, v) \in E \)

\( 0 \leq x_v \leq 1 \quad \forall v \in V \)

(3) "Round" function:

- Let \( x_v \) be the closest integer (break ties at \( \frac{1}{2} \rightarrow 1 \)) \( x_v \)

- \( x_v = 0.49 \rightarrow \) round it \( 0 \)
- \( x_v = 0.5 \rightarrow \) round it \( 1 \)
- \( x_v = 0.9 \rightarrow \) round it \( 1 \)

Claim: \( x_v \) is a vertex cover.

\[ \sum_{v \in V} x_v \leq 2 \sum_{v \in V} x_v \leq 2 \cdot \text{OPT} \]

\( |C| \leq 2 \cdot \text{OPT Size} \Rightarrow \text{algo is 2-approx} \)
Online Algorithms?

\[ \text{Input} = R_1, R_2, R_3, \ldots, R_t \]

- Imperfect Info
- Myopic decision making
- Sequential ""

\[
\max_I \frac{\text{cost}(\text{Alg}(I))}{\text{cost}(\text{OPT}(I))} = \text{(Competitive ratio)}_{\text{Alg}}
\]

Input \[ I \]

Requests \[ R_i \]

Actions \[ a_i \]

Cost \( \text{cost}(a_i) \)

\[
\frac{\text{cost} (\text{Alg})}{\text{cost} (\text{OPT})} = \sum
\]
Ski Rental:

Ski rental cost per day $1

- buys

Input:
- Y Y Y ... Y N
- can ski

T skis season ended

Observe:
- length of season $\geq B \Rightarrow$ buy
- $< B \Rightarrow$ rent

$$CR \leq \frac{2B-1}{B} \text{ small inputs. And input method is CR}=2\frac{1}{B}$$

Hedge your bets:

Rent until (a) season ends or (b) rented for $B$ days

then buy

If bought: paid $2B-1$

OPT $\geq B$

$\Rightarrow CR \leq 2$

If never bought: season $\leq B-1$

age = opts = season length $c.r=1 \leq 2$
distance $K$. 