15-451/651 Algorithms, Spring 2019

Homework #5 **Due: March 26, 2019**

(25 pts) 1. **(The Gates Elevators.)** Ever wonder whether you should wait for the elevators, or just take the stairs? If wait, how long before you give up? This is the ideal problem for you, the 451 student!!

We model this process as a zero-sum game. Taking the steps takes S units of time. The elevator takes E units of time. The elevator arrives at time 1 or 2 or 3, etc. If the elevator arrives at time $t \geq 1$ and you take it, you reach your destination at time $t + E$. However, if you have waited $w \geq 0$ units of time, and if the elevator has not yet arrived, you may decide to walk up, which takes S units of time, so you reach at time $w + S$. (Once you decide to start walking you cannot catch the elevator any more.)

Your actions are *"wait w time; if no elevator yet, walk up"*: one action for every integer $w \geq 0$. (So $w = 0$ means you don't wait at all.) The adversary's actions are: *"elevator arrives at time t"*, one for every positive integer $t \geq 1$.

For instance if $w = 2$ and $t \leq 2$ then you catch the elevator and arrive at time $t + E$, but if $w = 2, t = 3$ then you wait 2 time steps, (leave just before the elevator comes) and reaching at time $2 + S$.

Given the pair of actions (t, w), your pain (aka. the payoff to the adversary) is the ratio: (the time it took you) divided by (the optimum time it would take you if you knew when the elevator were to arrive).

(a) Let R be the matrix of payoffs to the adversary whose rows t are the adversary's actions and columns w are your actions, then for any $t \geq 1$ and $w \geq 0$, write down the mathematical expression for the payoff to the adversary.

Solution:

$$R_{tw} = \frac{\text{if } t \leq w \text{ then } t + E \text{ else } w + S}{\min(t + E, S)}$$

For the rest of the parts, assume that $E = 1$ and $S = 3$.

(b) This game has an infinite number of columns and an infinite number of rows. Let's add one more row, called "row ∞" for the scenario that the elevator never arrives. Argue that without loss of generality, we can assume the adversary chooses only rows $t = 1$ or $t = \infty$. Formally, argue that for any $t \geq 2$ we have $R_{tw} \leq R_{\infty w}$ for all t. This means that your strategy achieving expected payoff V in a world with only those two scenarios possible (elevator arrives at time $t = 1$, or it never arrives) will achieve also payoff V over the whole range of scenarios.

Solution: First, for $t \geq 2$, the denominator $\min(t + E, S) = \min(t + 1, 3) = 3$. We consider two cases:

Case 1($t \leq w$): In this case, $R_{tw} = \frac{t+1}{3}$ and $R_{\infty w} = \frac{w+3}{3}$. We know $\frac{t+1}{3} < \frac{w+3}{3}$ because $t \leq w$. Thus, $R_{tw} < R_{\infty w}$.

Case 2($w < t$): In this case, $R_{tw} = \frac{w+3}{3}$ and $R_{\infty w} = \frac{w+3}{3}$, so $R_{tw} = R_{\infty w}$.

Since for any $t \geq 2$, we have $R_{tw} \leq R_{\infty w}$, we can now assume the adversary chooses only rows $t = 1$ or $t = \infty$.

(c) Now that the game has just two rows, argue that the minimax-optimal strategy can safely put probability 0 on all columns except for $w \in \{0, 1\}$. Formally, argue that for any $w > 1$ we have $R_{tw} \geq R_{t1}$ for $t \in \{1, \infty\}$.

Solution: We will consider two cases based on the value of t (remember $w > 1$):

Case 1($t = 1$): In this case, $R_{tw} = \frac{2}{2} = 1$ and $R_{t1} = \frac{2}{2} = 1$. Thus, $R_{tw} = R_{t1}$.

Case 2($t = \infty$): In this case, $R_{tw} = \frac{w+3}{3}$ and $R_{t1} = \frac{1+3}{3} = \frac{4}{3}$. We know $\frac{w+3}{3} > \frac{4}{3}$ because $w > 1$. Thus, $R_{tw} > R_{t1}$.

Thus, we have proven that for any $w > 1$, we have $R_{tw} \geq R_{t1}$ for $t \in \{1, \infty\}$ and thus the minimax-optimal strategy can safely put probability 0 on all columns except for $w \in \{0, 1\}$.

(d) Now write down the 2-by-2 matrix that results. Write down the LP for the minimax-optimal strategy for you, the column player, and solve it to find the value of the game.

Solution: The 2-by-2 matrix is as follows:

t↓ w→	0	1
1	$\frac{3}{2}$	$\frac{2}{2} = 1$
∞	$\frac{3}{3} = 1$	$\frac{4}{3}$

Now if q and $1 - q$ are the probabilities that we pick columns $w = 0$ and $w = 1$, then we want to solve

$$\min_{0 \leq q \leq 1} \max(\frac{3}{2}q + (1-q), q + \frac{4}{3}(1-q))$$

This is not an LP, but we can rewrite this as:

$$\begin{aligned} \text{minimize} \quad & v \\ \text{subject to} \quad & (3/2)q + 1 - q \leq v \\ & q + (4/3)(1-q) \leq v \\ & 0 \leq q \leq 1 \end{aligned}$$

We can solve this by plotting this: we get $q = 2/5$ and the optimal value is $6/5$.

(e) Finally, suppose there is an elevator operator (row player) who controls the timing of the elevator, and is trying to cause you the most pain (get the highest payoff for herself). Solve for the row player's strategy.

Solution: Now if p and $1 - p$ are the probabilities that we pick rows $w = 1$ and $w = \infty$, then we want to solve

$$\max_{0 \leq p \leq 1} \min(\frac{3}{2}p + (1-p), p + \frac{4}{3}(1-p))$$

The LP looks very similar.

$$\begin{aligned} \text{maximize} \quad & v \\ \text{subject to} \quad & v \leq (3/2)p + 1 - p \\ & v \leq p + (4/3)(1-p) \\ 0 \leq\, & p \leq 1 \end{aligned}$$

Again the solution is $p = 2/5$ and the optimal value is $6/5$.

(25 pts) 2. **What are Widgets, Anyways?**

You are in charge of the supply chain for Widgets-r-Us, where you have to find ways to transport materials from their storage warehouses to the factories, subject to capacity constraints on the transportation network. You model it as a directed graph $G = (V, E)$, where each directed edge $e \in E$ represents a one-way road. (Roads capable of carrying traffic in both directions can be thought of as two one-way roads.)

There are k "warehouse" vertices, and k kinds of materials, where the i^{th} material is originally located at the warehouse vertex $s_i \in V$, and the total amount of material i is D_i. Each road (directed edge) e has a capacity u_e, such that the total amount of material (of all kinds) sent over this road must be at most u_e. You may assume that all quantities given as input in this problem are non-negative integers.

You also have ℓ "factory" vertices. There are ℓ factories producing widgets (the j^{th} factory is located at the j^{th} factory vertex $f_j \in V$). The factory f_j has a request vector $\mathbf{r}_j = (r_{1j}, r_{2j}, \ldots, r_{kj})$, such that to produce one unit of widget, it requires r_{ij} amounts of material i for every $i \in \{1, \ldots, k\}$. [1]

(a) Write an LP to figure out how to transport the material from their warehouses to the factories (respecting the road capacity constraints), to maximize the total amount of widgets produced, subject to these constraints. (It is OK if you produce fractional amounts of widgets.)

(b) You find out that widgets produced at different factories sell for different amounts of money: the price per unit of widget produced at factory j is p_j. You want to maximize your revenue. Change the LP from the previous part to handle this.

(c) You are informed that two of the roads $e_1 = (u, v)$ and $e_2 = (v, u)$ are special: they represent the two directions of traffic over a bridge. For structural reasons, you have the balance requirement that the absolute value of the difference between the amount of material send in the two directions can be at most $\delta \in \mathbb{Z}_{\geq 0}$. How can you add such a constraint to your LP.

Solution: We'll have a variable w_j for each factory $j \in [\ell]$ which represents the amount of widgets produced at factory f_j. And a variable f^i_{uv} for each $i \in [k]$ and every edge (u, v) representing the amount of material i flowing along this edge. For ease of writing, we will introduce some auxiliary variables along the way. First, let us define the *excess flow* z^i_v for

[1] Nodes like s_i and f_j can have both incoming and outgoing arcs, and the route taking material i from the warehouse s_i to some factory f_j is allowed to pass through some other warehouse vertex $s_{i'}$ or some other factory vertex $f_{j'}$. You may assume that all the $k + \ell$ warehouse and factory vertices are distinct.

a node v and material i to be the amount of material i entering it minus that leaving it. These are new variables. (You can write the LP without these variables, of course, but they make your life much easier.)

$$z_v^i = \sum_{u:(u,v)\in E} f_{uv}^i - \sum_{u:(v,u)\in E} f_{vu}^i. \qquad \forall i, v \tag{1}$$

Then for material i, we have

$$z_{s_i}^i \geq -D_i \qquad \forall i \tag{2}$$

since at warehouse i more material i can leave it than reaches it). At factories we can have positive excess, so

$$z_v^i \geq 0 \qquad \forall i, \forall v \in \{f_1, f_2, \ldots, f_\ell\} \tag{3}$$

At all other nodes we have conservation of material i:

$$z_v^i = 0 \qquad \forall i, \forall v \notin \{s_i, f_1, f_2, \ldots, f_\ell\} \tag{4}$$

Moreover, the amount of widgets produced is bounded above by the amount of excess flow of each material at the factories (scaled by the requests): we want to say that

$$w_j \leq \min_{i:r_{ij}>0} \frac{1}{r_{ij}} \cdot z_{f_j}^i.$$

But the $\min$ function is not a linear function, so we instead say the equivalent fact:

$$w_j \leq \frac{1}{r_{ij}} \cdot z_{f_j}^i \qquad \forall j \forall i : r_{ij} > 0 \tag{5}$$

We also need to account for the road capacity constraints:

$$\sum_i f_e^i \leq u_e \qquad \forall e \in E \tag{6}$$

Finally, the non-negativity constraints:

$$w_j, f_{uv}^i \geq 0. \tag{7}$$

The objective function for part (a) is $\sum_j w_j$, and for part (b) is $\sum_j p_j w_j$.

For (c), we want to balance the total flow between the two directions, and hence want:

$$\left| \sum_i f_{uv}^i - \sum_i f_{vu}^i \right| \leq \delta.$$

But again the absolute value function is not a linear function. So instead we realize that for any a,

$$|a| \leq \delta \qquad \Longleftrightarrow \qquad -\delta \leq a \leq \delta$$

and hence we add in the two constraints:

$$-\delta \leq \sum_i f_{uv}^i - \sum_i f_{vu}^i \leq \delta. \tag{8}$$

(25 pts) 3. **(Circuit Board Wiring)**

There are n terminals on a circuit board. Each terminal is either positive or negative. A wire will be attached to each terminal. It must be the case that for every pair of terminals of opposite polarity the two wires must be long enough so that they can be made to touch.

> E.g., if there's a positive terminal at point $a = (0, 0)$, and a negative terminal at $b = (0, 10)$, and another negative terminal at $c = (10, 0)$ then one solution is to put a wires of length 5 at all three terminals. Or to put a wire of length 10 at a and length zero at the others. Or, for any $x \in [0, 10]$, wires of length $\geq 10 - x$ at a, and length $\geq x$ at both b and c.

So given the locations of the terminals (and their polarities), the problem is to compute the shortest total length of wires needed to satisfy the stated requirement.

(a) Consider the following specific instance of the problem. All the terminals are along the x-axis. There are positive terminals at 0, 110, 111, 112. And there are negative terminals at 99 and 100. What's the optimal solution to the problem? (You don't need to prove your solution is optimal.)

(b) Formulate this problem as an LP. (Illustrate your LP by writing this for an example with two positive and two negative terminals.)

(c) Write the dual of the LP in part (b). (Again, write down this dual for the two-terminal example in part (b).)

(d) Show how to compute an optimal solution to the dual problem of part (c), without using a general LP solver.

Solution:

(a) There are several ways to do it, but you should be able to get a total length of 113. E.g., you can attach a wire of length 13 to position 99 and length 12 to position 100 (so that they both can reach the terminals to their right), and 88 to position 0 (so that it can reach the wires for the two negative terminals).

(b) Let w_i be the length of wire at terminal i, and let d_{ij} be the distance between terminals i and j. Let P and N be the sets of positive/negative terminals respectively. Then the LP is:

$$\begin{aligned} &\min \sum w_i \\ \text{subject to} \quad & w_i + w_j \geq d_{ij} && \forall i \in P, j \in N \\ & w_i \geq 0 && \forall i \end{aligned}$$

(c) For the dual, let y_{ij} be the variable corresponding to the constraint for i and j in the primal. Then the dual is:

$$\begin{aligned} &\max \sum y_{ij} d_{ij} \\ \text{subject to} \quad & \sum_{j \in N} y_{ij} \leq 1 && \forall i \in P \\ & \sum_{i \in P} y_{ij} \leq 1 && \forall j \in N \\ & y_{ij} \geq 0 && \forall i, j \end{aligned}$$

(d) We can represent this problem as a maximum-weight matching between the positive and negative terminals.

To solve this, we set up a flow graph. We have a vertex p_i for each positive terminal i and n_j for each negative terminal j. We also have the source node s and sink node t. We have edges from s to each p_i with cost 0 and capacity 1, edges from each n_j to t with cost 0 and capacity 1, and edges between each pair p_i and n_j with cost $-d_{ij}$ and infinite capacity. All edges are directed.

If we think of y_{ij} as the amount of flow along the edge from p_i to n_j, then the cost of the flow is $-\sum y_{ij} d_{ij}$, so minimizing this cost maximizes our objective function. The constraints in the dual above corresponds to the total outflow of a p_i node and the total inflow of a n_j node being at most 1, which is enforced in our graph by the capacities of the edges from the source and to the target.

At a higher level, the duality shows that the minimum length of the wiring problem equals the max-length of any matching between the positive and negative terminals. Think about it: any matching gives a lower bound, because the wires at the two ends of the matched edges have to touch each other for sure (e.g., a matching between 0 *and* 100*, and between* 99 *and* 112 *has cost* 113*). And by strong duality, the two are equal.*