# 15-451/651 Algorithms, Spring 2019
# Recitation #7 Worksheet

---

**Using Flows: Squares and Stars** You are given an $n \times n$ chessboard where some of the squares are colored red on them. You can place stars on the board, so that each star is placed on a red square, each row has at most one star, and each column has at most one star. Find the maximum number of stars you can place.

**Solution:** We solve this using max-flow. We construct source node $s$, a sink node $t$, nodes $R_i$ for each row $i$ and $C_j$ for each column $j$. There is an directed edge $(R_i, C_j)$ if the $(i, j)$ square is red. Finally, we add directed edges $(s, R_i)$ and $(C_j, t)$. All edges have unit capacity. Now we find a maximum (integer) flow from $s$ to $t$. Since the edges have unit capacity, this flow sends unit flow on some edges and zero on the others. For each red square $(i, j)$ we put a star on $(i, j)$ if there is unit flow on the edge $(R_i, C_j)$.

There are two parts to the proof. First, we show that if there is a solution to the squares-and-stars problem with $k$ stars, then we can send at least $k$ flow from $s$ to $t$. Then we show the converse: if we can send $\ell$ flow from $s$ to $t$, then the above process finds a solution to the squares-and-stars problem with $\ell$ stars. This shows that the two problems are equivalent.

For the first part, take a solution to S&S with $k$ stars: if a star is placed at $(i, j)$, send flow $s \to R_i \to C_j \to t$. Note that since each row and column has at most one star, edges $s \to R_i$ and $C_j \to t$ will not be used by any other star. And neither will edge $R_i \to C_j$, of course. So this is a feasible solution, one that respects all the edge capacities, that sends $k$ flow. So the max-flow is at least $k$.

Now we show the max-flow is *exactly* $k$ by proving the converse. Indeed, suppose the max-flow in the graph is $\ell$. Since all capacities are integer, we can assume this flow is integer valued. Suppose there is unit flow on the edge $(R_i, C_j)$, then this unit flow must have come via $(s, R_i)$, because that is the only edge directed into $R_i$. And this flow must leave via $(C_j, t)$ since that is the only edge directed out of $C_j$. And since these edges have unit capacity and we found an integer-valued flow, no other edges $(R_{i'}, C_j)$ or $(R_i, C_{j'})$ can have non-zero flow on them. So choosing $(i, j)$ satisfies the constraints of the problem: that square is red because there is an edge in the first place, and no other square in row $i$ or column $j$ has a star on it. This proves the converse. So the S&S problem answer and the max-flow value are the same, completing the proof.

**Using Flows: BaseBall Elimination.** We are trying to figure out if the Pittsburgh Pirates can make it to the playoffs. There are $n$ teams. We know, for each team $i$, the number of games $W_i$ they have already won. And for each pair of teams $(i, j)$, we know how many games $G_{ij}$ they still have to play. We want to find out if there are some outcomes possible for all the remaining games so that the Pirates (which we shall say is team #1) has the maximum number of final wins (possibly tied with someone else). Show how to solve this using max-flows. (Go Bucs!)

**Solution:** First, we can assume that the Pirates win all their remaining games, so at the end the Pirates will have $F_p := W_1 + \sum_j G_{1j}$ wins. So we want to find if there are outcomes for all the other games, such that each other team $i$ wins at most $F_p - W_i$ games.

Now set up a flow network, with a node $u_{ij}$ for each pair of teams $2 \le i < j \le n$. Also create nodes $v_j$ for each team $j \in \{2, \ldots, n\}$. Add infinite-capacity directed edges $(u_{ij}, v_i)$ and $(u_{ij}, v_j)$. Finally, add a node $s$ and edges $(s, u_{ij})$ for each $(i, j)$ with capacity $G_{ij}$. And a node $t$ and edges $(v_i, t)$ with capacity $F_p - W_i$. Now we find a max $s$-$t$ flow, and see if all the edges out of $s$ are saturated. If so, the Pirates can make it to the playoffs, else they cannot.

Why? If there is such a flow, there is a maximum integer-valued flow. In this flow, for each $(i, j)$, each of $G_{ij}$ flow is being split among the two teams $i$ and $j$, indicating who wins each of the $G_{ij}$ games. And now the fact that each $i$ receives $F_p - W_i$ flow, means team $i$ wins at most $F_p - W_i$ games. (And hence does not win more than the Pirates.)

To be a bit more formal, we really show that for each outcome where the Pirates make the playoffs, there is a flow in the network. And vice-versa. We leave the details for the reader.

**Rock-Paper-Scissors with a Twist** Suppose we have a non-standard game of rock-paper-scissors, which is still zero-sum, but with the following payoffs for the row player (Alice):

| | | Bob plays | | |
|---|---|---|---|---|
| | | $r$ | $p$ | $s$ |
| Alice plays | $r$ | 0 | −1 | 2 |
| | $p$ | 1 | 0.5 | −1 |
| | $s$ | −1 | 2 | −1 |

If Alice decides to play $\mathbf{p} = (p_1, p_2, 1 - p_1 - p_2)$ as her strategy, what should Bob play to minimize the payoff to Alice? What is Alice's payoff if he does this.

**Solution:** Bob can either play rock, or paper, or scissors. He'll play whichever gives the lowest payoff to Alice. All other mixed strategies are averages of these three pure strategies. In this case Alice's payoff will be

$$\min \Big\{ p_1 \cdot 0 + \quad p_2 \cdot 1 + \quad (1 - p_1 - p_2) \cdot (-1),$$
$$p_1 \cdot (-1) + \quad p_2 \cdot (0.5) + \quad (1 - p_1 - p_2) \cdot (2),$$
$$p_1 \cdot (2) + \quad p_2 \cdot (-1) + \quad (1 - p_1 - p_2) \cdot (-1) \Big\}$$

**Practice with Zero-Sum Games.** Consider a zero-sum game with payoffs:

```
(-1/2, 1/2) (3/4, -3/4)
(1, -1)     (-3/2, 3/2)
```

Show the minimax-optimal strategies are $\mathbf{p} = (2/3, 1/3), \mathbf{q} = (3/5, 2/5)$ and value of the game is 0.

Now consider the game with payoffs:

```
(-1/2, 1/2) (3/4, -3/4)
(1, -1)     (-2/3, 2/3)
```

Show that minimax-optimal strategies are $\mathbf{p} = (\frac{4}{7}, \frac{3}{7}), \mathbf{q} = (\frac{17}{35}, \frac{18}{35})$ and value of the game is $\frac{1}{7}$.

Now consider the game with payoffs:

```
(-1/2, 1/2)     (-1, 1)
(1, -1)         (2/3, -2/3)
```

Show that minimax-optimal strategies are $\mathbf{p} = (0, 1), \mathbf{q} = (0, 1)$ and value of game is $\frac{2}{3}$.