15-451/651 Algorithms, Spring 2019
Recitation #13 Worksheet

VCG and Pricing Advertisements

We saw the VCG mechanism for incentive-compatible auctions in Lecture. Let's use this for pricing online advertising slots. There are 2 ad slots that ElGogo wants to sell on a page, the first slot has a clickthru rate of 0.5, the second has a clickthru rate of 0.3. *Each bidder can get at most one slot.* There are 4 bidders, with the following valuations:

- A: \$10 per click (so, e.g., this bidder values the first slot at 10*0.5 = 5, and the second slot at 10*0.3 = 3.)
- B: \$8 per click
- C: \$7 per click
- D: \$2 per click

1. What is the social-welfare maximizing allocation?

   Solution: A gets the first slot, B gets the second. The total value to A is 5 and the value to B is 8*0.3 = 2.4. Total social welfare = 7.4.

2. What are the VCG payments?

   Solution: If A did not bid, B and C would get the first and second slots respectively. The social welfare would be 8*0.5 + 7*0.3 = 6.1. So A's payment is how much his presence caused B,C,D's welfare to fall, = (optimal welfare without A) - (optimal welfare of everyone else with A) = 6.1 - (2.4 + 0 + 0) = 3.7.

   If B did not bid, A and C would get the first and second slots respectively. The social welfare would be 10*0.5 + 7*0.3 = 7.1. So B's payment is how much his presence caused the others' welfare to decrease = (optimal welfare without B) - (optimal welfare of everyone else with B) = 7.1 - (5 + 0 + 0) = 2.1.

   C and D do not pay anything.

Combinatorial Auctions

VCG can be used even with complicated preferences. Suppose we have two identical hotel rooms in Las Vegas, a flight ticket f from PIT to LAS, and a concert ticket c in Vegas to auction off. In the following, a generic hotel room is denoted by h, and none of the people want two rooms.

- Buyer A: values $\{h\}$ at \$100, $\{f\}$ at \$200, $\{h, f\}$ at \$450, $\{h, f, c\}$ at \$440. (He hates the band in question so much, he gets *negative value* from getting c along with h, f.) All other sets are valued at \$0.

- Buyer B (doesn't care for the concert): values $\{h\}$ at \$50, $\{f\}$ at \$400, $\{h, f\}$ at \$500, and $\{h, f, c\}$ at \$501. All other sets are valued at \$0.
- Buyer C (lives in Vegas): values $\{c\}$ (and all sets containing c) at \$200.

What is the social-welfare maximizing allocation, and what are the VCG payments?

Solution: The allocation is A gets $\{h\}$, B gets $\{h, f\}$, and C gets $\{c\}$. This gives a total valuation (aka social welfare) of $\$100 + 500 + 200 = \$800$.

A pays 0, B pays \$350, C pays \$1.

Polynomials and Interpolation

Find a polynomial $P(x)$ of degree ~~at most 3~~ at most 2 that satisfies

$$P(1) = 3, P(59) = 81, P(10000) = 1.$$

You must explicitly write down the polynomial, but need not simplify it.

Solution: Using Lagrange interpolation, we can first get a polynomial that has $P(1) = 1$, and also $P(59) = P(10000) = 0$. This is

$$Q_1(x) = \frac{(x-59)(x-10000)}{(1-59)(1-10000)}.$$

Similarly we can find $Q_{59}(x) = \frac{(x-1)(x-10000)}{(59-1)(59-10000)}$ and $Q_{10000}(x) = \frac{(x-1)(x-59)}{(10000-1)(10000-59)}$. Then

$$P(x) = 3Q_1 + 81Q_{59} + Q_{10000}.$$

Several of you propose writing linear constraints and solving them. That does not give an explicit representation, and in general may take more time (since you have to solve a system of linear equations).

Recap and Practice Final

Please ask us questions about topics in the course, and try to solve problems on the practice final.