(Oh Spanner, my Spanner!) We are given an undirected, unweighted graph $G = (V, E_G)$ on n nodes. Say an edge $\{u, v\}$ of G is *heavy* if both of its endpoints u and v have degree greater than $n^{1/3}$ in G, otherwise the edge is *light*.

We show how to find a 6-*additive spanner* of G, which we denote $H = (V, E_H)$, having $\tilde{O}(n^{4/3})$ edges. Here is the algorithm:

i. Include all edges in H which are incident to vertices of degree at most $n^{1/3}$ in G.

ii. Next we sample a set A of $O(n^{2/3} \log n)$ vertices from V uniformly at random. For each vertex $v \in V$, if v is adjacent to one or more vertices in A, then include an edge $\{v, a\}$ in H for exactly one arbitrary vertex $a \in A$.

iii. Moreover, for $i = 0, 1, 2, \ldots, O(\log n)$, we also sample a set B^i of $O(2^{-i} n^{2/3} \log n)$ vertices from V independently and uniformly at random.

iv. For each i, for each pair of vertices $u \in A$ and $v \in B^i$, among the paths beween u and v in G containing $O(2^i)$ heavy edges, we choose the shortest one P and include in our spanner H all heavy edges along it (if there are no paths containing $O(2^i)$ heavy edges we do nothing).

To argue H is a 6-additive spanner with the claimed number of edges, answer the questions below:

(a) How many edges are in H?

(b) Consider a shortest path P between an arbitrary pair u and v of vertices in G. Suppose there are k heavy edges in P. Argue that there is a set $S(P)$ of $\Omega(kn^{1/3})$ distinct vertices v *adjacent* to P. That is, for every vertex $s \in S(P)$, there should be a vertex u on P such that $\{u, s\}$ is an edge.

(c) For each pair a, b of vertices in G, fix $P_{a,b}$ to be an arbitrary shortest path between them in G. Argue that with probability at least $1 - 1/\text{poly}(n)$, the following statement holds:

> For all $a, b \in V$, if the number of heavy edges along $P_{a,b}$ is in the range $[2^{i-1}, 2^i)$, then the set $B^i \cap S(P)$ is non-empty.

(d) Argue that H is a 6-additive spanner with probability at least $1 - 1/\text{poly}(n)$.

Hint: For each pair a, b consider the path $P_{a,b}$ and look at the number of heavy edges on it.