Algorithm
Design and Analysis

Computational Geometry (Incremental Algorithms)

Goals for today

- Apply **randomized incremental algorithms** to geometry
- Give randomized incremental algorithms for two key problems:
  - The **closest pair** problem
  - The **smallest enclosing circle** problem

Closest Pair

The closest pair problem

Problem (closest pair): Given n points P, define $CP(P)$ to be the closest distance, i.e.

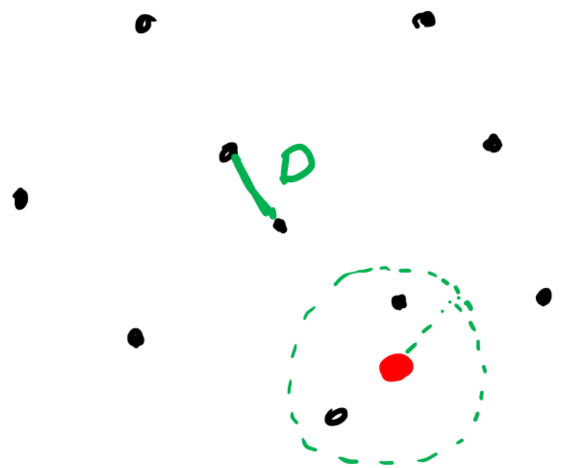
$$CP(P) = \min_{p,q \in P} ||p - q||$$

Brute force solution: Try all pairs → $O(n^2)$

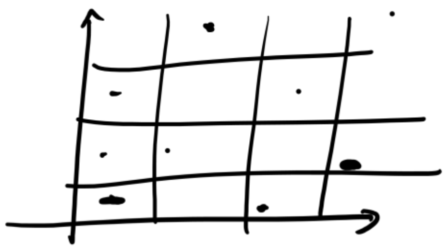
Improving brute force: incremental

- Brute force reuses no information whatsoever
- Geometry problems often have a lot of reusable information!

- Suppose I know the closest pair among the first i points…

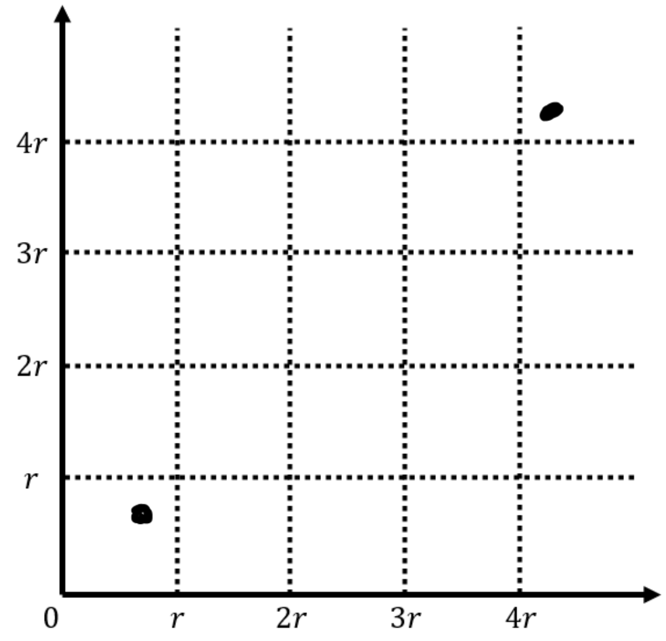


The problem



New Question: How do we find the set of points within distance d of the new point?

Put points into buckets using a grid?



A grid data structure!

- If the grid size is sufficiently large, closest pair will be in same cell, or in neighboring cells
- If the grid size is too large, there will be too many points per cell…

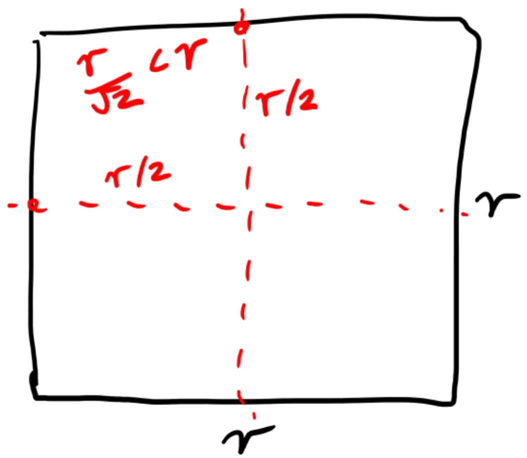
Goal: Choose the right grid size.

- Want few points per cell, so that looking in a cell is fast
- Want the closest pair to be in neighboring cells so we find them fast

The right grid size

Claim (the right grid size): Given a grid with points P and grid size $r = CP(P)$, no cell contains more than four points

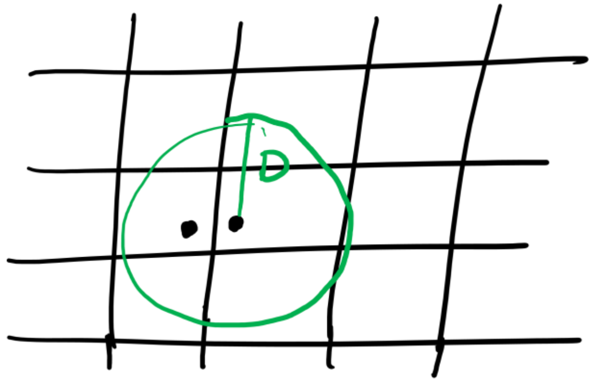
Proof:



The incremental approach

Key idea (incremental): Add the points one at a time

- Check neighboring cells to see if there's a new closest pair
- If so, rebuild the grid with the new size
- Otherwise keep going



A grid data structure

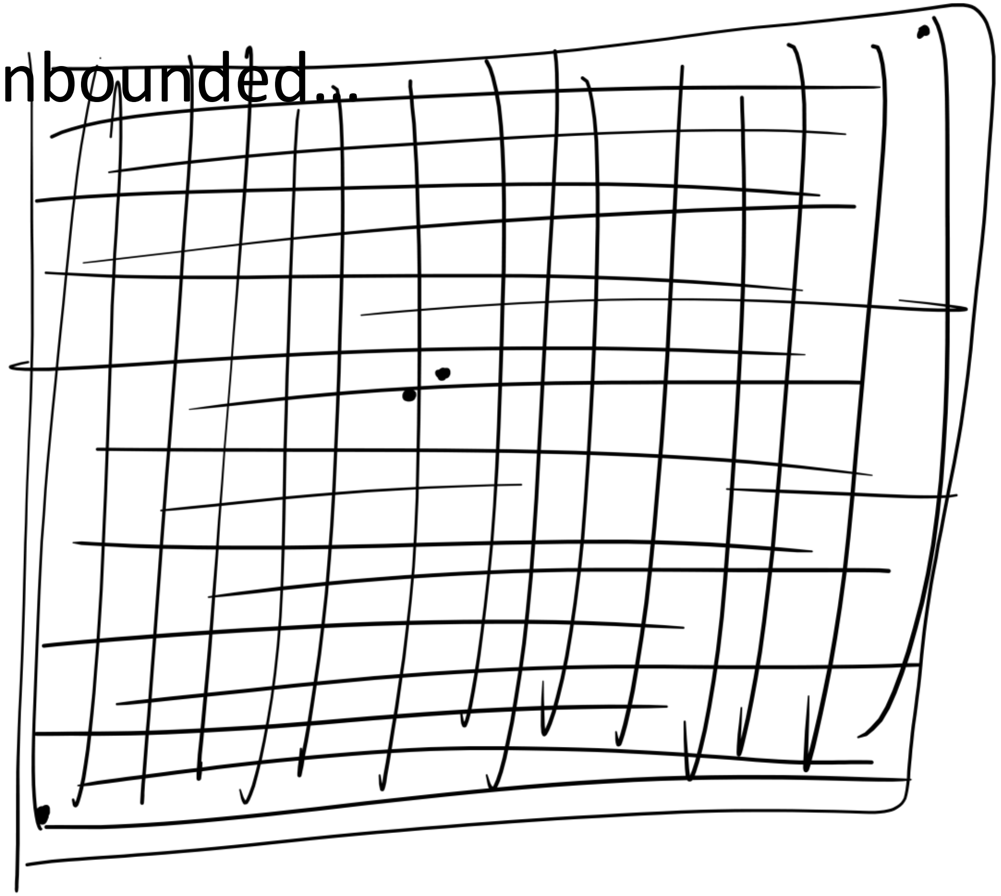
Invariant (grid size): Given a grid containing a set of points P, we want the grid size r to always equal $CP(P)$

- MakeGrid(p, q): Make a grid containing p and q, with $r = ||p - q||$
- Lookup(G, p): Given a grid G and point p (not currently in the grid), we want to know whether p is part of a new closest pair
- Insert(G, p): Given a grid G and point p, inserts p and returns the grid size (which may have changed because of p)

Implementing the grid

Issue: The number of grid cells could be unbounded…

Hashtable!



Implementing the grid

Implement MakeGrid*(p, q)*:

$r = \|p - q\|$

Put p & q in grid

Implementing the grid

Implement Lookup(G, q):

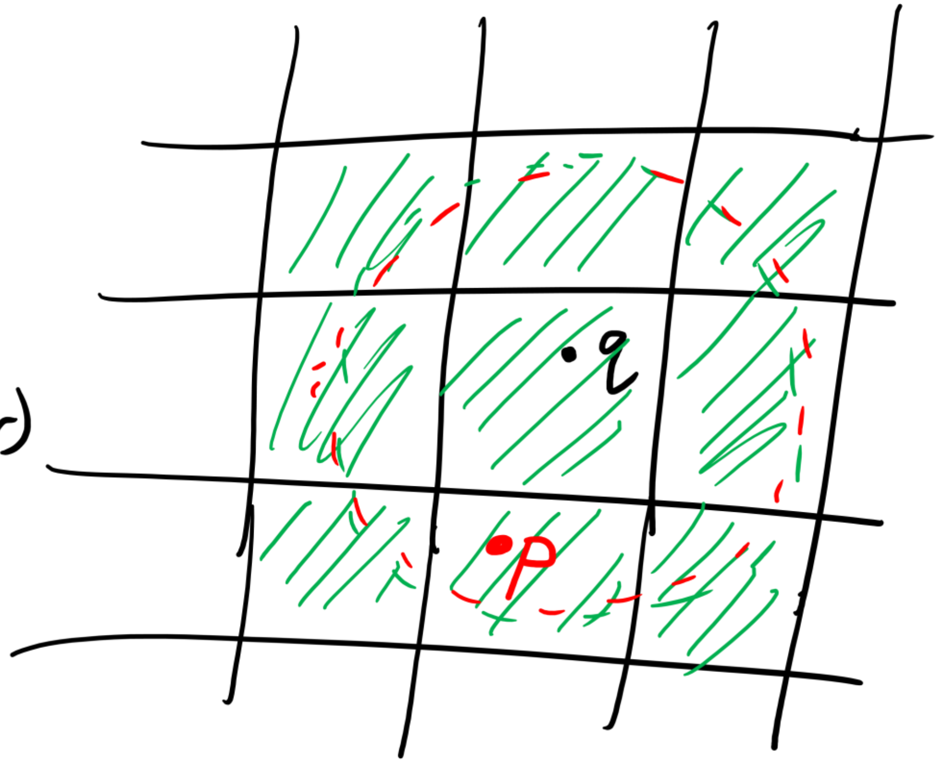
Search neighbouring grid cells

$\leq$ 36 pts

If $\|p - q\| < r$ (new answer)

  return $\|p - q\|$

return None



Implementing the grid

Implement Insert(G, q):

Lookup (q)

If distance changes (Lookup returns not None)

Build from scratch on current points

Else

put q in grid

Runtime

Claim (runtime): The worst-case runtime of the incremental grid algorithm is $O(n^2)$

Proof:

$$\text{Cost} = O\left(\sum_{i=2}^{n} i\right) = O(n^2)$$

Randomization to the rescue!!!

Randomized runtime

Claim (randomized incremental is fast): Randomly shuffle the points, then run the incremental algorithm, it takes $O(n)$ time in expectation

Proof:

$$P_i = \langle p_{\pi_1}, p_{\pi_2}, \ldots, p_{\pi_i} \rangle$$

$$X_i = \begin{cases} 1 & \text{if } CP(P_i) \neq CP(P_{i-1}) \\ 0 & \text{otherwise} \end{cases}$$

$$T = \sum_{i=2}^{n} (1 + X_i \cdot i)$$

$$\mathbb{E}[T] = O(n) + \sum_{i=2}^{n} \mathbb{E}[X_i] \cdot i$$

$$\Pr[X_i = 1] = \Pr[CP(P_i) \neq CP(P_{i-1})]$$

Randomized runtime (continued)

We need to bound $\Pr[X_i = 1]$... *(i.e.,* $\Pr[CP(P_i) \neq CP(P_{i-1})]$*)*

Call a point q "critical" if $CP(P_i \setminus \{q\}) \neq CP(P_i)$

≤ 2 critical pts

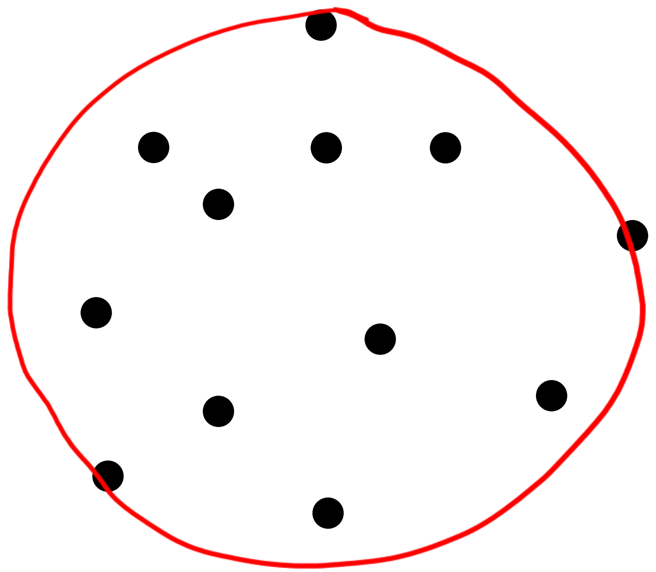
$\Pr[X_i = 1] \leq 2/i$

$\mathbb{E}[T] = O(n) + \sum_{i=2}^{n} \frac{2}{i} O(i) = O(n)$ expected!!

Smallest enclosing circle

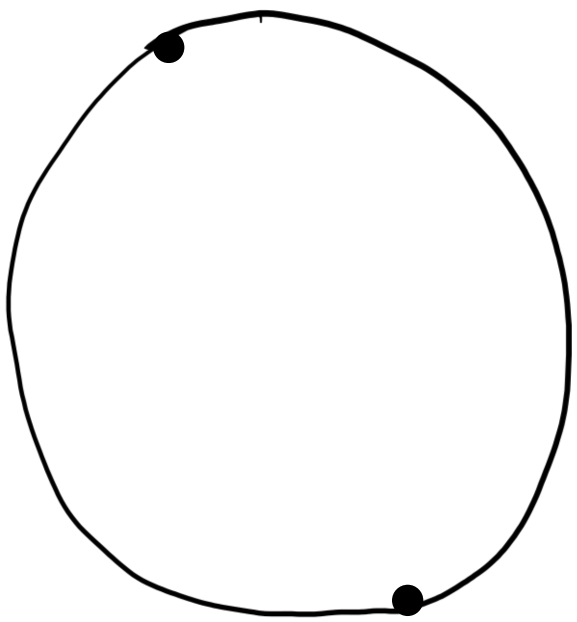
The smallest enclosing circle

Problem (Smallest enclosing circle): Given $n \geq 2$ points in two dimensions, find the smallest circle that contains all of them



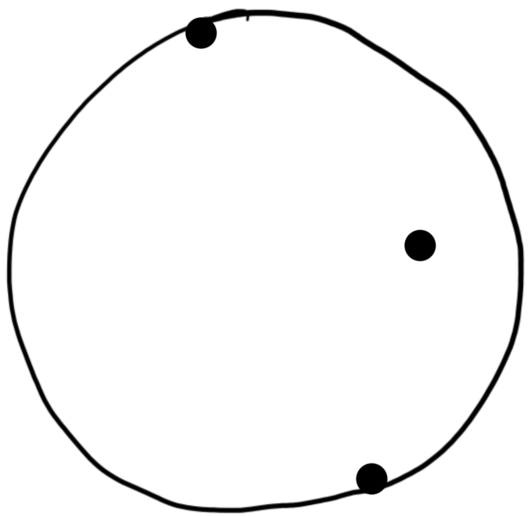
Base cases

Base case (two points):

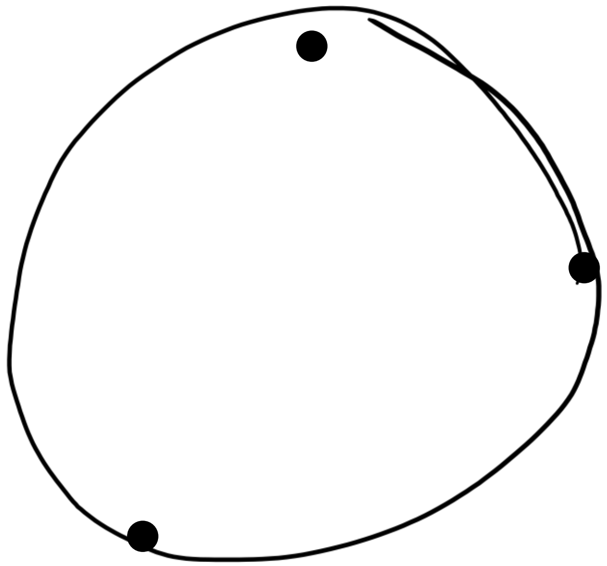


Base cases

Base case (three points):



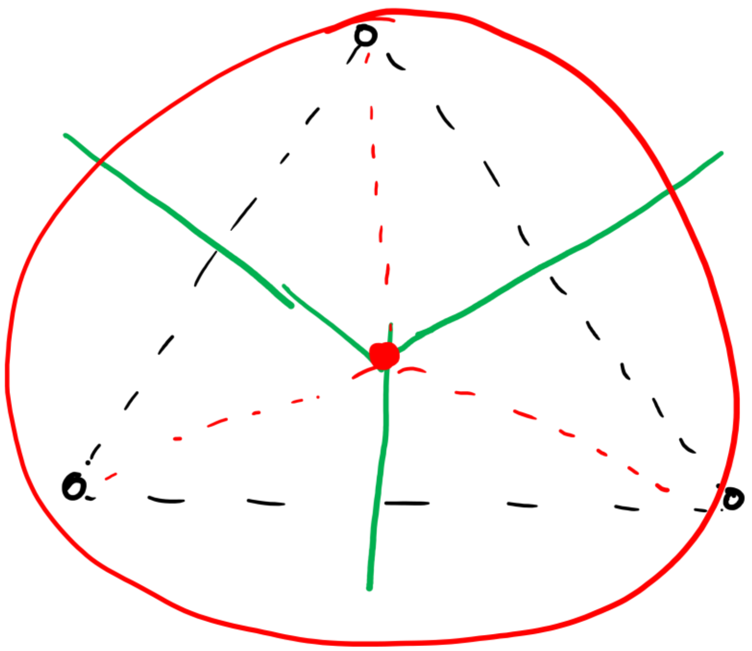
Case 1: Obtuse angle



Case 2: Acute angle

Three points and a circle

Fact (unique circle): Given three non-colinear points, there is a unique circle that goes through them



The general case

Given $n > 3$ points, how many circles do we need to consider?

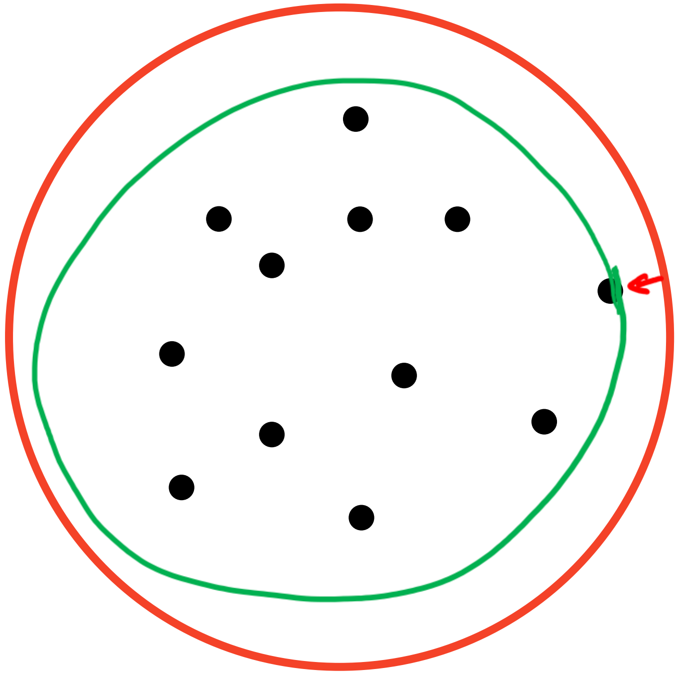
Theorem (three points is always enough): For any set of points, the smallest enclosing circle either touches two points p_i, p_j at a diameter, or touches three points p_i, p_j, p_j forming an ***acute*** triangle

In other words: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

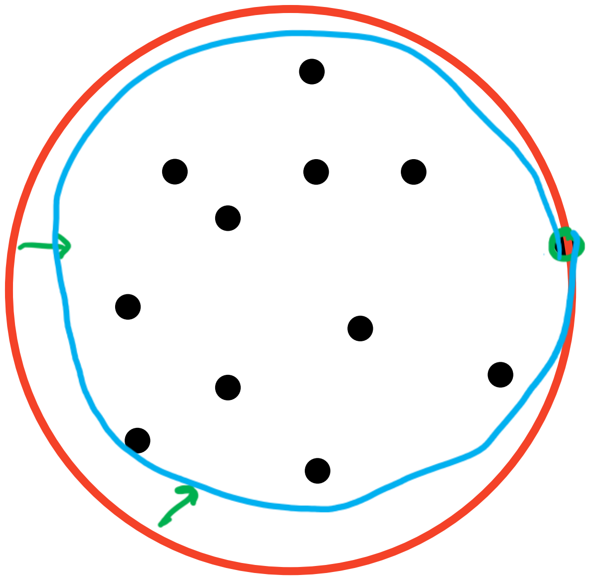
Proof of theorem

Case 1 (no points):



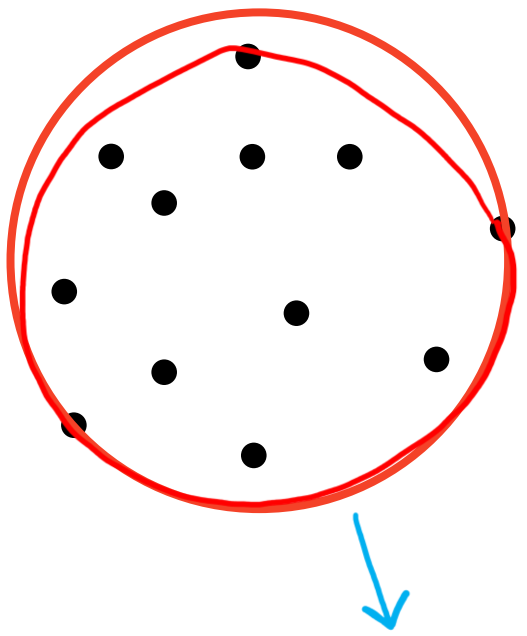
Proof of theorem

Case 2 (one point):



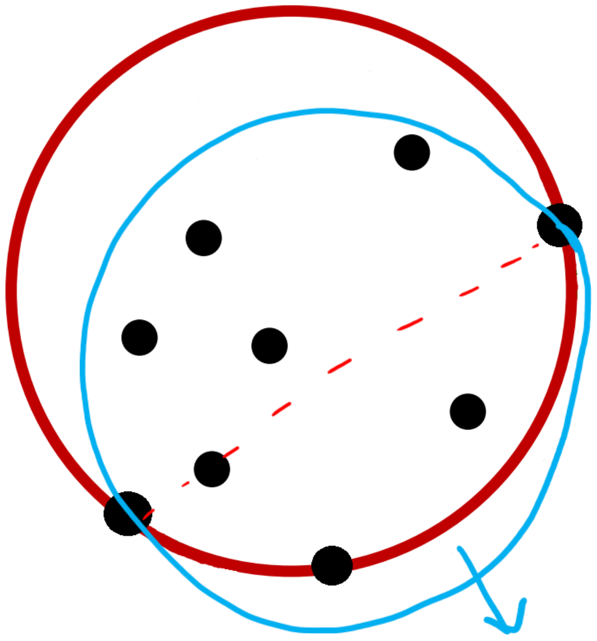
Proof of theorem

Case 3 (two points, not on a diameter):



Proof of theorem

Case 4 (three points, no acute angle):



We just proved

Theorem: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

- Either two points at a diameter, or
- Three points forming an acute triangle

Brute force algorithms

Algorithm 1 (brute force): Try all triples of points and find their smallest enclosing circle. Check whether this circle contains every point. Returns the smallest such circle.

Algorithm 2 (better brute force): Try all triples of points and find their smallest enclosing circle. Return the **largest** such circle.

Beating brute force: incremental

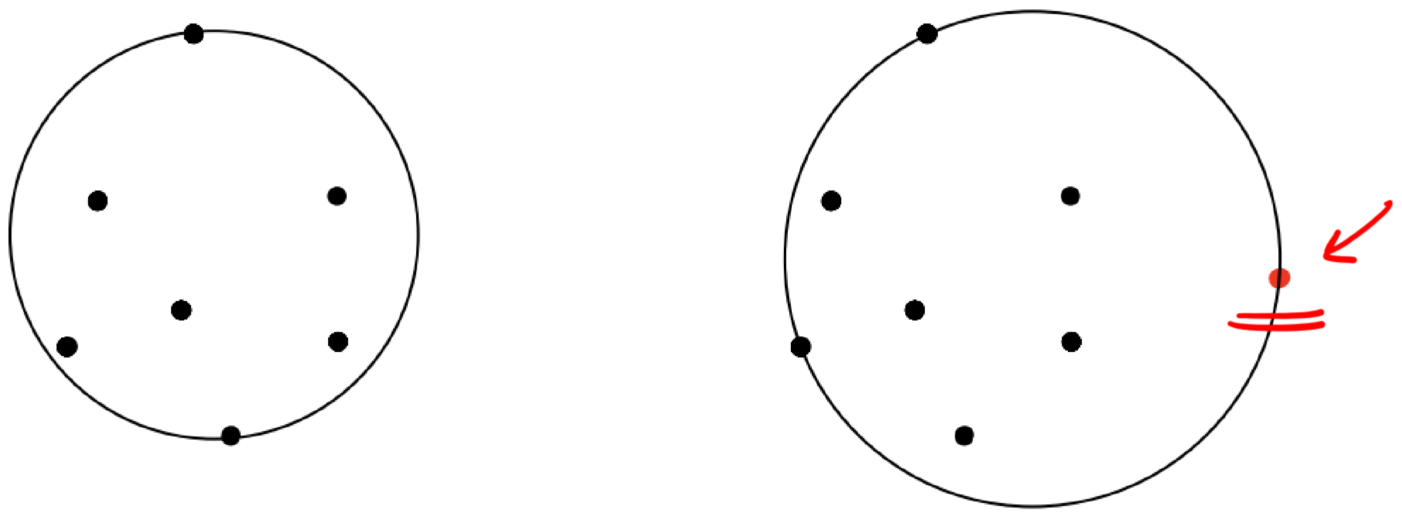
Incremental approach: Insert points one by one and maintain the smallest enclosing circle

When inserting p_i:

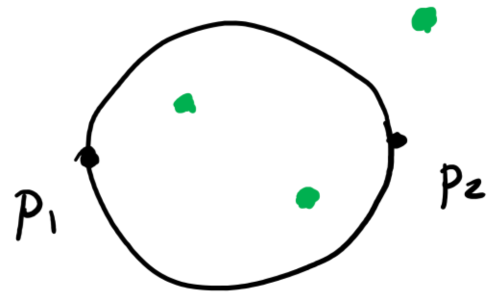
- ***Case 1:*** p_i is inside the current circle. Great, do nothing!
- ***Case 2:*** p_i is outside the current circle. Need to find the new one

Making incremental fast

Observation: When we add p_i, if it is not in the current circle, then it is on the boundary of the new circle



Incremental algorithm



SEC($[p_1, p_2, \ldots, p_n]$) = {

Let C be the smallest circle enclosing p_1 and p_2

for *i = 3 to n do* {

if p_i is not inside C **then** $C =$ SEC1($[p_1, p_2, \ldots, p_{i-1}]$, p_i)

1 point is fixed — p_i locked in

}

return C

}

Incremental algorithm continued

1 point is fixed

SEC1$([p_1, p_2, \dots, p_k], q)$ = {

Let C be the smallest circle enclosing p_1 and q

for *i = 2 to k do* {

if p_i is not inside C **then** $C =$ SEC2 ([p1, p2, ... pi-1], pi, q)

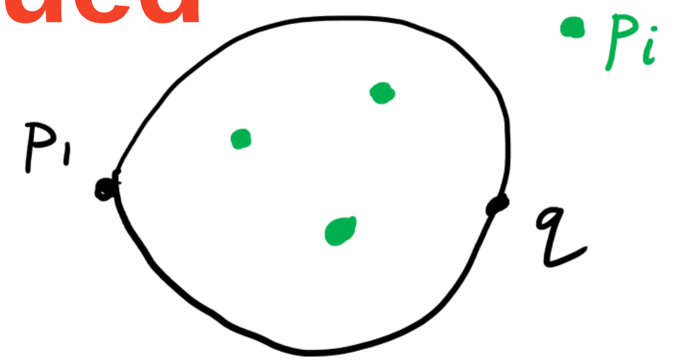
2 points fixed

2 points locked in

}

return C

}



Incremental algorithm deeper again

2 points locked in

SEC2($[p_1, p_2, \dots, p_k], q_1, q_2$) = {

Let C be the smallest circle enclosing q_1 and q_2

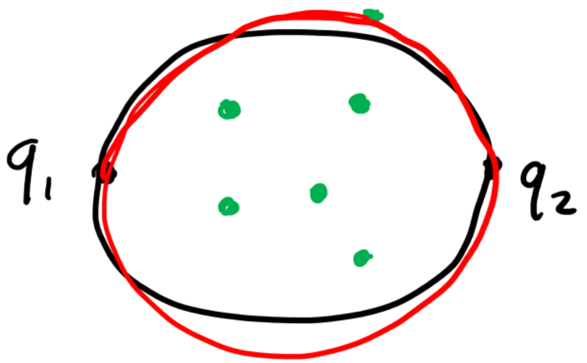
for *i = 1 to k do* {

if p_i is not inside C **then** $C =$ SEC of p_i, q_1, q_2

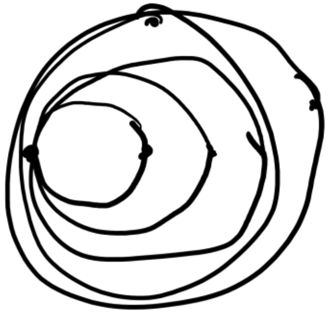
}

return C

}



Runtime



Runtime (SEC2): SEC2 runs in $O(k)$ time

Runtime (SEC1): In the worst case, SEC1 runs in $O(k^2)$ time

Runtime (SEC): In the worst case, SEC runs in $O(n^3)$ time

If answer changes every time!

Randomization to the rescue!!!

Claim (randomized SEC is fast): If we randomly shuffle the points in SEC and SEC1, then SEC1 runs in $O(k)$ expected time and SEC runs in $O(n)$ expected time

Call q is "critical" if $SEC(P_i \setminus \{q\}) \neq SEC(P_i)$

≤ 3 critical points

$Pr[q \text{ is critical}] \leq 3/i$

Same math $E[T] = O(k) + \sum_{i=1}^{k} 1 + \frac{3}{i} O(i) = O(k)$ [SEC1]

$E[T] = O(n) + \sum_{i=2}^{n} 1 + \frac{3}{i} O(i) = O(n)$ [SEC]

Summary

- **Randomized incremental algorithms** are pretty great. We can turn slow brute force algorithms into expected linear-time algorithms!
- We got $O(n)$ time for **closest pair** and **smallest enclosing circle**