Algorithm Design and Analysis

Linear Programming Part II: Duality

Roadmap for today

- **Reminder: Midterm Two is next week (see the form on Ed that you should fill out if you require a make up exam!)**
- “Standard Form” for linear programs
- Linear program duality
- Weak and strong duality theorems
- Examples of duality

Review: Formal definition

Definition (Linear program): A linear program consists of

- n real-valued **variables** $x_1, x_2, \dots, x_n$
- A linear ***objective function***, e.g., minimize/maximize $2x_1 + 3x_2 + x_3$
- m linear **inequalities**, e.g., $3x_1 + 4x_2 \leq 6$, or $0 \leq x_1 \leq 3$

Goal: Find values for x's that satisfy the constraints and minimize/maximize the objective

Standard form

- The same LP can be written in many ways
- It is convenient to have a "standard way" to write an LP

Definition (Standard Form): An LP with n variables $x_1, \dots, x_n$ and m constraints in **standard form** is written with constants $c_1, \dots, c_n$, $b_1, \dots b_m$, $a_{11}, \dots, a_{mn}$

Objective must be max, not min →

$$\textbf{maximize} \quad c_1 x_1 + \cdots + c_n x_n$$

Constraints are all $\leq$ constant →

$$\begin{aligned} \textbf{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + \cdots + a_{2n} x_n \leq b_2 \\ & \vdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m \end{aligned}$$

All variables are non-negative. (These do not count towards the m number of constraints!) →

$$x_i \geq 0 \quad \text{for all } i$$

=

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Converting to standard form

Claim: Every LP that is not written in standard form can be converted to an equivalent LP in standard form

$3x + 4y \le 10$

$3x^+ - 3x^- + 4y \le 10$

$x^+ = 0 \quad x^- = 3$

$x^+ = 1 \quad x^- = 4$

- ***How to convert minimization to maximization?***

  negate objective

- ***How to convert a $\geq$ constraint?***

  negate it

- ***How to convert an $=$ constraint?***

  convert into $\le$ and $\ge$

- ***How to convert a variable x_i which could negative?***

  two variables x_i^+ x_i^-. Change $x_i = x_i^+ - x_i^-$

Objective must be max, not min

Constraints are all $\leq$ constant

All variables are non-negative.

Motivating problem: The carpenter

- You are a carpenter. You make tables, chairs, and shelves out of wood, nails, and paint.

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	$50
Chair	4	15	3	$30
Shelf	3	5	3	$20
Stock	**100**	**300**	**80**	

- How many of each item should you make for maximum profit (ignoring rounding errors)

Motivating problem: The carpenter

$$\text{maximize} \quad 50x + 30y + 20z$$

$$8x + 4y + 3z \leq 100$$
$$20x + 15y + 5z \leq 300$$
$$5x + 3y + 3z \leq 80$$
$$x, y, z \geq 0$$

Let x = #tables, y = #chairs, z = #shelves

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	$50
Chair	4	15	3	$30
Shelf	3	5	3	$20
Stock	**100**	**300**	**80**	

Solution: $x \approx 1.82$ $y = 14.5$ $z = 9.09$

$709.09

Along comes a merchant

- Along comes a traveling merchant willing to purchase your stock of wood, nails, and paint, for a fair price.
- What is a fair price for wood, nails, and paint?
- You are not willing to sell your materials for less than the amount you could make by turning them into items

Along comes a merchant

minimize $100w + 300s + 80p$

$$8w + 20s + 5p \geq 50$$
$$4w + 15s + 3p \geq 30$$
$$3w + 5s + 3p \geq 20$$
$$w, s, p \geq 0$$

Let w = \$wood , s = \$nails, p = \$paint

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	$50
Chair	4	15	3	$30
Shelf	3	5	3	$20
Stock	**100**	**300**	**80**	

Solution: $w \approx 2.73$, $s = 0.73$, $p = 2.73$

$709.09

A tale of two LPs

$$\begin{aligned} \text{maximize} \quad & 50x + 30y + 20z \\ \text{subject to} \quad & 8x + 4y + 3z \le 100 \\ & 20x + 15y + 5z \le 300 \\ & 5x + 3y + 3z \le 80 \\ & x, y, z \ge 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 100w + 300s + 80p \\ \text{subject to} \quad & 8w + 20s + 5p \ge 50 \\ & 4w + 15s + 3p \ge 30 \\ & 3w + 5s + 3p \ge 20 \\ & w, s, p \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 20 & 15 & 5 \\ 5 & 3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 300 \\ 80 \end{bmatrix}, c = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & b^T y \\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

The dual program

Definition (Dual): Given a standard-form LP, its **dual** is

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & A^T\boldsymbol{y} \geq \boldsymbol{c} \\ & \boldsymbol{y} \geq 0 \end{array}$$

- The original problem is called the **primal problem**
- If the primal has n variables and m constraints, the dual has m variables and n constraints, i.e., variables and constraints swap roles!

Exercise: Show that the dual of the dual is the primal. This shows that which one you call the primal and which you call the dual is arbitrary

Theorems

Primal LP

maximize $\boldsymbol{c}^T\boldsymbol{x}$
subject to $A\boldsymbol{x} \le \boldsymbol{b}$ ←
$\boldsymbol{x} \ge 0$

Dual LP

minimize $\boldsymbol{b}^T\boldsymbol{y}$
subject to $A^T\boldsymbol{y} \ge \boldsymbol{c}$ ←
$\boldsymbol{y} \ge 0$

Theorem (Weak Duality): If x is any feasible solution to the primal LP and y is any feasible solution to the dual LP

$$c^T x \le b^T y$$

Proof. $\underline{c^T x} \le (A^T y)^T x = (y^T A) x = y^T (A x) \le y^T b = \underline{b^T y}$

Theorems

Primal LP

maximize $\boldsymbol{c}^T\boldsymbol{x}$
subject to $A\boldsymbol{x} \leq \boldsymbol{b}$
$\boldsymbol{x} \geq 0$

Dual LP

minimize $\boldsymbol{b}^T\boldsymbol{y}$
subject to $A^T\boldsymbol{y} \geq \boldsymbol{c}$
$\boldsymbol{y} \geq 0$

Theorem (Strong Duality): If the primal problem is feasible and bounded, then the dual is feasible and bounded. If x^* is an optimal solution to the primal LP and y^* is an optimal solution to the dual LP

$$c^T x^* = b^T y^*$$

Proof: Too long.

Consequences

$-\infty$

$c^T x \le b^T y$

∞

- Suppose the primal problem is unbounded, what can we say about the dual? Dual is infeasible

- Similarly, suppose the dual problem is unbounded, what can we say about the primal? Primal infeasible

- ***Consequence***: It is impossible for both the primal and dual to be unbounded. "$\infty \le -\infty$"

Application

Zero-sum games, again

$$Ax \le b$$

Variables: $p_1, \ldots, p_n$ and v.
Objective: Maximize v.
Constraints:

- $p_i \geq 0$ for all $1 \leq i \leq n$,
- $\sum_{i=1}^{n} p_i = 1$. (the p_i form a probability distribution)
- $\sum_{i=1}^{n} p_i R_{ij} \geq v$ for all columns $1 \leq j \leq m$

- Let R denote the payoff matrix
- Assume WLOG $R_{ij} \geq 0$
- Solution of this LP is

$$\text{lb}^* = \max_{\boldsymbol{p}} \min_j \sum_i p_i R_{ij}$$

$$v - \sum_{i=1}^{n} p_i R_{ij} \leq 0 \quad \forall j$$

$$\sum_{i=1}^{n} p_i \leq 1$$

$$\boldsymbol{x} = \begin{pmatrix} v \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} 1 & \\ 1 & \\ 1 & -R^T \\ 1 & \\ 1 & \\ 0 & 1 \; 1 \; \cdots \; 1 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Zero-sum games, the dual

LP for lb^* is dual of LP for ub^*

$$x = \begin{bmatrix} v \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & & -R^T & \\ 1 & & & \\ \vdots & & & \\ 0 & 1 & \cdots & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dual LP

minimize $\mathbf{b}^T \mathbf{y}$
subject to $A^T \mathbf{y} \geq \mathbf{c}$
$\mathbf{y} \geq 0$

Call the dual variables $\mathbf{y} = [q_1, q_2, \ldots, q_m, v']^T$

minimize v'

$$A^T = \begin{bmatrix} 1 \; 1 \cdots & \cdots 1 & 0 \\ & & 1 \\ -R & & \vdots \\ & & 1 \end{bmatrix}$$

$\rightarrow q_1 + q_2 + \ldots + q_m \geq 1$

$-q_1 R_{i1} - q_2 R_{i2} - \ldots + v' \geq 0 \quad \forall i$

minimize v'

$\sum q_i = 1$

$\sum q_j R_{ij} \leq v' \quad \forall$ rows i

$q_i \geq 0$

Corollary: Minimax theorem

Theorem (Minimax): Given a finite 2-player zero-sum game with row payoff matrix R

$$\mathrm{lb}^* = \max_{\boldsymbol{p}} \min_{j} \sum_{i} p_i R_{ij} = \min_{\boldsymbol{q}} \max_{i} \sum_{j} q_i R_{ij} = \mathrm{ub}^*$$

Proof: Strong duality of the LPs from the last two slides!

Take-home messages

- ***Duality*** gives us a powerful tool to prove see a problem in an equivalent but different form
- The ***strong and weak duality theorems*** tell us about the relationship between the primal and dual problem
- Duality can be used to prove equivalence between two problems (e.g., proving the ***minimax theorem***)