Algorithm Design and Analysis

Linear Programming Part II: Duality

Roadmap for today

- Reminder: Midterm Two is next week (see the form on Ed that you should fill out if you require a make up exam!)
- "Standard Form" for linear programs
- Linear program duality
- Weak and strong duality theorems
- Examples of duality

Review: Formal definition

Definition (Linear program): A linear program consists of

- n real-valued variables $x_1, x_2, ..., x_n$
- A linear *objective function*, e.g., minimize/maximize $2x_1 + 3x_2 + x_3$
- m linear inequalities, e.g., $3x_1 + 4x_2 \le 6$, or $0 \le x_1 \le 3$

Goal: Find values for x's that satisfy the constraints and minimize/maximize the objective

Standard form

- The same LP can be written in many ways
- It is convenient to have a "standard way" to write an LP

Definition (Standard Form): An LP with n variables x_1, \dots, x_n and m constraints in **standard form** is written with constants $c_1, \ldots, c_n, b_1, \ldots b_m, a_{11}, \ldots, a_{mn}$

Objective must be max, not min

maximize $c_1x_1 + \cdots + c_nx_n$

Constraints are $all \leq constant$

subject to $a_{11}x_1 + \cdots + a_{1n}x_n \le b_1$ $a_{21}x_1 + \dots + a_{2n}x_n \le b_2$

 $a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$

maximize $c^T x$ subject to $Ax \leq b$ $x \ge 0$

All variables are nonnegative. (These do not count towards the m. number of constraints!)

 $x_i \ge 0$ for all i

Converting to standard form

Claim: Every LP that is not written in standard form can be converted to an equivalent LP in standard form

$$3x + 4y \le 10$$

$$3x^{+} - 3x^{-} + 4y \le 10$$

$$x^{+} = 0 \quad x^{-} = 3$$
Objective must

x+=1 x=4

- How to convert minimization to maximization?
- negate Objective

 How to convert a > constraint?
- How to convert a ≥ constraint?
 Megate if
- How to convert an = constraint?
- How to convert a variable x_i which could negative?

two variables
$$c_i^{\dagger}$$
 c_i^{\dagger} . Change $c_i^{\dagger} = c_i^{\dagger} - c_i^{\dagger}$

Constraints are all < constant

All variables are nonnegative.

Motivating problem: The carpenter

 You are a carpenter. You make tables, chairs, and shelves out of wood, nails, and paint.

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	\$50
Chair	4	15	3	\$30
Shelf	3	5	3	\$20
Stock	100	300	80	

 How many of each item should you make for maximum profit (ignoring rounding errors)

Motivating problem: The carpenter

maximize
$$50x + 30y + 20z$$

 $8x + 4y + 3z \le 100$
 $20x + 15y + 5z \le 300$
 $8x + 3y + 3z \le 80$
 $8x + 3y + 3z \le 80$

Let
$$x = \#$$
tables, $y = \#$ chairs, $z = \#$ shelves

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	\$50
Chair	4	15	3	\$30
Shelf	3	5	3	\$20
Stock	100	300	80	

Solution:
$$x \approx 1.82$$
 $y = 14.5$ $z = 9.09$ $$\$ 709.09$

Along comes a merchant

- Along comes a traveling merchant willing to purchase your stock of wood, nails, and paint, for a fair price.
- What is a fair price for wood, nails, and paint?
- You are not willing to sell your materials for less than the amount you could make by turning them into items

Along comes a merchant

minimize
$$100w + 300s + 80p$$

 $8w + 20s + 5p > 50$
 $4w + 15s + 3p > 30$
 $3w + 5s + 3p > 20$
 $w, s, p > 0$

Let
$$w = \$ wood$$
, $s = \$ nails$, $p = \$ paint$

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	\$50
Chair	4	15	3	\$30
Shelf	3	5	3	\$20
Stock	100	300	80	

Solution:
$$\omega \approx 2.73$$
, $S = 0.73$, $P = 2.73$
\$709.09

A tale of two LPs

maximize
$$50x + 30y + 20z$$

subject to $8x + 4y + 3z \le 100$
 $20x + 15y + 5z \le 300$
 $5x + 3y + 3z \le 80$
 $x, y, z \ge 0$

minimize
$$100w + 300s + 80p$$

subject to $8w + 20s + 5p \ge 50$
 $4w + 15s + 3p \ge 30$
 $3w + 5s + 3p \ge 20$
 $w, s, p \ge 0$

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 20 & 15 & 5 \\ 5 & 3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 300 \\ 80 \end{bmatrix}, \quad c = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$m \sim X \subset X$$

$$A \times \leq b$$

$$\times > 0$$

min
$$b^{T}y$$

$$A^{T}y \geqslant c$$

$$y \geqslant 0$$

The dual program

Definition (Dual): Given a standard-form LP, its dual is

maximize
$$c^T x$$
 minimize $b^T y$ subject to $Ax \le b$ subject to $A^T y \ge c$ $x \ge 0$

- The original problem is called the primal problem
- If the primal has n variables and m constraints, the dual has m variables and n constraints, i.e., variables and constraints swap roles!

Exercise: Show that the dual of the dual is the primal. This shows that which one you call the primal and which you call the dual is arbitrary

Theorems

Primal LP

maximize
$$c^T x$$
 minimize $b^T y$ subject to $Ax \le b \leftarrow$ subject to $A^T y$ $x \ge 0$ $y \ge$

Dual LP

subject to $Ax \le b \leftarrow$ subject to $A^Ty \ge c \leftarrow$ $y \ge 0$

Theorem (Weak Duality): If x is any feasible solution to the primal LP and y is any feasible solution to the dual LP

$$c^T x \le b^T y$$

Proof.
$$C^T x \leq (A^T y)^T x = (y^T A) x = y^T (A x) \leq y^T b = b^T y$$

Theorems

Primal LP

maximize $c^T x$ minimize $b^T y$ subject to $Ax \leq b$ $x \ge 0$

Dual LP

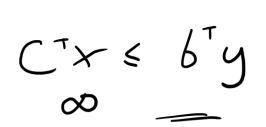
subject to $A^T y \ge c$ $y \ge 0$

Theorem (Strong Duality): If the primal problem is feasible and bounded, then the dual is feasible and bounded. If x^* is an optimal solution to the primal LP and y^* is an optimal solution to the dual LP

$$c^T x^* = b^T y^*$$

Proof: Too long.

Consequences



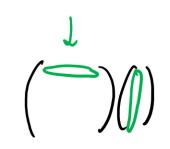
- Suppose the primal problem is unbounded, what can we say about the dual?

 Dual & infersible
- Similarly, suppose the dual problem is unbounded, what can we say about the primal?

 Primal inferible
- **Consequence**: It is impossible for both the primal and dual to be unbounded. " → ≤ ∞ "

Application

Zero-sum games, again (f) () Ax ≤ b



Variables: p_1, \ldots, p_n and v.

Objective: Maximize v.

Constraints:

- $p_i \ge 0$ for all $1 \le i \le n$,
- $\sum_{i=1}^{n} p_i = 1$. (the p_i form a probability distribution)
- $-\sum_{i=1}^{n} p_i R_{ij} \ge v \qquad \text{for all columns } 1 \le j \le m$

- Let *R* denote the payoff matrix
- Assume WLOG $R_{ij} \ge 0$
- Solution of this LP is

$$lb^* = \max_{p} \min_{j} \sum_{i} p_i R_{ij}$$

$$v - \sum_{i=1}^{n} \underline{p_i R_{ij}} \le \underline{0} \qquad \forall$$

$$\sum_{i=1}^{n} p_i \le \underline{1}$$

$$x = \begin{pmatrix} V \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & -\mathbf{R}^{\mathsf{T}} \\ \mathbf{1} & -\mathbf{R}^{\mathsf{T}} \\ \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix} \mathbf{b}$$

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

LP for 18 × 1s dual of LP for ub ×

Zero-sum games, the dual

$$\boldsymbol{x} = \begin{bmatrix} v \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \quad 1 \quad \cdots \quad 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dual LP

minimize
$$\boldsymbol{b}^T \boldsymbol{y}$$
 subject to $A^T \boldsymbol{y} \ge \boldsymbol{c}$ $\boldsymbol{y} \ge 0$

Call the dual variables
$$\mathbf{y} = [q_1, q_2, ..., q_m, v']^T$$
 minimize \mathbf{V}'

Call the dual variables
$$y = [q_1, q_2, ..., q_m, v]^T$$

$$A^T = \begin{bmatrix} 1 & 1 & ... & 1 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & &$$

Corollary: Minimax theorem

Theorem (Minimax): Given a finite 2-player zero-sum game with row payoff matrix R

$$\underline{\underline{lb}^*} = \max_{p} \min_{j} \sum_{i} p_i R_{ij} = \min_{q} \max_{i} \sum_{j} q_i R_{ij} = \underline{\underline{ub}^*}$$

Proof: Strong duality of the LPs from the last two slides!

Take-home messages

- Duality gives us a powerful tool to prove see a problem in an equivalent but different form
- The *strong and weak duality theorems* tell us about the relationship between the primal and dual problem
- Duality can be used to prove equivalence between two problems (e.g., proving the *minimax theorem*)