Algorithm Design and Analysis

Network Flows Part I: Flows, Cuts, and Matchings

Roadmap for today

- Define the ***maximum network flow*** problem
- Learn the ***Ford-Fulkerson*** algorithm for maximum flow
- Define the related ***minimum cut*** problem
- See and prove the ***min-cut max-flow theorem***
- Study an application of network flow ***(bipartite matchings)***

Definition: Network flow

Motivation (Flow network): Consider a network of pipes, each able to handle a certain number of liters of water per minute.

How much water can you send from s to t?

Definition (Value of a flow): The ***value*** of a flow is the net flow leaving s

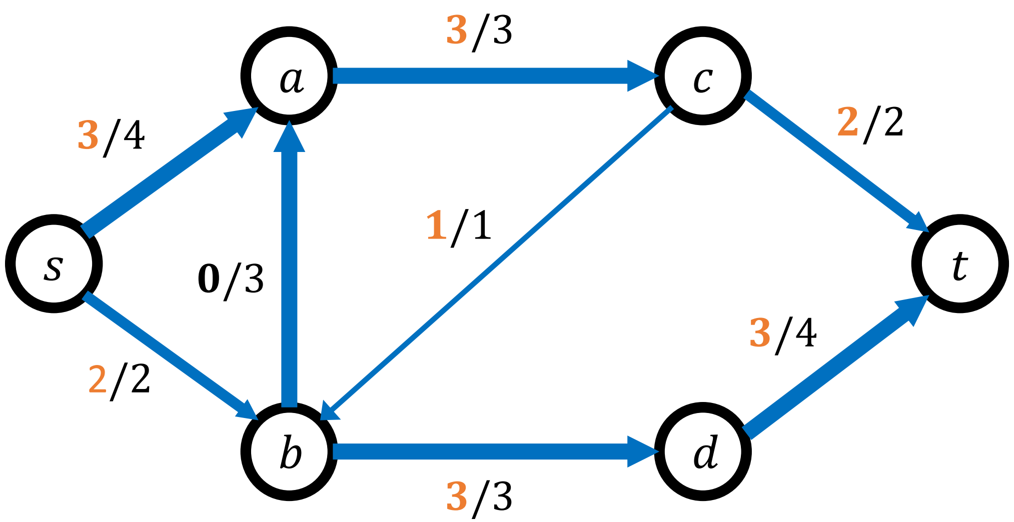
Problem (Maximum flow): Find a feasible flow with the maximum possible value of all such flows.

Definition (Flow network): A directed graph with

- Edge ***capacities*** $c(u, v)$
- A ***source*** vertex s
- A ***sink*** vertex t

Definition (A flow): A quantity of flow on each edge, $f: E \to \mathbb{R}$, called ***feasible*** if:

- **Conservation:** Flow in = Flow out $\forall v \notin \{s, t\}$
- **Capacity:** $0 \le f(u, v) \le c(u, v)$



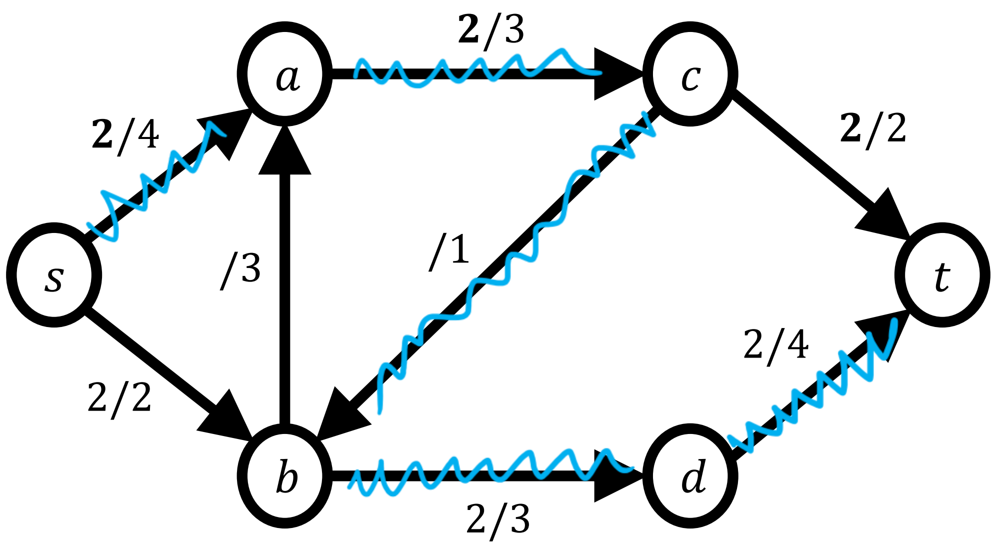
Improving a flow: *s*-*t* paths

- Is the flow on the right optimal?

s → a → c → b → d → t

has capacity available!

+1 flow



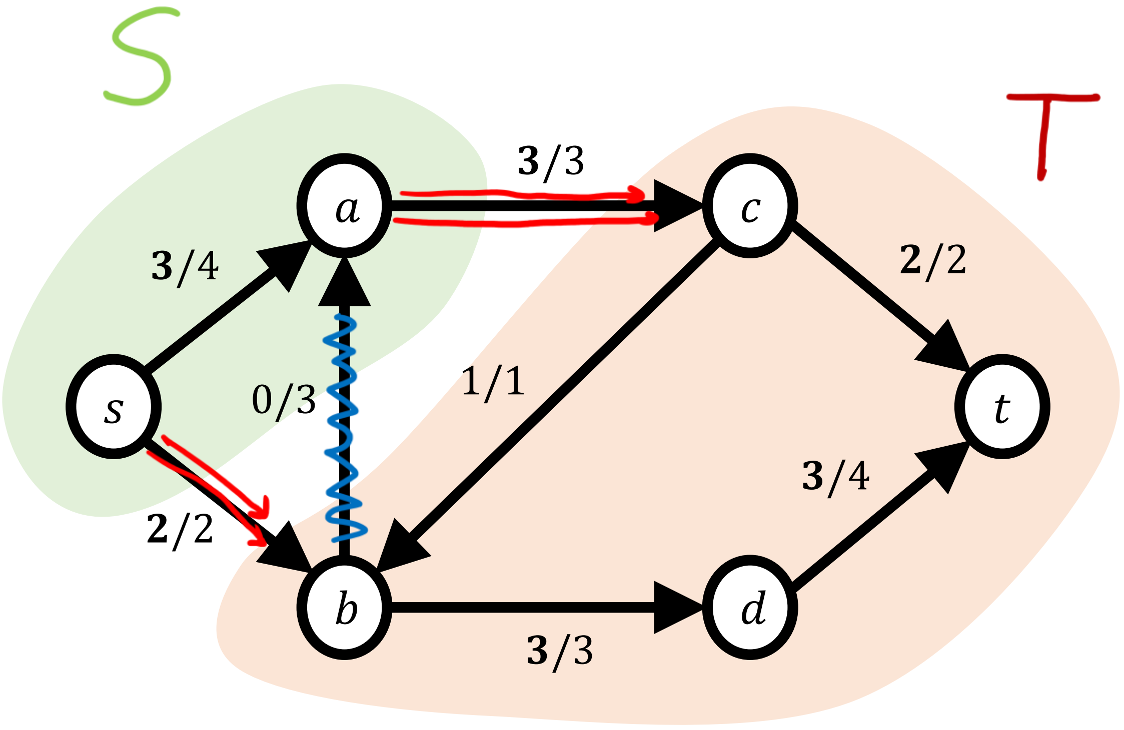
Certifying optimality: *s*-*t* cuts

- Is the flow on the right optimal?

Definition (*s*-*t* Cut): An ***s*-*t* cut** is a partition of the vertices into two disjoint sets (S, T) such that $s \in S$ and $t \in T$

Definition (Capacity): The ***capacity*** of an s-t cut (S, T) is the total capacity on edges (u, v) where $u \in S$ and $v \in T$:

$$\text{cap}(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$



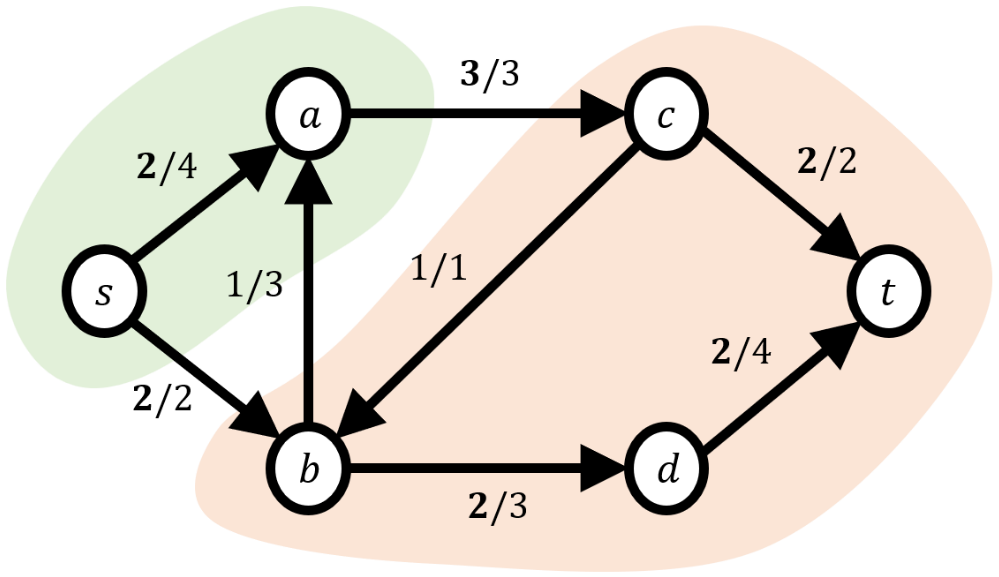
Net flow across a cut

Definition (Net flow): The ***net flow*** across an s-t cut (S,T) is the amount of flow moving from S to T:

$$f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u)$$

Observe: The value of a flow (which we defined as the net flow out of s) is the net flow across the cut $(\{s\}, V\setminus\{s\})$

Theorem: For any s-t cut (S,T), the net flow across the cut equals the value of the flow!



Proof: Algebra using the definitions.

Net flow theorem

Theorem: For any s-t cut (S, T):

$$f(S,T) \le \text{cap}(S,T)$$

Proof:

$$f(S,T) = \sum_u \sum_v f(u,v) - \sum_u \sum_v f(v,u)$$

$$\le \sum_u \sum_v c(u,v) - \sum \sum f(v,u)$$

$$\le \sum \sum c(u,v) = \text{cap}(S,T)$$

Corollary: max-flow $\le$ min-cut

Proof: any flow $\le$ max flow $\le$ (=) min cut $\le$ any cut

Definition (Capacity): The ***capacity*** of an s-t cut (S, T) is the total capacity on edges (u, v) where $u \in S$ and $v \in T$:

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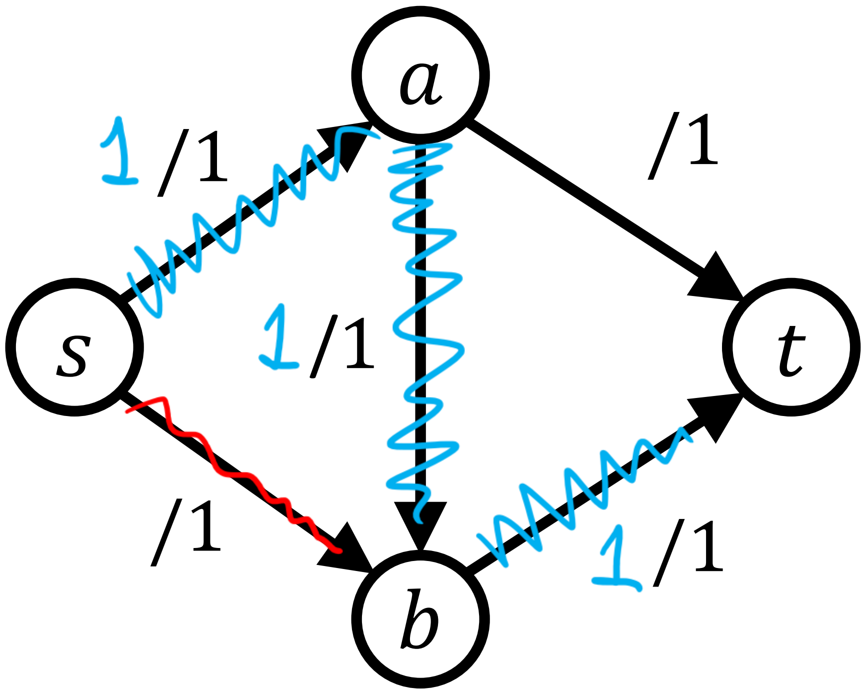
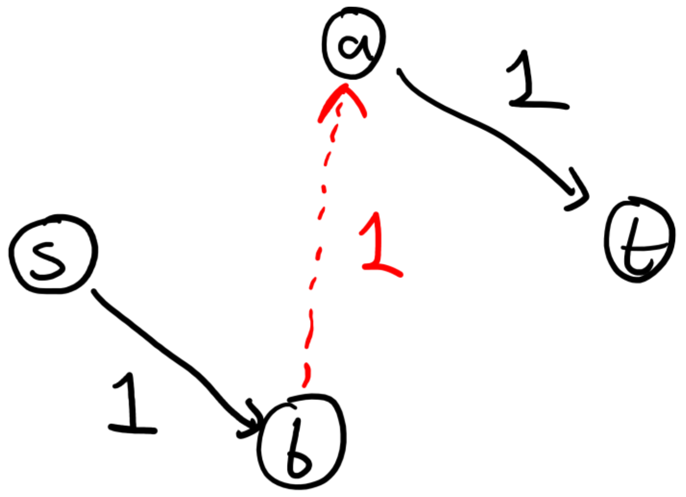
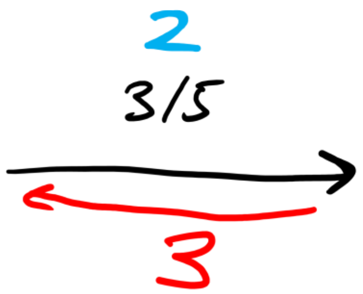
Is greedy optimal?



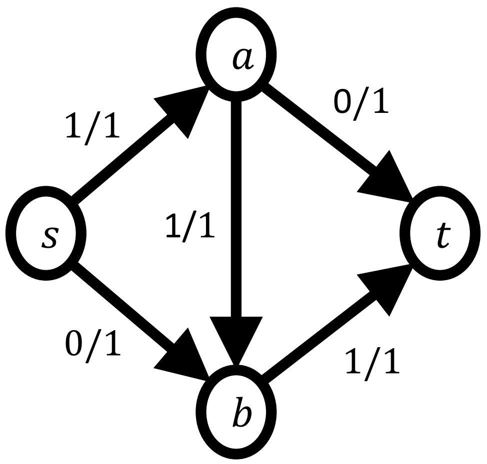


The residual graph

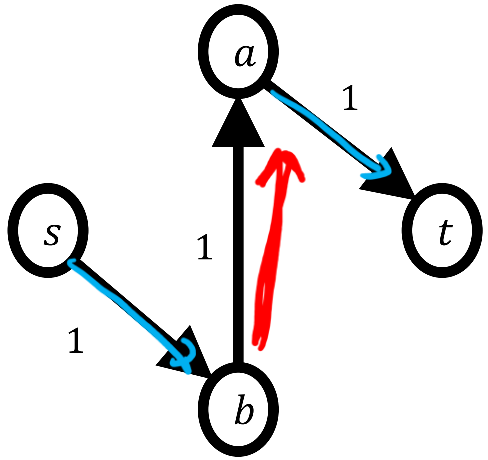


Definition (residual capacity): An edge (u, v) with capacity $c(u, v)$ and current flow $f(u, v)$ has ***residual capacity***

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v), & (u, v) \in E \\ f(v, u), & (v, u) \in E \end{cases}$$

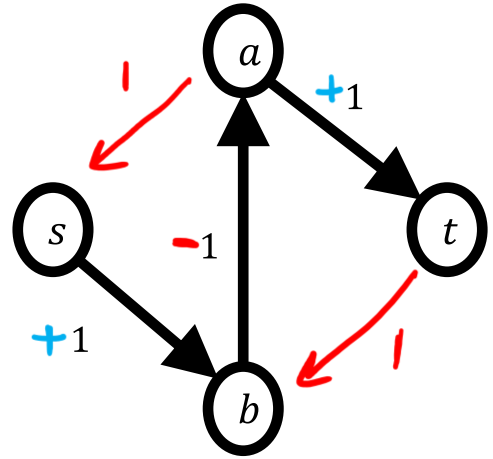


Definition (residual network): Given a flow network G and a current flow f, the ***residual network*** G_f is a flow network whose capacities are the residual capacities $c_f(u, v)$

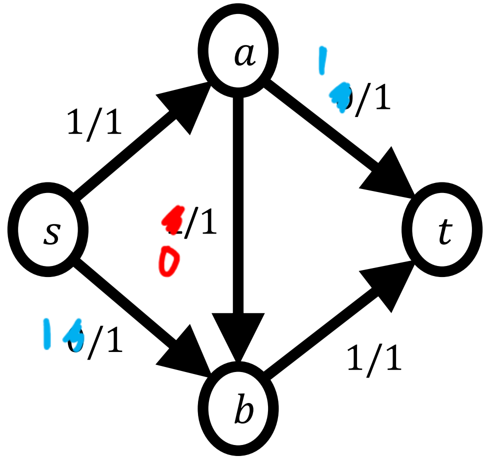


Augmenting paths

G_f:



G:



Definition (augmenting path): An ***augmenting path*** is a path from s to t of non-zero capacity in the residual network.

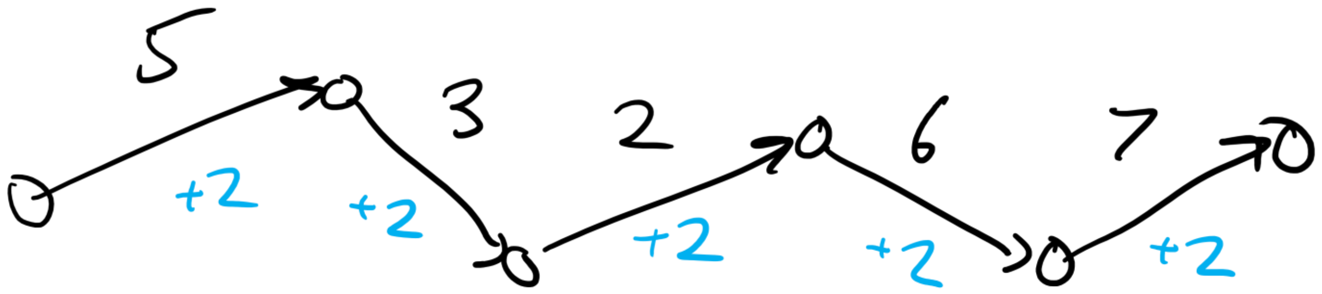
Key idea (reverse edges): Augmenting along a ***reverse edge*** removes that amount of flow from the edge.

The Ford-Fulkerson Algorithm

Algorithm (Ford-Fulkerson):

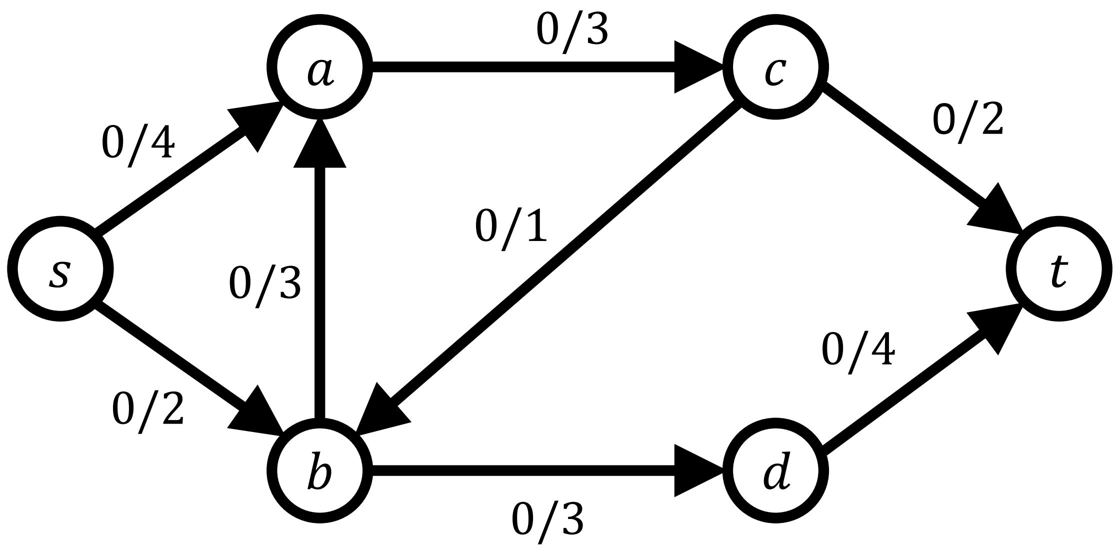
while ∃ an augmenting path (using DFS or BFS)
    add flow to it (as much as possible)

"bottleneck"

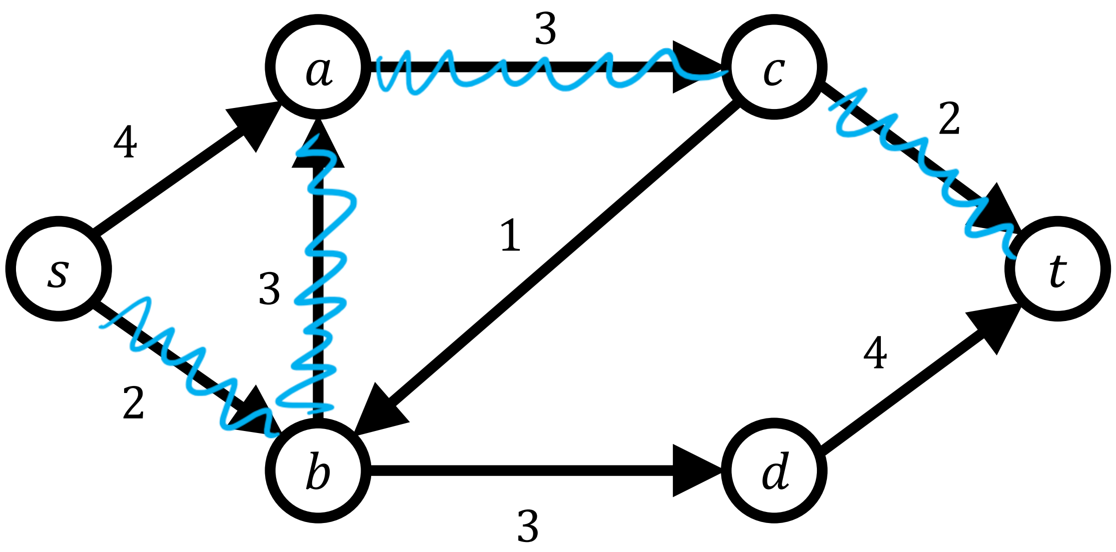


Example

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E. \end{cases}$$



Flow network G



Residual network G_f

Example

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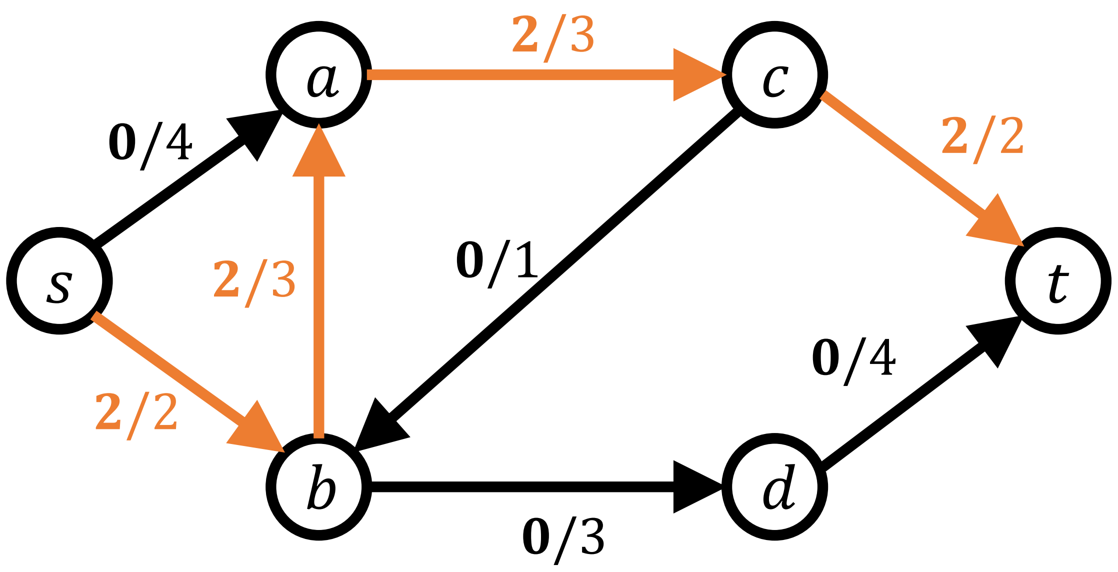

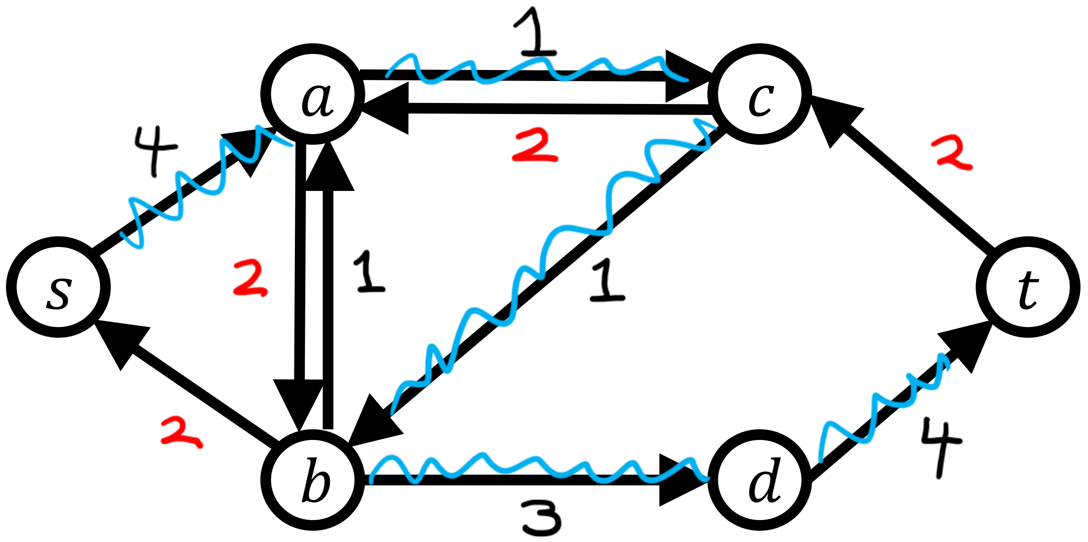

Flow network G

Residual network G_f

Example

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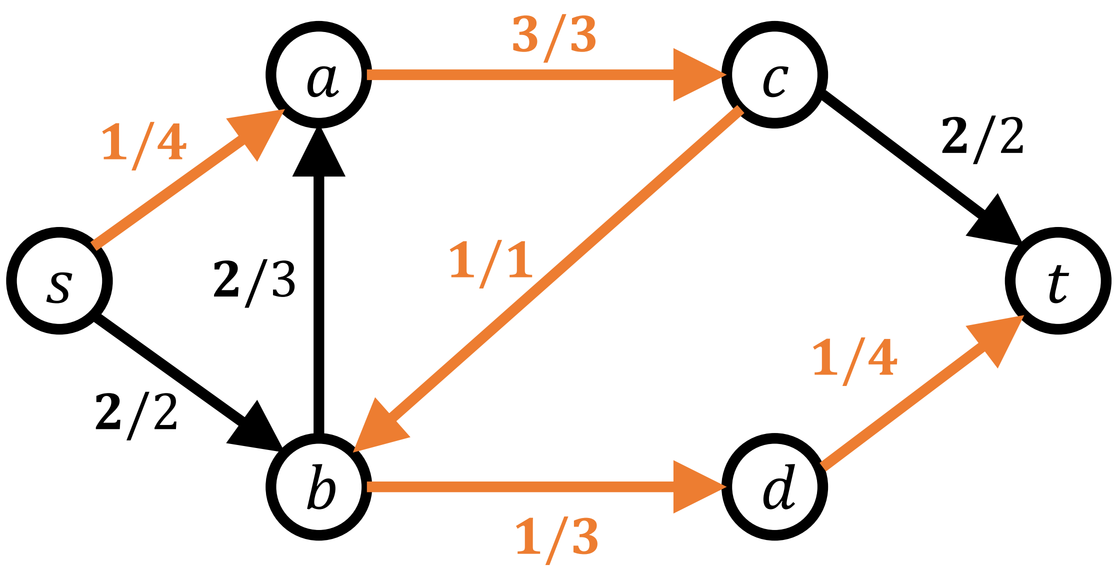
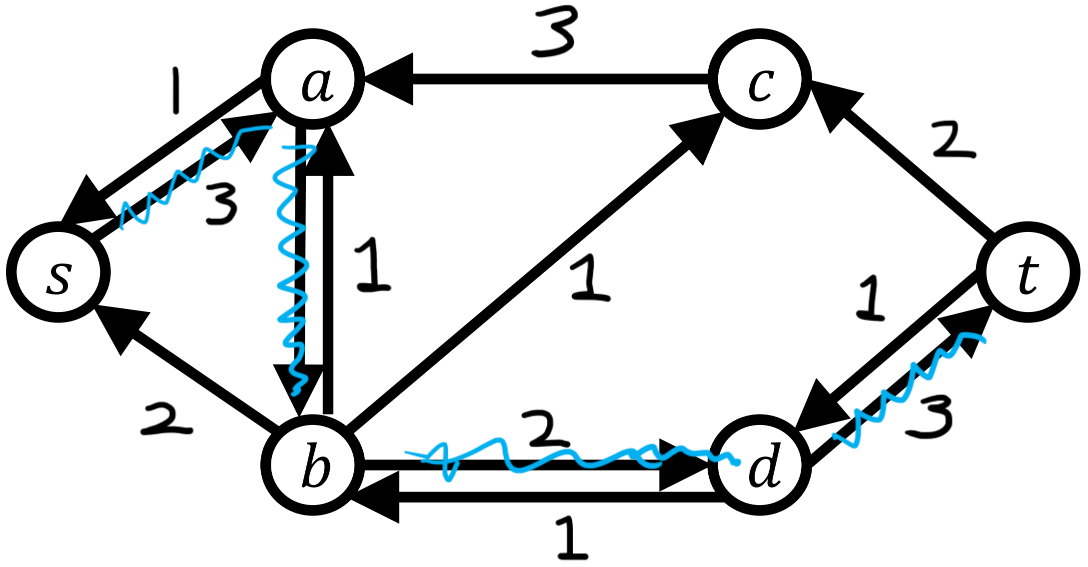


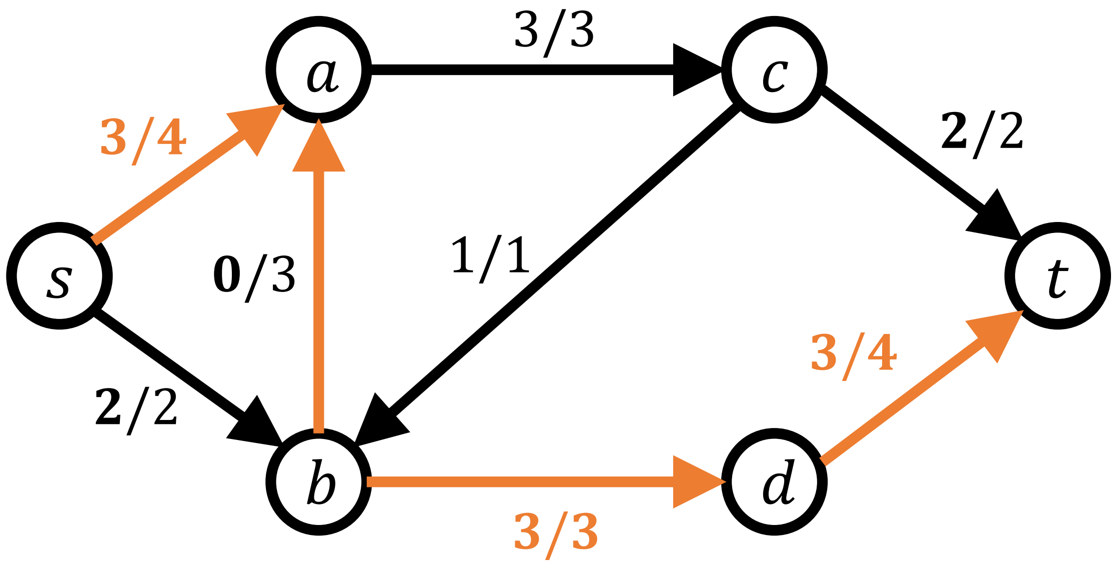
Flow network G



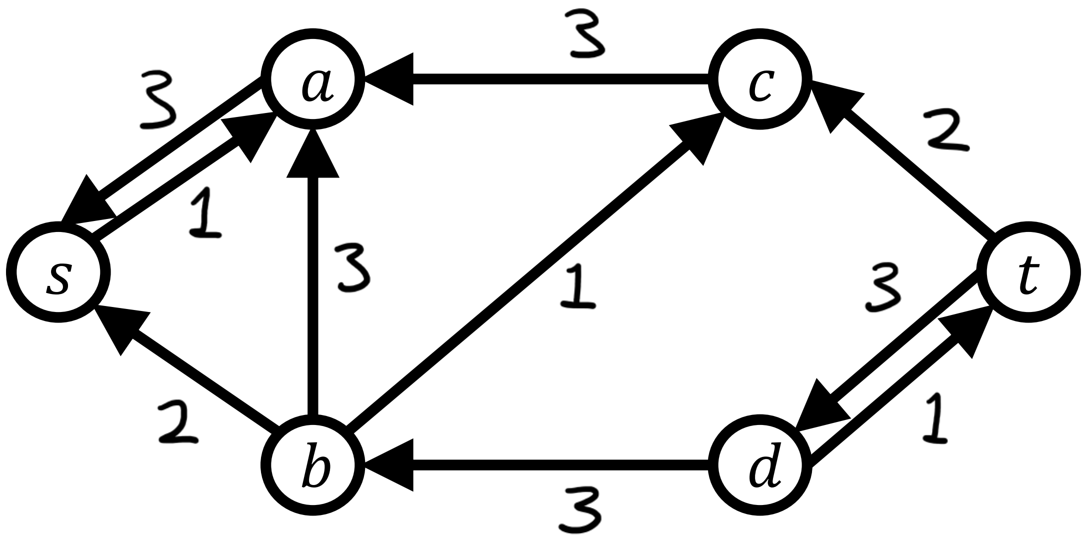
Residual network G_f

Example

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E. \end{cases}$$



Flow network G



Residual network G_f

Analysis

Analysis

Theorem (runtime): ***Assuming integer capacities,*** Ford-Fulkerson runs in $O(mF)$ time, where F is the value of the maximum s-t flow

Proof:

+1 flow per iteration

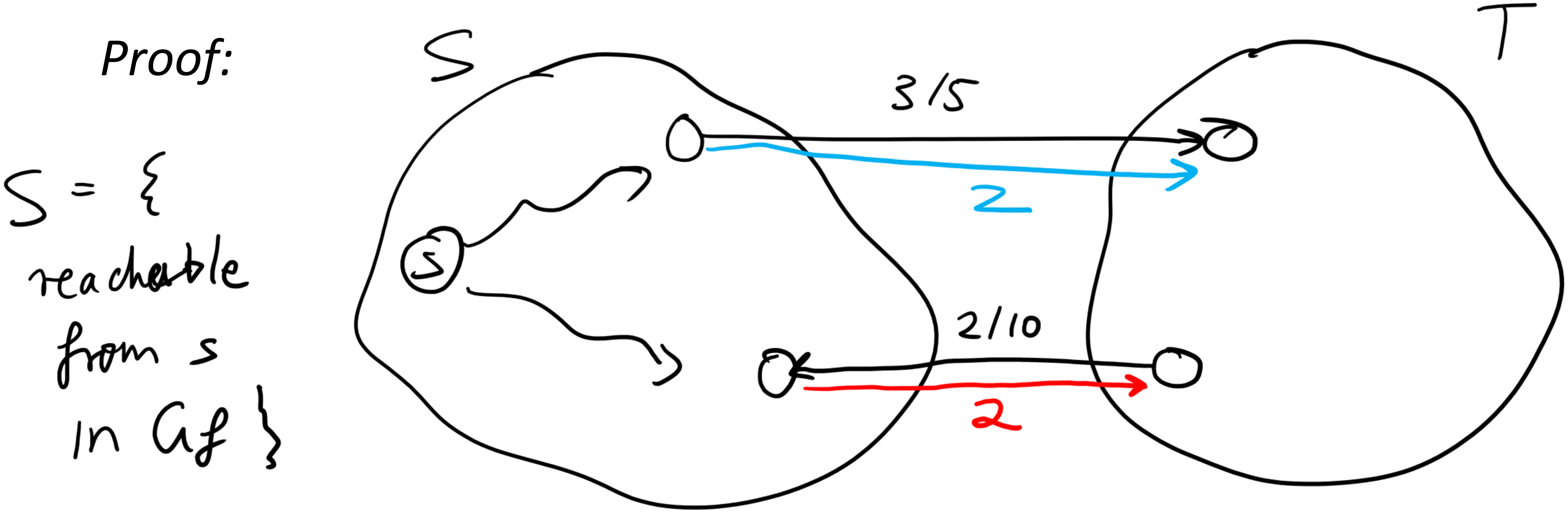
=> $\leq F$ iterations

$O(n+m)$ $O(m)$ for connected

Analysis

Theorem (maximality): Ford-Fulkerson finds a flow whose value is equal to the capacity of the minimum cut.

Proof:



Analysis

Corollary (*Min-cut Max-flow theorem*): For any flow network, the value of the maximum s-t flow is equal to the capacity of the minimum s-t cut

Proof:

The projector stopped working at this point. Please read notes!

Analysis

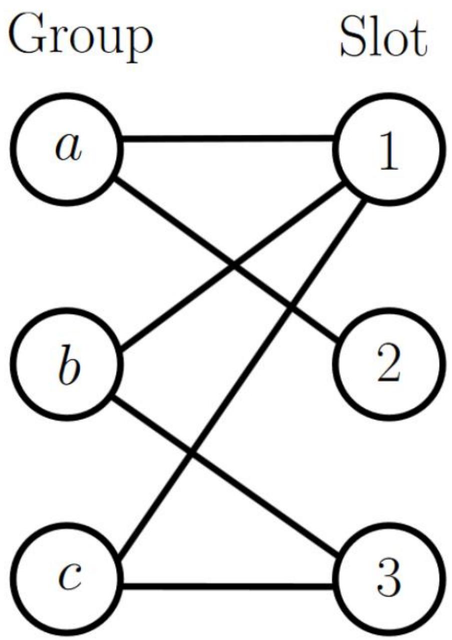
Theorem (Integral flows): For any flow network with integer capacities, there exists a maximum flow in which the flow on every edge is an integer

Proof:

Applications

Bipartite Matching

Problem (Bipartite matching): Given a bipartite graph G, find a largest possible set of edges with no endpoints in common.



Analysis of matching

Important (flow model proofs): When modeling problems with flow, you need to prove that the reduction is correct! This usually consists of a bidirectional proof.

*Claim #1 Given a matching M in the original graph, there exists a flow f in our flow network of value $|M|$ ($\Rightarrow$ **max flow** $\geq$ **max-matching**)*

Analysis of matching

Important (flow model proofs): When modeling problems with flow, you need to prove that the reduction is correct! This usually consists of a bidirectional proof.

*Claim #2: Given a flow f in our flow network, there exists a matching M of size $|f|$ in the original graph ($\Rightarrow$ **max-flow** $\leq$ **max-matching**)*

Summary of flow fundamentals

- Lots of definitions! Take time to review them.
  - Flow networks, flows, s-t cuts, residual networks, augmenting paths
- The Ford-Fulkerson algorithm
  - Finds a maximum flow (in an integer-capacity network) in $O(mF)$ time
- Modeling with flows
  - Bipartite matching
  - Remember that the proofs must show both directions of correspondence!