# Topic 2: Concrete Models and Tight Upper and Lower Bounds

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# Theme: Tight Upper and Lower Bounds

- Number of comparisons to sort an array

- Number of exchanges to sort an array

- Number of comparisons needed to find the largest and second-largest elements in an array

# Formal Model

- Look at models which specify exactly which operations may be performed on the input, and what they cost
  - E.g., performing a comparison, or swapping a pair of elements

- An upper bound of f(n) means the algorithm takes at most f(n) steps on any input of size n

- A lower bound of g(n) means for any algorithm there exists an input for which the algorithm takes at least g(n) steps on that input

# Sorting in the Comparison Model

- In the comparison model, we have n items in some initial order

  An algorithm may compare two items (asking is $a_i > a_j$?) at a cost of 1
  - Moving the items is free

- No other operations allowed, such as XORing, hashing, etc.

- Sorting: given an array a = $[a_1, \ldots, a_n]$, output a permutation π so that $[a_{\pi(1)}, \ldots, a_{\pi(n)}]$ in which the elements are in increasing order

# Sorting Lower Bound

- Theorem: Any deterministic comparison-based sorting algorithm must perform at least $\lg_2(\mathrm{n}!)$ comparisons to sort n elements in the worst case

- I.e., for any sorting algorithm A and $\mathrm{n} \geq 2$, there is an input I of size n so that A makes $\geq \lg(\mathrm{n}!) = \Omega(\mathrm{n} \log \mathrm{n})$ comparisons to sort I.

- Need to rule out any possible algorithm

- Proof is information-theoretic

# Encoding Argument

- Proof: a deterministic algorithm is a sequence of comparisons " is $a_i \leq a_j$?"

- Encode the result of the i-th comparison as a bit $b_i$

- Since the algorithm is deterministic, the result of previous comparisons determines what the next comparison asked is

- Encode the results of comparisons as $b_1, b_2, \ldots, b_\ell$, where $\ell$ is the total number of comparisons. If $\ell < \lceil \log_2 n! \rceil$, then there are $2^\ell < n!$ possible bitstrings. So, two different input permutations $\pi \neq \pi'$ on {1, 2, 3, …, n} result in the same bitstring

- But there is no output permutation that is correct on both $\pi$ and $\pi'$. QED

# Encoding Argument Notes

- You could have non-distinct values $a_1, a_2, \ldots, a_n$ in the sorting problem but our lower bound only considered the case when $a_1, a_2, \ldots, a_n$ were distinct

- That's okay, since a correct sorting algorithm must in particular sort any input for which $a_1, a_2, \ldots, a_n$ are distinct

- For any problem, we can choose a subset S of all possible inputs to the problem and prove a lower bound for any algorithm which is correct on inputs in S
  - The lower bound we obtain then holds for algorithms correct on all inputs (not only those in S)

# Sorting Lower Bound

- Information-theoretic: need $\lg(n!)$ bits of information about the input before we can correctly decide on the output

- $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) < n \lg n$
- $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) > \left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right) = \Omega(n \lg n)$

- $n! \in [\left(\frac{n}{e}\right)^n, n^n]$, so $n \lg n - n \lg e < \lg(n!) < n \lg n$
  $n \lg n - 1.443n < \lg(n!) < n \lg n$

- $\lg(n!) = (n \lg n)(1 - o(1))$

# Sorting Upper Bounds

- Suppose for simplicity n is a power of 2

- Binary insertion sort: using binary search to insert each new element, the number of comparisons is $\sum_{k=2,\ldots,n} \lceil \lg k \rceil \leq n \lg n$
  - Note: may need to move items around a lot, but only counting comparisons

- Mergesort: merging two sorted lists of n/2 elements requires at most n-1 comparisons
  - Unrolling the recurrence, total number of comparisons is

$$(n-1) + 2\left(\frac{n}{2} - 1\right) + 4\left(\frac{n}{4} - 1\right) + \cdots + \frac{n}{2}(2-1) = n \lg n - (n-1) < n \lg n$$

# Selection in the Comparison Model

- How many comparisons are necessary and sufficient to find the maximum of n elements in the comparison model?

- Claim: n-1 comparisons are sufficient
- Proof: scan from left to right, keep track of the largest element so far

- For lower bounds, what does our earlier information-theoretic argument give?
  - Only $\Omega(\log n)$, which is too weak

- Also, we have to look at all elements, otherwise we may have not looked at the largest, but that can be done with n/2 comparisons, also not tight

# Lower Bound for Finding the Maximum

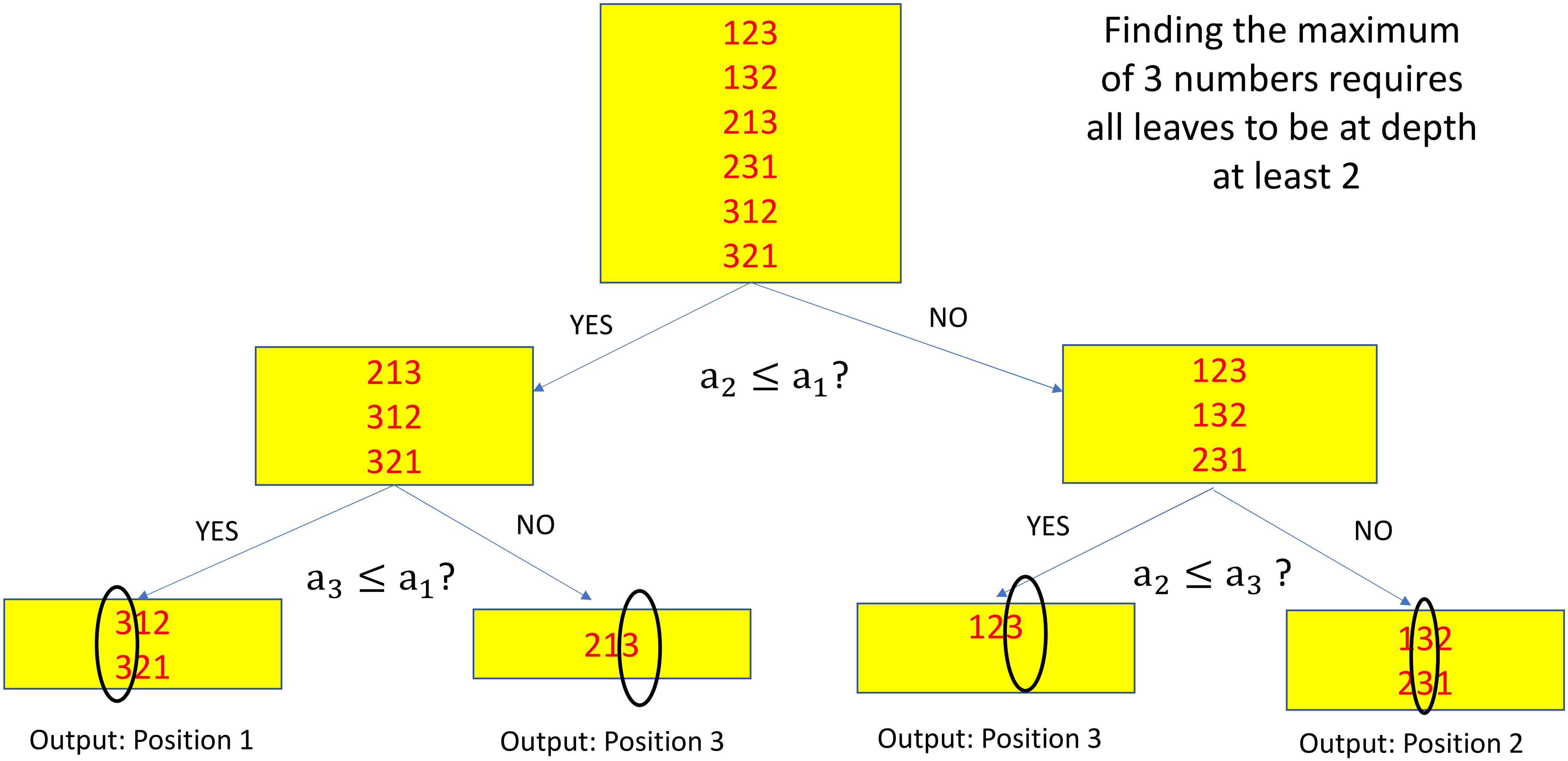
- Claim: n-1 comparisons are needed in the worst-case to find the maximum of n elements

- Proof: each time a comparison is made, one item “loses” and one item “wins”

- Every item that is not the maximum must lose at least one comparison. Otherwise, suppose both $a_i$ and $a_j$ never lost
  - If algorithm outputs $a_i$, then you can make $a_j$ arbitrarily large and preserve all comparisons. Similarly, if algorithm outputs $a_j$, can make $a_i$ arbitrarily large

- Thus, n-1 comparisons are necessary

# Decision Tree Lower Bound for Maximum




Finding the maximum of 3 numbers requires all leaves to be at depth at least 2

# Lower Bound for Finding the Maximum

- Recap: upper and lower bounds match at n-1

- Argument different from information-theoretic bound for sorting

- Instead,
  - if algorithm makes too few comparisons on some input In and outputs Out,
  - find another input In’ where the algorithm makes the same comparisons and also outputs Out,
  - but Out is not a correct output for In’

# An Adversary Argument

- If algorithm makes “too few” comparisons, fool it into giving an incorrect answer

- Any deterministic algorithm sorting 3 elements requires at least 3 comparisons
  - If < 2 comparisons, some element not looked at and the algorithm is incorrect
  - After first comparison, 3 elements are w, l, and z, the winner and loser of the first comparison, as well as the uninvolved item
  - If the second query is between w and z, say
    - w is larger
  - If the second query is between l and z, say
    - l is smaller
  - Algorithm needs one more comparison for correctness

- Goal: answer comparisons so that (a) answers consistent with some input In, (b) answers make the algorithm perform “many” comparisons

# First and Second Largest of n Elements

- How many comparisons are necessary (lower bound) and sufficient (upper bound) to find the first and second largest of n distinct elements?

- Claim: n-1 comparisons are needed in the worst-case

- Proof: need to at least find the maximum

# What about Upper Bounds?

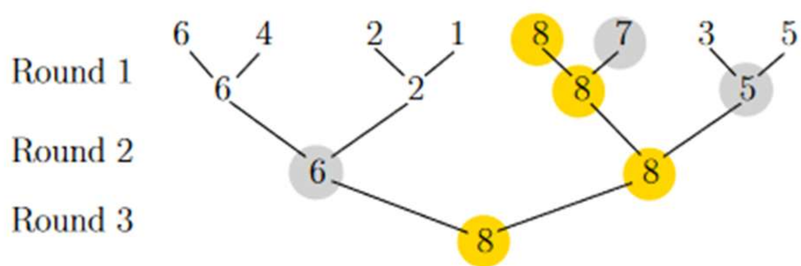
- Claim: 2n-3 comparisons are sufficient to find the first and second-largest of n elements

- Proof: find the largest using n-1 comparisons, then find the largest of the remainder using n-2 comparisons, so 2n-3 total

- Upper bound is 2n-3, and lower bound n-1, both are $\Theta(n)$ but can we get tight bounds?

# Second Largest of n Elements Upper Bound

- Claim: $n + \lg n - 2$ comparisons are sufficient to find the first and second-largest of n elements
- Proof: find the maximum element using n-1 comparisons by grouping elements into pairs, finding the maximum in each pair, and recursing




- What can we say about the second maximum?
  - Must have been directly compared to the maximum and lost, so lg(n)-1 additional comparisons suffice. Kislitsyn (1964) shows this is optimal

# Sorting in the Exchange Model

- Consider a shelf containing n unordered books to be arranged alphabetically. How many swaps do we need to order them?

- In the exchange model, you have n items and the only operation allowed on the items is to swap a pair of them at a cost of 1 step

    - All other work is free, e.g., the items can be examined and compared

- How many exchanges are necessary and sufficient?

# Sorting in the Exchange Model

- Claim: n-1 exchanges is sufficient
- Proof: here's an algorithm:
- In first step, swap the smallest item with the item in the first location
- In second step, swap the second smallest item with the item in the second location
- In k-th step, swap the k-th smallest item with the item in the k-th location
  - If no swap is necessary, just skip a given step
- No swap ever undoes our previous work
- At the end, the last item must already be in the correct location

# Lower Bound for Sorting in Exchange Model

- Claim: n-1 exchanges are necessary in the worst case
- Proof: create a directed graph in which the edge (i,j) means the book in location i must end up in location j

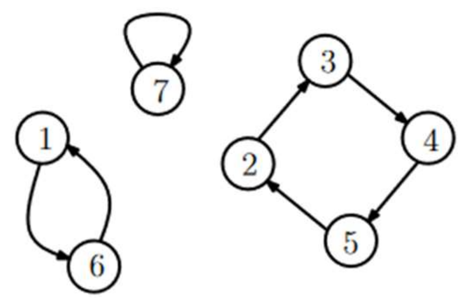
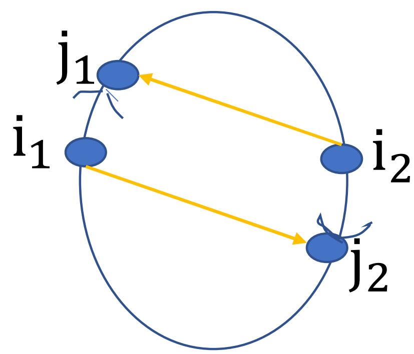



Figure 1: Graph for input [f c d e b a g]

- Graph is a set of cycles
  - Indegree and Outdegree of each node is 1

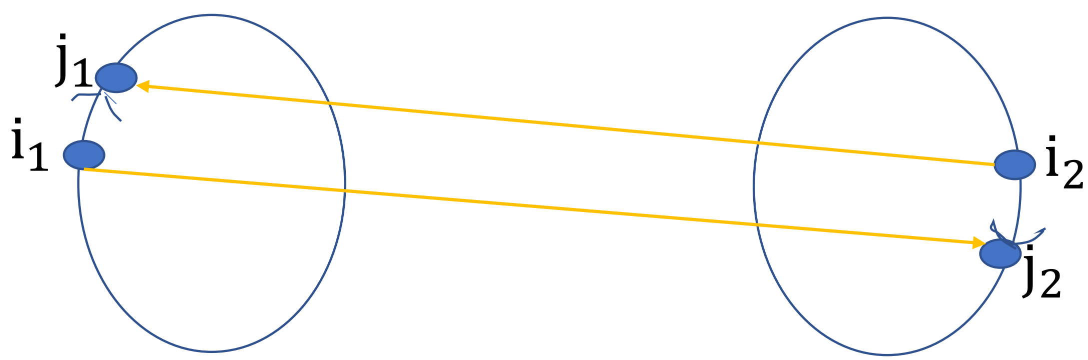
# Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in the same cycle?
  - Suppose we have edges $(i_1, j_1)$ and $(i_2, j_2)$ and swap elements in locations $i_1$ and $i_2$
  - This replaces these edges with $(i_2, j_1)$ and $(i_1, j_2)$ since now the item in position $i_2$ need to go to $j_1$ and item in position $i_1$ need to go to $j_2$
  - Since $i_1$ and $i_2$ in the same cycle, now we get two disjoint cycles



# Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in different cycles?
  - If we swap elements $i_1$ and $i_2$ in different cycles, similar argument shows this merges two cycles into one cycle



# Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in the same cycle?
  - Get two disjoint cycles
- What is the effect of exchanging any two elements in different cycles?
  - Merges two cycles into one cycle
- How many cycles are in the final sorted array?
  - n cycles
- Suppose we begin with an array [n, 1, 2, …, n-1] with one big cycle
- Each step increases the number of cycles by at most 1, so need n-1 steps

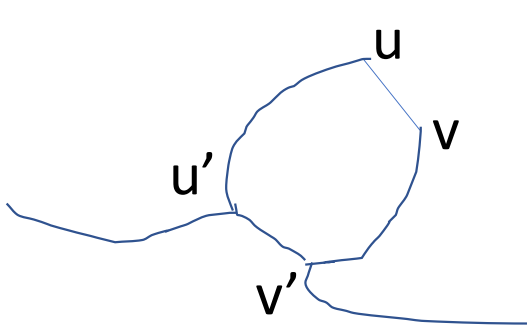
# Query Models and Evasiveness

- Let G be the adjacency matrix of an n-node graph
  - G[i,j] = 1 if there is an edge between i and j, else G[i,j] = 0
- In 1 step, we can query any element of G. All other computation is free
- How many queries do we need to tell if G is connected?
- Claim: n(n-1)/2 queries suffice
- Proof: Just query every pair {i,j} to learn G, then check if G is connected

- *What about lower bounds?*

# Connectivity is an Evasive Graph Property

- Theorem: n(n-1)/2 queries are necessary to determine connectivity
- Proof: adversary strategy: given a query G[u,v], answer 0 *unless* the graph consistent with all of your responses so far, which also satisfies G[u', v'] = 1 for each unasked pair {u',v'}, is disconnected
- Invariant: for any unasked pair {u,v}, the graph revealed so far has no path from u to v
- Reason: consider the last edge {u',v'} revealed on that path. Could have answered 0 and kept same connectivity by having edge {u,v} be present



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- Suppose there is some unasked pair {u,v} by the algorithm
  - If algorithm says "connected", we place all 0s on unasked pairs
  - If algorithm says "disconnected", we place all 1s on unasked pairs
- So algorithm needs to query every pair