Matrix Games

Today, we'll talk about game theory and some of its connections to computer science. Game theory is the study of how people behave in social and economic interactions, and how they make decisions in these settings. It is an area originally developed by economists, but given its general scope, it has applications to many other disciplines, including computer science.

Objectives of this lecture

In this lecture, we will:

- Define two-player zero-sum games, and the concept of minimax-optimal strategies.
- Some techniques for solving two-player zero-sum games.
- See the connection of two-player zero-sum games to randomized algorithms.

1 Introduction to Game Theory

A clarification at the very beginning: a *game* in game theory is not just what we traditionally think of as a game (chess, checkers, poker, tennis, or football), but is much more inclusive — a game is any interaction between parties, each with their own interests. And game theory studies how these parties make decisions during such interactions.[1]

Since we very often build large systems in computer science, which are used by multiple users, whose actions affect the performance of all the others, it is natural that game theory would play an important role in many CS problems. For example, game theory can be used to model routing in large networks, or the behavior of people on social networks, or auctions on Ebay, and then to make qualitative/quantitative predictions about how people would behave in these settings.

In fact, the two areas (game theory and computer science) have become increasingly closer to each other over the past two decades — the interaction being a two-way street — with game-theorists proving results of algorithmic interest, and computer scientists proving results of interest to game theory itself.

[1] Robert Aumann, Nobel prize winner, has suggested the term "interactive decision theory" instead of "game theory".

2 Definitions and Examples

In a game, we have

- A collection of participants, often called ***players***.
- Each player has a set of choices, called ***actions***, from which players choose about how to play (i.e., behave) in this game.
- Their combined behavior leads to ***payoffs*** (think of this as the "happiness" or "satisfaction level") for each of the players.

In this lecture we'll consider a class of games called *matrix games*. These games are powerful enough to model interesting problems, but also amenable to theoretical analysis. These involve just two players, and a *payoff* matrix.

2.1 The Shooter-Goalie Game

This game abstracts what happens in a game of soccer, when some team has a penalty shot. There are two players in this game. One called the *shooter*, the other is called the *goalie*.

The shooter has two choices: either to shoot to her left, or shoot to her right. The goalie has two choices as well: either to dive to the shooter's left, or to the shooter's right. Hence, in this case, both the players have two actions, denoted by the set $\{\mathsf{L},\mathsf{R}\}$.[2]

Now for the payoffs. If both the shooter and the goalie choose the same strategy (say both choose L, or both choose R) then the goalie makes a save. Note this is an abstraction of the game: for now we assume that the goalie always makes the save when diving in the correct direction. This brings great satisfaction for the goalie, not so much for the shooter. On the other hand, if they choose different strategies, then the shooter scores the goal (again, we are modeling a perfect shooter). This brings much happiness for the shooter, but the goalie is disappointed.

Being mathematically-minded, suppose we say that the former two choices lead to a payoff of $+1$ for the goalie, and -1 for the shooter. And the latter two choices lead to a payoff of -1 for the goalie, and $+1$ for the shooter. We can write it in a matrix (called the *payoff matrix* M) thus:

		goalie L	**goalie** R
shooter	L	$(-1,1)$	$(1,-1)$
shooter	R	$(1,-1)$	$(-1,1)$

The rows of the game matrix are labeled with actions for one of the players (in this case the shooter), and the columns with the actions for the other player (in this case the goalie). The entries are pairs of numbers, indicating who wins how much: e.g., the L, L entry contains $(-1,1)$,

[2]Note carefully: we have defined things so that left and right are with respect to the shooter. From now on, when we say the goalie dives left, it should be clear that the goalie is diving to *the shooter's left*.

the first entry is the payoff to the row player (shooter), the second entry the payoff to the column player (goalie). In general, the payoff is (r, c) where r is the payoff to the row player, and c the payoff to the column player.

In this case, note that for each entry (r, c) in this matrix, the sum $r+c=0$. Such a game is called a **zero-sum game**. The zero-sum-ness captures the fact that the game is the amount that one player wins is the amount that the other player loses. (Matrix games that are not zero-sum are discussed in section 6 of these notes.)

Note that being zero-sum does not mean that the game is "fair" in any sense—a game where the payoff matrix has $(1,-1)$ in all entries is also zero-sum, but is clearly unfair to the column player.

Lastly, for 2-player games, we will define the *row-payoff matrix* R which consists of the payoffs to the row player, and the *column-payoff matrix* C consisting of the payoffs to the column player. The tuples in the payoff matrix M contain the same information, i.e.,

$$M_{ij} = (R_{ij}, C_{ij}).$$

The game being zero-sum now means that $R = -C$, or $R + C = 0$. In the example above, the matrix R is

		goalie L	R
shooter	L	-1	1
	R	1	-1

Note that the row payoff matrix R has all of the information about the game, since we can deduce the column player's payoff by taking the negative of the row player's payoff. We will generally omit the C values when writing a matrix for a zero-sum game.

2.2 Pure and Mixed Strategies

Now given a game with payoff matrix M, the two players have to choose strategies, i.e., decide how to play.

One strategy would be for the row player to decide on a row to play, and the column player to decide on a column to play. Say the strategy for the row player was to play row I and the column player's strategy was to play column J, then the payoffs would be given by the tuple (R_{IJ}, C_{IJ}) in location I, J:

$$\text{payoff } R_{IJ} \text{ to the row player, and } C_{IJ} \text{ to the column player}$$

In this case both players are playing deterministically. (E.g., the goalie decides to always go left, etc.) A strategy that decides to play a single action is called a *pure strategy*.

Definition: Pure strategy

A pure strategy for a player is one in which the player deterministically selects a single action to always play, e.g., always shoot left.

But as will become obvious vary soon, pure strategies in these games are not what we play. We are trying to compete with the worst adversary, and we may like to "hedge our bets". Hence we may use a randomized algorithm: e.g., the players dive/shoot left or right with some probability, or when playing the classic game of Rock-Paper-Scissors the player choose one of the options with some probability. This is called a *mixed strategy*

Definition: Mixed strategy

A mixed strategy for a player consists of a non-negative real probability p_i for each action such that $\sum_i p_i = 1$, i.e., a probability distribution over the actions for the player.

When talking about a pair of mixed strategies (one for the row player and one for the column player), we will usually write p_i for each row, such that $\sum_i p_i = 1$ for the row player, and similarly, a $q_i \geq 0$ for each column, such that $\sum_i q_i = 1$ for the column player. The probability distributions $\mathbf{p}, \mathbf{q}$ are called the *mixed strategies* for the row and column player respectively. And then we look at the *expected payoff*

Claim: Expected payoff

The expected payoff to the *row player* is

$$V_R(\mathbf{p},\mathbf{q}) := \sum_{i,j} \Pr[\text{row player plays } i \text{ and column player plays } j] \cdot R_{ij} = \sum_{ij} p_i q_j R_{ij},$$

and the expected payoff to the *column player* is

$$V_C(\mathbf{p},\mathbf{q}) := \sum_{i,j} p_i q_j C_{ij}$$

This being a two-player zero-sum game, we know that $V_R(\mathbf{p},\mathbf{q}) = -V_C(\mathbf{p},\mathbf{q})$, so we will just mention the payoff to one of the players (say the row player). For instance, if $\mathbf{p} = (0.5, 0.5)$ and $\mathbf{q} = (0.5, 0.5)$ in the shooter-goalie game, then $V_R = 0$, whereas $\mathbf{p} = (0.75, 0.25)$ and $\mathbf{q} = (0.6, 0.4)$ gives $V_R = 0.45 - 0.55 = -0.1$.

2.3 Minimax-Optimal Strategies

What does the row player want to do? She wants to find a vector $\mathbf{p}^\star$ that maximizes the expected payoff to her, over all choices of the opponent's strategy $\mathbf{q}$. The mixed strategy that maximizes the minimum payoff.

Definition: Lower bound for the row player

Consider a strategy **p** for the row player. After picking **p** we then allow the column player to pick **q**, the best strategy for the column player who knows the row player is playing **p**. So we write:

$$\mathsf{lb}(\mathbf{p}) := \min_{\mathbf{q}} V_R(\mathbf{p}, \mathbf{q})$$

This is the minimum amount that the row player is guaranteed to achieve using strategy **p**. We can then define:

$$\mathsf{lb}^* := \max_{\mathbf{p}} \mathsf{lb}(\mathbf{p})$$

which is the highest possible lower bound over all row strategies.

Make sure you parse this correctly:

$$\mathsf{lb}^* := \overbrace{\max_{\mathbf{p}} \quad \underbrace{\min_{\mathbf{q}} V_R(\mathbf{p}, \mathbf{q})}_{\substack{\text{payoff when opponent} \\ \text{plays the optimal response} \\ \text{against our choice p}}}}^{\substack{\text{mixed strategy that maximizes} \\ \text{the minimum expected payoff}}}$$

Loosely, *the row player can guarantee to herself this much payoff no matter what the column player does.* The quantity lb is a **lower bound on the row-player's payoff**.

What about the column player? She wants to find some $\mathbf{q}^*$ that maximizes her own expected payoff, over all choices of the opponent's strategy **p**. She wants to optimize

$$\max_{\mathbf{q}} \min_{\mathbf{p}} V_C(\mathbf{p}, \mathbf{q})$$

But this is a zero-sum game, so this is the same as

$$\max_{\mathbf{q}} \min_{\mathbf{p}} (-V_R(\mathbf{p}, \mathbf{q}))$$

And pushing the negative sign through, we get the column player is trying to optimize her own worst-case payoff, which is

$$-\min_{\mathbf{q}} \max_{\mathbf{p}} V_R(\mathbf{p}, \mathbf{q})$$

So the payoff in this case to the row player is

Definition: Upper bound for the row player

Consider a strategy $\mathbf{q}$ for the column player. After picking $\mathbf{q}$ we then allow the row player to pick $\mathbf{p}$, the best strategy for the row player who knows the column player is playing $\mathbf{q}$. So we write:

$$\mathsf{ub}(\mathbf{q}) := \max_{\mathbf{p}} V_R(\mathbf{p}, \mathbf{q})$$

This is the maximum amount that the row player can achieve, given that the column player is using strategy $\mathbf{q}$. We can then define:

$$\mathsf{ub}^* := \min_{\mathbf{q}} \mathsf{ub}(\mathbf{q})$$

which is the lowest possible upper bound over all column strategies.

The column player can guarantee that the row player does not get more than ub^* *in payoff, no matter what the row-player does.* This is an **upper bound on the row player's payoff**.

Claim 1: Lower bounds are below upper bounds

For any $\mathbf{p}$ and any $\mathbf{q}$ we have:

$$\mathsf{lb}(\mathbf{p}) \quad \leq \quad \mathsf{ub}(\mathbf{q})$$

Proof. The left hand side is an amount that the row player can achieve. The right hand side is an amount that the row player cannot exceed. The result follows. □

This is a very useful approach, as we'll see later. In order to actually evaluate a game we'll find two strategies $\mathbf{p}$ and $\mathbf{q}$ such that $\mathsf{lb}(\mathbf{p}) = \mathsf{ub}(\mathbf{q})$, which will be the value of the game.

Now we can make a powerful observation which will make it much easier to actually evaluate these lower and upper bounds.

Claim 2: A pure response is optimal

To evaluate the $\mathsf{lb}(\mathbf{p})$ we can assume that the column player plays a pure strategy. That is:

$$\mathsf{lb}(\mathbf{p}) \;=\; \min_{\mathbf{q}} V_R(\mathbf{p}, \mathbf{q}) \;=\; \min_{j} \sum_{i} p_i R_{i,j}$$

Proof. Once the row player fixes a mixed strategy $\mathbf{p}$, the column player then has no reason to randomize: her payoffs will be some average of the payoffs from playing the individual columns, so she can just pick the best column for her.

More fomally we have:

$$\begin{aligned}\mathsf{lb}(\mathbf{p}) &= \min_{\mathbf{q}} V_R(\mathbf{p},\mathbf{q}) = \min_{\mathbf{q}} \sum_{i,j} p_i q_j R_{i,j} \\ &= \min_{\mathbf{q}} \sum_j q_j \left(\sum_i p_i R_{i,j} \right) = \min_j \sum_i p_i R_{i,j}\end{aligned}$$

The last equation follows because the term in parentheses is just a function of j. It is the value of column j to the row player. So the optimum choice for $\mathbf{q}$ is to have $q_j = 1$ for the index j where formula in parentheses is minimum, and zero for other indices. □

Corollary: Alternate definition for upper and lower bounds

The quantity lb^* can be equivalently defined as

$$\mathsf{lb}^* = \max_{\mathbf{p}} \min_j \sum_i p_i R_{ij}.$$

Similarly ub^* can be written as

$$\mathsf{ub}^* = \min_{\mathbf{q}} \max_i \sum_j q_j R_{ij}.$$

2.3.1 The Shooter-Goalie Game Example

For the shooter-goalie game, we claim that the minimax-optimal strategies for both players is $(0.5, 0.5)$.

Example: Lower and upper bounds for the shooter-goalie game

Row Player: For the row player (shooter), suppose $\mathbf{p} = (p_1, p_2)$ is the mixed strategy. Note that $p_1 \geq 0, p_2 \geq 0$ and $p_1 + p_2 = 1$. So it is easier to write the strategy as $\mathbf{p} = (p, 1-p)$ with $p \in [0,1]$.

OK. If the column player (goalie) plays L, then this strategy gets the shooter a payoff of

$$p \cdot (-1) + (1-p) \cdot (1) = 1 - 2p.$$

If the column player (goalie) plays R, then this strategy gets the shooter a payoff of

$$p \cdot (1) + (1-p) \cdot (-1) = 2p - 1.$$

So we want to choose some value $p \in [0,1]$ to maximize

$$\mathsf{lb}(p) = \min(1 - 2p, 2p - 1)$$

In this case, this maximum is achieved at $p = 1/2$. (One way to see it is by drawing these two lines.) And the minimax-optimal expected payoff to the shooter is $\mathsf{lb}^* = 0$.

Column Player: The calculation for the column player (goalie) is similar in this case. We let q be the probability of the goalie going left, and $1-q$ the probability of going right.

$$\mathsf{ub}(q) = \max(1-2q, 2q-1)$$

We take the minimum of this for $q \in [0,1]$. This is achieved at $q = 1/2$, and the value is 0, so $\mathsf{ub}^* = 0$.

An observation: the shooter can guarantee a payoff of $\mathsf{lb}^* = 0$, and the goalie can guarantee that the shooter's payoff is never more than $\mathsf{ub}^* = 0$. Since $\mathsf{lb} = \mathsf{ub}$, in this case the *"value of the game"* is said to be $\mathsf{lb} = \mathsf{ub} = 0$.

2.3.2 An Asymmetric Goalie Example

Let's change the game slightly. Suppose the goalie is weaker on the left. For example, what happens if the payoff matrix is now:

		goalie L	**goalie** R
shooter	L	$-\frac{1}{2}$	1
shooter	R	1	-1

Example: Asymmetric shooter-goalie game

Row Player: For the row player (shooter), suppose $\mathbf{p} = (p, 1-p)$ is the mixed strategy, with $p \in [0,1]$. If the column player (goalie) plays L, then this strategy gets the shooter a payoff of

$$p \cdot (-1/2) + (1-p) \cdot (1) = 1 - (3/2)p.$$

If the column player (goalie) plays R, then this strategy gets the shooter a payoff of

$$p \cdot (1) + (1-p) \cdot (-1) = 2p - 1.$$

So we want to choose some value $p \in [0,1]$ to maximize

$$\mathsf{lb}(p) = \min(1-(3/2)p, 2p-1)$$

In this case, this maximum is achieved at $p = 4/7$. And the minimax-optimal expected payoff to the shooter is $\mathsf{lb}^* = 1/7$. Note that the goalie being weaker means the shooter's payoff increases.

Column Player: What about the calculation for the column player (goalie)? If her strategy is $\mathbf{q} = (q, 1-q)$ with $q \in [0,1]$, then if the shooter plays L then the shooter's payoff is

$$q \cdot (-1/2) + (1-q) \cdot (1) = 1 - (3/2)q.$$

If she plays R, then it is $2q-1$. So the goalie will try to minimize

$$\mathsf{ub}(q) = \max(1-(3/2)q, 2q-1)$$

which will again give $(4/7, 3/7)$ and guarantees that the expected loss is never more than $\mathsf{ub}^* = 1/7$.

Again, the shooter guarantees a payoff of $\mathsf{lb}^* = 1/7$, and the goalie can guarantee that the shooter's payoff is never more than $\mathsf{ub}^* = 1/7$. In this case the value of the game is said to be $\mathsf{lb}^* = \mathsf{ub}^* = 1/7$.

Problem 1. What if both players have somewhat different weaknesses? What if the payoffs are:

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{3}{2} \end{bmatrix}$$

Show that minimax-optimal strategies are $\mathbf{p} = (2/3, 1/3), \mathbf{q} = (3/5, 2/5)$ and value of game is 0.

Problem 2. For the game with payoffs:

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{2}{3} \end{bmatrix}$$

Show that minimax-optimal strategies are $\mathbf{p} = (\frac{4}{7}, \frac{3}{7}), \mathbf{q} = (\frac{17}{35}, \frac{18}{35})$ and the value of the game is $\frac{1}{7}$.

Problem 3. For the game with payoffs:

$$\begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{2}{3} \end{bmatrix}$$

Show that minimax-optimal strategies are $\mathbf{p} = (0, 1), \mathbf{q} = (0, 1)$ and value of game is $\frac{2}{3}$.

3 Von Neumann's Minimax Theorem

In all the above examples of 2-player zero-sum games, we saw that the row player has a strategy $\mathbf{p}^\star$ that guarantees some payoff lb^* for her, no matter what strategy $\mathbf{q}$ the column player plays. And the column player has a strategy $\mathbf{q}^\star$ that guarantees that the row player cannot get payoff more than ub^*, no matter what strategy $\mathbf{p}$ the row player plays. The remarkable fact in the examples was that $\mathsf{lb}^* = \mathsf{ub}^*$ in all these cases! Was this just a coincidence? No: a celebrated result of von Neumann[3] shows that we always have $\mathsf{lb}^* = \mathsf{ub}^*$ in (finite) 2-player zero-sum games.

Theorem 1: Minimax Theorem (von Neumann, 1928)

Given a finite 2-player zero-sum game with payoff matrices $R = -C$,

$$\mathsf{lb}^* = \max_{\mathbf{p}} \min_{\mathbf{q}} V_R(\mathbf{p}, \mathbf{q}) = \min_{\mathbf{q}} \max_{\mathbf{p}} V_R(\mathbf{p}, \mathbf{q}) = \mathsf{ub}^*.$$

This common value is called the value of the game.

[3]John von Neumann, mathematician, physicist, and polymath.

The theorem implies that in a zero-sum game, both the row and column players can even "publish" their minimax-optimal mixed strategies (i.e., tell the strategy to the other player), and it does not hurt their expected performance as long as they play optimally.[4]

Von Neumann's Minimax Theorem is an important result in game theory, but it has beautiful implications to computer science as well — as we see in the next section.

4 Techniques for Solving Games

If you're confronted with a zero-sum matrix game how do you solve it? Sometimes it's possible by trial and error, and intuition, or luck, to come up with strategies $(\mathbf{p}, \mathbf{q})$ such that $\mathsf{lb}(\mathbf{p}) = \mathsf{ub}(\mathbf{q})$.

In the upcoming lectures on linear programming, we show how to solve games using an LP solver. In the remainder of this section, we give some methods that are often effective at solving small games by hand.

4.1 Removing Dominated Rows or Columns

Suppose there's a row that is dominated by another row, that is, every element of row j is at least the corresponding element of row k. Then we can just delete row k from the game, and its value stays the same. For example:

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 5 \\ 7 & 1 & 8 \end{bmatrix}$$

We can immediately delete the first row, because the 2nd one dominates it.

The same idea applies to columns. So, after deleting the first row of the above matrix, look at the resulting matrix. Now it's clear that the first column. is always better for the column player than the 3rd column. So the 3rd column can be deleted.

4.2 Convex Combinations of Rows or Columns

Consider the following game:

$$\begin{bmatrix} 10 & 0 \\ 0 & 10 \\ 3 & 3 \end{bmatrix}$$

In this case, no one row dominates another. However adding $\frac{1}{2}$ of the first row to $\frac{1}{2}$ of the second row is equivalent to having $(5\ \ 5)$ as a row. This dominates the last row $(3\ \ 3)$, so we can just delete it. Any solution that puts any weight on the $(3\ \ 3)$ row would be better off putting half that weight on the $(10\ \ 0)$ and the other half on the $(0\ \ 10)$ row.

[4] It's like telling your rock-paper-scissors opponent that you will play each action with equal probability, it does not buy them anything to know your strategy. This is not true in general non-zero-sum games; there if you tell your opponent the mixed-strategy you're playing, she may be able to do better. Note carefully that you are not telling them the actual random choice you will make, just the distribution from which you will choose.

In general, a row can be removed it is dominated by a convex combination of the other rows.

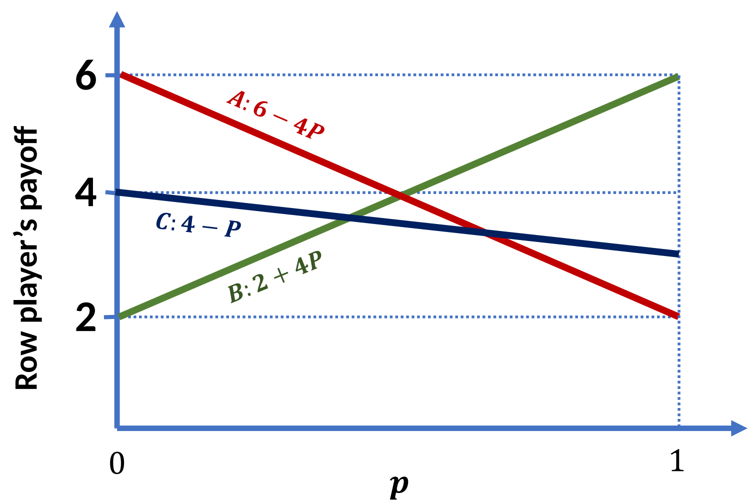
4.3 A General Method for Solving 2-Row Games

In this section we give a general method for solving games with two rows (or, by symmetry, two columns). We'll develop the method with the following example:

		column player		
		A	**B**	**C**
row	**1**	2	6	3
player	**2**	6	2	4

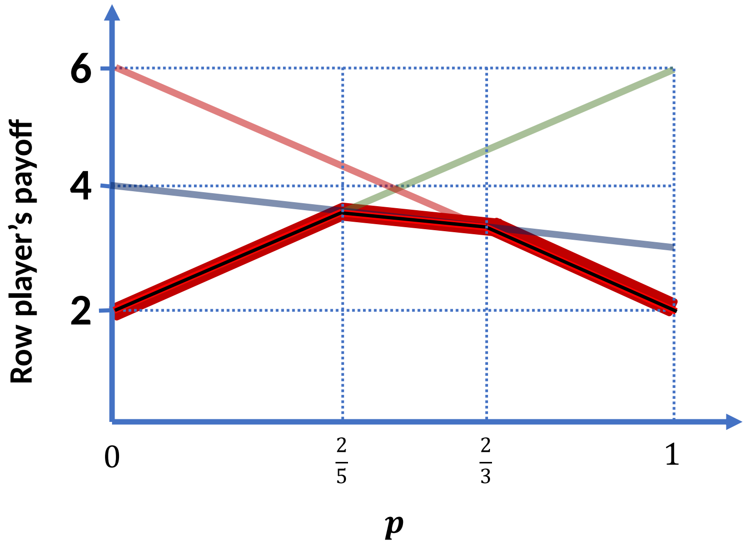
The game we'll analyze is shown in the 2×3 matrix above with values for the row player. We'll call the three options for the column player A, B, and C, and the two options for the row player 1 and 2.

The general mixed strategy for the row player is to choose option 1 with probability p and option 2 with probability $1-p$. For any choice of p, the column player can respond with A, B, or C. We can plot the payoff to the row player with respect to every possible choice of p and use this to analyze the game.



In the diagram above, the horizontal axis represents p, and the vertical axis represents the payoff to the row player. The three lines in the diagram correspond to the three options for the column player. For example, the line labeled B, which is the graph of the function $2+4p$, is the expected value of the game for the row player (as a function of p) if the column player chooses column B.

So what does the actual payoff consist of for the row player? Whatever value of p they pick, their opponent, if playing optimally, will play whichever of A, B and C gives the lowest payoff, so the row player's payoff at any point looks like the *lowest of the three lines*. We can represent this on the diagram as follows. Its called the *lower envelope* of the three options for the column player.



This concave function represents the expected payoff for the row player if he chooses option 1 with probability p, assuming that the column player plays optimally knowing the value of p. Thus, it represents a lower bound on the value of the game for the row player for each value of p. By inspection of this graph, the concave function achieves its maximum value at the point of intersection between B and C. This point is $p = \frac{2}{5}$, and with that choice of p the value of the game is $3 + \frac{3}{5}$. That's the lower bound on the game's value.

What's a good strategy for the column player? Consider the convex combination of B and C chosen such that the result is a horizontal line. This is $\frac{4}{5}C + \frac{1}{5}B$, and corresponds to a mixed strategy of picking B with probability $1/5$ and C with probability $4/5$. With this mixed strategy, the value of the game for the row player is $3 + \frac{3}{5}$, no matter what the row player does. So this is an upper bound on the value of the game for the row player. Since the lower and upper bound are equal, we know that this is the value of the game.

The same technique can be applied to any game with two rows. The value of the game is obtained by constructing the concave function and finding where it achieves its maximum.

5 Lower Bounds for Randomized Algorithms

In order to prove lower bounds, we thought of us coming up with algorithms, and the adversary coming with some inputs on which our algorithm would perform poorly—take a long time, or make many comparisons, etc. We can encode this as a zero-sum game with row-player payoff matrix R. We will, unsurprisingly, use sorting in the comparison model as an example.

- The columns are all *deterministic* algorithms for sorting n elements.
- The rows are all the possible inputs (all $n!$ of them).

- The entry R_{ij} is the cost of the algorithm j on the input i (say the number of comparisons).

This may be a huge matrix, but we'll never actually write it down; it's just a conceptual guide which we will use to reason about lower bounds. It may be huge, but for it to be a valid matrix it does have to be finite. Is that the case here?

Remark: There are a finite number of (sensible) algorithms

Remember that in the comparison model, every deterministic algorithm can be encoded as a *decision tree* which encodes the comparisons made by the algorithm. Although there are technically an infinite number of possible algorithms and hence decision trees for the problem, there are only a finite number of them that never perform any redundant comparisons. This is because after performing $\binom{n}{2}$ comparisons, one has compared every pair of elements and definitely knows the answer, so the decision trees can have depth at most $\binom{n}{2}$.

Since this is finite, the number of possible decision tree configurations is finite. Lastly, since every algorithm that performs redundant comparisons only has a strictly worse cost than one that does not, we can ignore these algorithms when thinking about lower bounds and efficient algorithms.

So, we know that the matrix is finite and hence valid to interpret as a zero-sum game. What does it tell us? A lot, as it turns out.

- A deterministic algorithm with good worst-case guarantee is a column that does well against all rows: all entries in this column are small.
- A randomized algorithm with good expected guarantee is a probability distribution $\mathbf{q}$ over columns, such that the expected cost for each row i is small. This is a mixed strategy for the column player. It gives an upper bound.

This second claim is actually subtle and not at all obvious. Why can we say that a randomized algorithm is just a distribution over deterministic algorithms?

Remark: Randomized algorithms $\equiv$ randomly choosing a deterministic algorithm

A randomized algorithm is just a deterministic algorithm that may also read from a source of randomness and base some of its decisions on that. If we imagine pre-generating the results of all randomness in the algorithm and "hardcode" it in, the algorithm just becomes a deterministic algorithm! So, given the distribution of the source of randomness, we can infer a distribution on resulting (deterministic) algorithms.

So, a randomized algorithm is essentially a *mixed strategy* $\mathbf{q}$ for the column player in this zero-sum game! How would we find the *best* randomized algorithm? That would correspond to an upper bound strategy $\mathbf{q}^*$ (which would be the minimax-optimal strategy)!

What is a lower bound for randomized algorithms? It is a mixed-strategy over rows (a probability distribution $\mathbf{p}$ over the inputs) such that for every column (i.e., deterministic algorithm j), the expected cost of j (under distribution $\mathbf{p}$) is high.

Key Idea: Lower bounds

To prove a lower bound for randomized algorithms, if suffices to show that lb^* is high for this game. i.e., give a strategy for the row player (a distribution over inputs) such that every column (deterministic algorithm) incurs a high cost on it.

5.1 A Lower Bound for Sorting Algorithms

Recall from Lecture 2 we showed that any deterministic comparison-based sorting algorithm must perform $\log_2 n! = n \log_2 n - O(n)$ comparisons in the worst case. The next theorem extends this result to randomized algorithms.

Theorem

Let $\mathscr{A}$ be any randomized comparison-based sorting algorithm (that always outputs the correct answer). Then there exist inputs on which $\mathscr{A}$ performs $\Omega(\lg n!)$ comparisons in expectation.

Proof. Suppose we construct a matrix R as above, where the columns are possible (deterministic) sorting algorithms for n elements, the rows are the $n!$ possible inputs, and entry R_{ij} is the number of comparisons algorithm j makes on input i. We claim that the value of this game is $\Omega(\lg n!)$: this implies that the best distribution over columns (i.e., the best randomized algorithm) must suffer at least this much cost on some column (i.e., input).

To show the value of the game is large, we show a probability distribution over the rows (i.e., inputs) such that the expected cost of every column (i.e., every deterministic algorithm) is $\Omega(\lg n!)$.

This probability distribution is the uniform distribution: each of the $n!$ inputs is equally likely. Now consider any deterministic algorithm: as in Lecture 2, this is a decision tree with at least $n!$ leaves. No two inputs go to the same leaf.

In this tree, how many leaves can have depth at most $(\lg n!) - 10$? At most the number of nodes at depth at most $(\lg n!) - 10$ in a complete binary tree. Which in turn is

$$1 + 2 + 4 + 8 + \ldots + 2^{(\lg n!)-10} \leq 1 + 2 + 4 + 8 + \ldots + \frac{n!}{1024} \leq \frac{n!}{512}$$

So $\frac{511}{512} \geq 0.99$ fraction of the leaves in this tree have depth more than $(\lg n!) - 10$. In other words, if we pick a random input, it will lead to a leaf at depth more than $(\lg n!) - 10$ with probability 0.99. Which gives the expected depth of a random input to be $\geq 0.99((\lg n!) - 10) = \Omega(\lg n!)$. □

6 General-Sum Two-Player Games

Optional content — Not required knowledge for the exams

In general-sum games, we don't deal with purely competitive situations, but cases where there are win-win and lose-lose situations as well. For instance, the coordination game of "chicken", a.k.a. *what side of the street to drive on?* It has the payoff matrix:

		Bob	
		L	R
Alice	L	$(1,1)$	$(-1,-1)$
	R	$(-1,-1)$	$(1,1)$

Note that we are now using the convention that a player choosing L is driving on *their* left. Note that if both players choose the same side, then both win. And if they choose opposite sides, both crash and lose. (Both players can choose to drive on the left—like Britain, India, etc.—or both on the right, like the rest of the world, but they must coordinate. Both these are stable solutions and give a payoff of 1 to both parties.)

Consider another coordination game that we call "which movie?" Two friends are deciding what to do in the evening. One wants to see *Citizen Kane*, and the other *Dumb and Dumber*. They'd rather go to a movie together than separately (so the strategy profiles C,D and D,C have payoffs zero to both), but C,C has payoffs $(8,2)$ and D,D has payoffs $(2,8)$.

		Bob	
		C	D
Alice	C	$(8,2)$	$(0,0)$
	D	$(0,0)$	$(2,8)$

Finally, yet another game is "Prisoner's Dilemma" (or "to pollute or not?") with the payoff matrix:

		Bob	
		C	D
Alice	C	$(2,2)$	$(-1,3)$
	D	$(3,-1)$	$(0,0)$

6.1 Nash Equilibria

In this case, a good notion is to look for a *Nash Equilibrium*[5] which is a stable set of (mixed) strategies for the players. Stable here means that given strategies $(\mathbf{p}^*, \mathbf{q}^*)$, neither player has any incentive to unilaterally switch to a different strategy. I.e., for any other mixed strategy $\mathbf{p}$ for the row player

$$\text{row player's new payoff } = \sum_{ij} p_i q_j^* R_{ij} \le \sum_{ij} p_i^* q_j^* R_{ij} = \text{ row player's old payoff}$$

and for any other possible mixed strategy $\mathbf{q}$ for the column player

$$\text{column player's new payoff } = \sum_{ij} p_i^* q_j C_{ij} \le \sum_{ij} p_i^* q_j^* C_{ij} = \text{ column player's old payoff.}$$

Here are some examples of Nash equilibria:

- In the chicken game, both $(\mathbf{p}^* = (1,0), \mathbf{q}^* = (1,0))$ and $(\mathbf{p}^* = (0,1), \mathbf{q}^* = (0,1))$ are Nash equilibria, as is $(\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2}), \mathbf{q}^* = (\frac{1}{2}, \frac{1}{2}))$.
- In the movie game, the only Nash equilibria are $(\mathbf{p}^* = (1,0), \mathbf{q}^* = (1,0))$ and $(\mathbf{p}^* = (0,1), \mathbf{q}^* = (0,1))$.
- In prisoner's dilemma, the only Nash equilibrium is to defect (or pollute). So we need extra incentives for overall good behavior.

It is easy to come up with games where there are no stable *pure* strategies—this is even true for zero-sum games. But what about mixed-strategies? The main result in this area was proved by Nash in 1950 (which led to his name being attached to this concept)

Theorem 2: Existence of Stable Strategies

Every finite player game (with each player having a finite number of strategies) has at least one (mixed-strategy) Nash equilibrium.

This theorem implies the Minimax Theorem (Theorem 1) as a corollary.

Proof. Take any two-player zero-sum game. Let $(\mathbf{p}^*, \mathbf{q}^*)$ be a Nash equiblibrum. We have:

$$\mathsf{lb}(\mathbf{p}^*) = \min_{\mathbf{q}} V_R(\mathbf{p}^*, \mathbf{q}) = V_R(\mathbf{p}^*, \mathbf{q}^*)$$

The latter equality follows from the fact that there is no better option for the column player than $\mathbf{q}^*$.

Similarly we have:

$$\mathsf{ub}(\mathbf{q}^*) = \max_{\mathbf{p}} V_R(\mathbf{p}, \mathbf{q}^*) = V_R(\mathbf{p}^*, \mathbf{q}^*)$$

It immediately follows that $\mathsf{lb}(\mathbf{p}^*) = \mathsf{ub}(\mathbf{q}^*)$, which proves the minimax theorem. □

[5]Named after John Nash: CMU graduate, mathematician, and Nobel prize winner.