Lectures 6: The Data Stream Model

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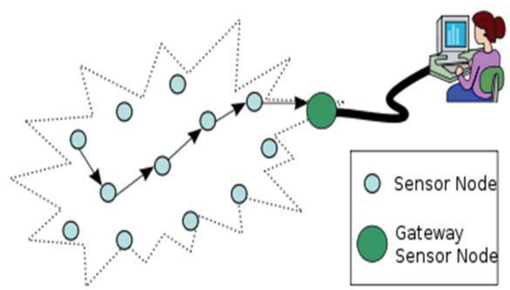
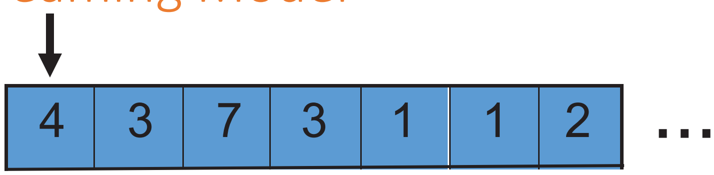
Data Streams

- A data stream is a sequence of data, that is too large to be stored in available memory
- Examples
  - Internet search logs
  - Network Traffic
  - Sensor networks



  - Scientific data streams (astronomical, genomics, physical simulations)…

Streaming Model



- Stream of elements $a_1, \ldots, a_i, \ldots$ each from an alphabet Σ and taking b bits to represent
- Single or small number of passes over the data
- Almost all algorithms are randomized and approximate
  - Usually necessary to achieve efficiency
  - Randomness is in the algorithm, not the input
- Goals: minimize space complexity (in bits), processing time

Example Streaming Problems

- Let $a_{[1:t]} = < a_1, \ldots, a_t >$ be the first t elements of the stream

- Suppose $a_1, \ldots, a_t$ are integers in {$-2^b + 1$, $-2^b + 2$, …, -1, 0, 1, 2, …, 2^b-1}
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32

- How many bits do we need to maintain $f(a_{[1:t]}) = \sum_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 4, 21, 25, 16, 48, 149, 152, -570, -567, 333, 337, 379, …
  - O(b + log t)

- How many bits do we need to maintain $f(a_{[1:t]}) = \max_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 3, 17, 17, 32, 101, 101, 101, 101, 900, 900, 900, …
  - O(b) bits

Example Streaming Problems

- What about the median of all the numbers we've stored so far?
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32
  - Median: 3, 1, 3, 3, 3, 3, 4, 3, …
  - This seems harder…
- What about the number of distinct elements we've seen so far?
  - Outputs on example: 1, 2, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9, 9, …
- What about the elements that have appeared at least an ϵ-fraction of the time? These are called ϵ-heavy hitters
  - Cover this today

Many Applications

- Internet router may want to figure out which IP connections are "elephants", that is, the heavy hitters, e.g., the ones that use more than .01% of your bandwidth
- Or maybe the router wants to know the median (or 90-th percentile) of the file sizes being transferred
- Hashing is a key technique

Finding ϵ-Heavy Hitters

- S_t is the multiset of items at time t, so $S_0 = \emptyset, S_1 = \{a_1\}, \ \ldots, S_i = \{a_1, \ldots, a_i\}$,
$$\text{count}_t(e) = \{i \in \{1, 2, \ldots, t\} \text{ such that } a_i = e\}$$

- $e \in \Sigma$ is an ϵ-heavy hitter at time t if $\text{count}_t(e) > \epsilon \cdot t$

- Given $\epsilon > 0$, can we maintain a data structure to output the ϵ-heavy hitters?
    - More precisely, let's output a set of size $\frac{1}{\epsilon}$ containing all the ϵ-heavy hitters

- Note: can output "false positives" but not allowed to output "false negatives", i.e., not allowed to miss any heavy hitter, but could output non-heavy hitters

Finding ϵ-Heavy Hitters

- Example: E, D, B, D, D_5 D, B, A, C, B_{10} B, E, E, E, E_{15}, E
  (the subscripts are just to help you count)

- At time 5, the element D is the only 1/3-heavy hitter
- At time 11, both B and D are 1/3-heavy hitters
- At time 15, there is no 1/3-heavy hitter
- At time 16, only E is a 1/3-heavy hitter

Can't afford to keep counts of all items, so how to maintain a short summary to output the ϵ-heavy hitters?

Finding a Majority Element

- First find a .5-heavy hitter, that is, a majority element:
  memory ← empty and counter ← 0
  when element a_t arrives
    if (counter == 0)
      memory ← a_t and counter ← 1
    else
      if a_t = memory
        counter + +
      else
        counter - -
        (discard a_t)
- At end of the stream, return the element in memory

Analysis of Finding a Majority Element

- If there is no majority element, we output a false positive, which is OK
- If there is a majority element, we will output it. Why?
  - When we discard an element a_t, we throw away a different element
  - Every time we throw away a copy of a majority element, we throw away another element, but majority element is more than half the total number of elements, so can't throw away all of them

Extending to ϵ-Heavy Hitters

Set $\mathrm{k} = \left\lceil \frac{1}{\epsilon} \right\rceil - 1$
Array T[1, …, k], where each location can hold one element from Σ
Array C[1, …, k], where each location can hold a non-negative integer
C[i] ← 0 and $\mathrm{T}[\mathrm{i}]$ ←⊥ **for** all i

If there is j ∈ $\{1, 2, \dots, \mathrm{k}\}$ such that $\mathrm{a_t} = \mathrm{T[j]}$, **then** $\mathrm{C[j]}++$
Else if some counter C[j] = 0 **then** $\mathrm{T[j]} \leftarrow \mathrm{a_t}$ and C[j] $\leftarrow 1$
Else decrement all counters by 1 (and discard element $\mathrm{a_t}$)

$\mathrm{est_t(e)} = \mathrm{C[j]}$ if $\mathrm{e} == \mathrm{T[j]}$, and $\mathrm{est_t(e)} = 0$ otherwise

Analyzing Counts

- Lemma: $0 \leq \mathrm{count}_t(\mathrm{e}) - \mathrm{est}_t(\mathrm{e}) \leq \frac{\mathrm{t}}{\mathrm{k}+1} \leq \epsilon \cdot \mathrm{t}$

- Proof: $\mathrm{count}_t(\mathrm{e}) \geq \mathrm{est}_t(\mathrm{e})$ since we never increase a counter for e unless we see e

If we don't increase $\mathrm{est}_t(\mathrm{e})$ by 1 when we see an update to e, we decrement k counters and discard the current update to e

So we drop k+1 distinct stream updates, but there are t total updates, so we won't increase $\mathrm{est}_t(\mathrm{e})$ by 1, when we should, at most $\frac{\mathrm{t}}{\mathrm{k}+1} \leq \epsilon \cdot \mathrm{t}$ times

Heavy Hitters Guarantee

- At any time t, all ϵ-heavy hitters e are in the array T. Why?
- For an ϵ-heavy hitter e, we have $\mathrm{count}_t(\mathrm{e}) > \epsilon \cdot \mathrm{t}$
- But $\mathrm{est}_t(\mathrm{e}) \geq \mathrm{count}_t(\mathrm{e}) - \epsilon \cdot \mathrm{t}$
- So $\mathrm{est}_t(\mathrm{e}) > 0$, so e is in array T
- Space is O(k (log(Σ) + log t)) = O(1/ ϵ) (log(Σ) + log t) bits

Heavy Hitters with Deletions

- Suppose we have a stream which allows deleting elements e that have already appeared
- Example: (add, A), (add, B), (add, A), (del, B), (del, A), (add, C)
- Multisets at different times:
  $S_0 = \emptyset, S_1 = \{A\}, S_2 = \{A, B\}, S_3 = \{A, A, B\}, S_4 = \{A, A\}, S_5 = \{A\},$
  $S_6 = \{A, C\}, \ldots$
- "active" set S_t has size $|S_t| = \sum_{e \in \Sigma} \mathrm{count}_t(e)$ and can grow and shrink

Data Structure for Approximate Counts

- Query "What is $\mathrm{count}_t(e)$?", should output $\mathrm{est}_t(e)$ with:
$$\Pr[|\mathrm{est}_t(e) - \mathrm{count}_t(e)| \leq \epsilon|S_t|] \geq 1 - \delta$$
- Want space close to our previous O(1/ ϵ) (log(Σ) + log t) bits
- Let $h: \Sigma \to \{0,1,2, \dots, k-1\}$ be a hash function (will specify later)
- Maintain an array A[0, 1, …, k-1] to store non-negative integers

when update a_t arrives:

if $a_t = (\mathrm{add}, e)$ **then** $A[h(e)]++$

else $a_t = (\mathrm{del}, e)$, and $A[h(e)]--$

- $\mathrm{est}_t(e) = A[h(e)]$

Data Structure for Approximate Counts

- $\mathrm{A[h(e)]} = \sum_{\mathrm{e}' \in \Sigma} \mathrm{count_t(e')} \cdot \mathbf{1}(\mathrm{h(e')} = \mathrm{h(e)})$, where **1**(condition) evaluates to 1 if the condition is true, and evaluates to 0 otherwise

- $\mathrm{A[h(e)]}$ = $\mathrm{count_t(e)} + \sum_{\mathrm{e}' \neq \mathrm{e}} \mathrm{count_t(e')} \cdot \mathbf{1}(\mathrm{h(e')} = \mathrm{h(e)})$,

- $\mathrm{est_t(e)} - \mathrm{count_t(e)} = \sum_{\mathrm{e}' \neq \mathrm{e}} \mathrm{count_t(e')} \cdot \mathbf{1}(\mathrm{h(e')} = \mathrm{h(e)})$

- Since we have a small array A with k locations, there are likely many $\mathrm{e}' \neq \mathrm{e}$ with h(e') = h(e), but can we bound the expected error?

Data Structure for Approximate Counts

- Recall: Family H of hash functions h: U -> {0, 1, …, M-1} is universal if for all $x \neq y$,
$$\Pr_{h \leftarrow H}[h(x) = h(y)] \leq \frac{1}{M}$$
- Gave a simple family where h can be specified using O(log $|\Sigma|$) bits

- $E[\mathrm{est}_t(e) - \mathrm{count}_t(e)] = E[\sum_{e' \neq e} \mathrm{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))]$
$= \sum_{e' \neq e} \mathrm{count}_t(e') \cdot E[\mathbf{1}(h(e') = h(e))]$
$= \sum_{e' \neq e} \mathrm{count}_t(e') \cdot \Pr[h(e') = h(e)]$
$\leq \sum_{e' \neq e} \mathrm{count}_t(e') \cdot \left(\frac{1}{k}\right)$
$= \frac{|S_t| - \mathrm{count}_t(e)}{k} \leq \frac{|S_t|}{k}$

k = $1/\epsilon$ makes this at most $\epsilon \cdot |S_t|$. Space is $O(\frac{1}{\epsilon})$ counters plus storing hash function

High Probability Bounds

- Have $0 \leq \mathrm{est}_t(e) - \mathrm{count}_t(e) \leq |S_t|/k$ in expectation
  - With probability 1/2, $\mathrm{est}_t(e) - \mathrm{count}_t(e) \leq 2|S_t|/k$ Why?

- Can we amplify the success probability to 1-δ?
  - Independent repetition: pick m hash functions $h_1, \ldots, h_m$ with $h_i: \Sigma \to \{0, 1, 2, \ldots, k-1\}$ independently from H. Create array A_i for h_i

    when update a_t arrives:

    for each i from 1 to m

    if $a_t = (\mathrm{add}, e)$ **then** $A_i[h_i(e)] + +$

    else $a_t = (\mathrm{del}, e)$ and $A_i[h_i(e)] - -$

High Probability Bounds and Overall Space

What is our new estimate of $\mathrm{count}_t(e)$?

$$\mathtt{best}_t(e) := \min_{i=1}^{m} A_i[h_i(e)].$$

- Intuition: each $A_i[h_i(e)]$ is an *overestimate* to $\mathrm{count}_t(e)$
- By independence, Pr[for all i, $A_i[h_i(e)] \geq 2|S_t|/k] \leq \left(\frac{1}{2}\right)^m$
- For $k = \frac{2}{\epsilon}$ and m = $\log_2\left(\frac{1}{\delta}\right)$, the error is at most $\epsilon|S_t|$ with probability 1-δ
- Space: $m \cdot k = O(\frac{\log\left(\frac{1}{\delta}\right)}{\epsilon})$ counters each of O(lg t) bits

  $m \cdot O(\log|\Sigma|) = O(\log\left(\frac{1}{\delta}\right)\log|\Sigma|)$ bits to store hash functions

ϵ-Heavy Hitters

- Our new estimate $\mathrm{best}_t(e)$ satisfies
$$\Pr[|\mathrm{best}_t(e) - \mathrm{count}_t(e)| \leq \epsilon |S_t|] \geq 1 - \delta$$
  and uses $O(\frac{\log\left(\frac{1}{\delta}\right) \log t}{\epsilon} + \log\left(\frac{1}{\delta}\right) \log |\Sigma|)$ bits of space

- What if we want with probability 1/10, simultaneously for all e, $|\mathrm{best}_t(e) - \mathrm{count}_t(e)| \leq \epsilon |S_t|$?

- Set $\delta = \frac{1}{10|\Sigma|}$ and apply a union bound over all $e \in \Sigma$