Lectures 5: Hashing

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Hashing

- Useful tool for
  - Dictionary data structures
  - Cryptography
  - Complexity theory
  - Streaming algorithms
- Today
  - Universal hashing
  - Perfect hashing

Maintaining a Dictionary

- Large universe of "keys" denoted by U
  - U could be all strings of ASCII characters of length at most 80
- Much smaller "dictionary", which is a subset S of U
  - S could be the set of all English words
- Want to support operations to maintain the dictionary
  - Insert(x): add the key x to S
  - Query(x): is the key x in S?
  - Delete(x): remove the key x from S

Dictionary Settings

- Static: don't support insert and delete operations, just want to optimize your data structure for fast query operations
  - For example, the English dictionary does not change (or only very gradually)
  - Could use a sorted array with binary search
- Insertion-only: just support insert and query operations
- Dynamic: support insert, delete, and query operations
  - Could use a balanced search tree (AVL trees) to get O(log |S|) time per iteration
- Hashing gives an alternative approach, often the fastest and most convenient way to solve these problems

Formal Hashing Setup

- Universe U is very large
  - E.g., set of ASCII strings of length 80 is 128^{80}
- Care about a small subset $S \subset U$. Let N = |S|.
  - S could be the names of all students in this class
- Our data structure is an array A of size M and a "hash function" h: U → $\{0, 1, \ldots, M\text{-}1\}$.
  - Typically $M \ll U$, so can't just store each key x in A[x]
  - Insert(x) will try to place key x in A[h(x)]
- But what if h(x) = h(y) for $x \neq y$? We let each entry of A be a linked list.
  - To insert an element x into A[h(x)], insert it at the top of the list
  - Hope linked lists are small

Implementation Details

- Hashing easy to implement
- Query(x): compute i = h(x) and walk down the list A[i] until you find x or walk off the list
- Insert(x): place x at the top of the list A[i]
- Delete(x): remove x from the list A[i] by walking down the list

How to Choose the Hash Function h?

- Want it to be unlikely that h(x) = h(y) for different keys x and y
- Want our array size M to be of size O(N), where N is number of keys
- Want to quickly compute h(x) given x
  - For now, we will treat this computation as O(1) time
- How long do Query(x) and Delete(x) take?
  - O(length of list A[h(x)]) time
- How long does Insert(x) take?
  - O(1) time no matter what
- *So how long can these lists A[h(x)] be?*

Bad Sets Exist for any Hash Function

- Claim: For any hash function h: U -> {0, 1, 2, …, M-1}, if $|\mathrm{U}| \geq (\mathrm{N}-1)\mathrm{M}+1$, there is a set S of N elements of U that all hash to the same location
- Proof: If every location had at most N-1 elements of U hashing to it, then we would have $|\mathrm{U}| \leq (\mathrm{N}-1)\mathrm{M}$
- So there's no good hash function h that works for every S. Thoughts?
- *Universal Hashing: Let's randomly choose h!*
  - We show for *any* sequence of insert, query, and delete operations, the expected number of operations, over a random choice of h, will be small

Universal Hashing

- Definition: A randomized algorithm H for constructing a hash function h: U -> {0, 1, 2, …, M-1} is universal if for all $x \neq y$,

$$\Pr_{h \leftarrow H}[h(x) = h(y)] \leq \frac{1}{M}$$

- We also say a set H of hash functions is a universal hash function family if choosing $h \in H$ uniformly at random is universal
- Note the condition holds for every x ≠ y, and the randomness is only over the choice of h from H
- Equivalently, for every $x \neq y$, we have: $\frac{|h \in H \; | h(x) = h(y)|}{|H|} \leq \frac{1}{M}$

Universal Hashing Examples

Example 1: The following three hash families with hash functions mapping the set $\{a, b\}$ to $\{0, 1\}$ are universal, because at most $1/M$ of the hash functions in them cause a and b to collide, were $M = |\{0, 1\}|$.

	a	b
h_1	0	0
h_2	0	1

	a	b
h_1	0	1
h_2	1	0

	a	b
h_1	0	0
h_2	1	0
h_3	0	1

Examples that are Not Universal

	a	b
h_1	0	0
h_3	1	1

	a	b	c
h_1	0	0	1
h_2	1	1	0
h_3	1	0	1

- Note that a and b collide with probability more than 1/M = 1/2

Universal Hashing Example

- The following hash function is universal with M = |{0,1,2}|

	a	b	c	
h_0	0	0	0	← Note!
h_1	0	1	2	
h_2	1	2	0	
h_3	2	0	1	

Using Universal Hashing

- Theorem: If H is universal, then for any set $S \subseteq U$ with |S| = N, for any $x \in S$, if we choose h at random from H, the **expected** number of collisions between x and other elements in S is at most N/M.

- Proof: For $y \in S$ with $y \neq x$, let $C_{xy} = 1$ if h(x) = h(y), otherwise $C_{xy} = 0$

  Let $C_x = \sum_{y \neq x} C_{xy}$ be the total number of collisions with x

  $E[C_{xy}] = \Pr[h(x) = h(y)] \leq \frac{1}{M}$

  By linearity of expectation, $E[C_x] = \sum_{y \neq x} E[C_{xy}] < \frac{N}{M}$

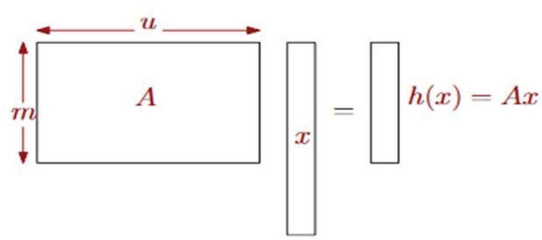
Using Universal Hashing

- Corollary: If H is universal, for any **sequence** of L insert, query, and delete operations in which there are at most M keys in the data structure at any time, the expected cost of the L operations for a random $h \in H$ is O(L)
  - Assumes the time to compute h is O(1)

- Proof: For any operation in the sequence, its expected cost is O(1) by the last theorem, so the expected total cost is O(L) by linearity of expectation

But how to Construct a Universal Hash Family?

- Suppose $|U| = 2^u$ and $M = 2^m$
- Let A be a random m x u binary matrix, and h(x) = Ax mod 2



- Claim: for $x \neq y$, $\Pr_x[h(x) = h(y)] = \frac{1}{M} = \frac{1}{2^m}$

But how to Construct a Universal Hash Family?

- Claim: For $x \neq y$, $\Pr_{x}[h(x) = h(y)] = \frac{1}{M} = \frac{1}{2^m}$
- Proof: $A \cdot x \bmod 2 = \sum_i A_i x_i$ mod 2, where A_i is the i-th column of A

  If h(x) = h(y), then Ax=Ay mod 2, so A(x-y) = 0 mod 2

  If $x \neq y$, there exists an i^* for which $x_{i^*} \neq y_{i^*}$

  Fix A_j for all $j \neq i^*$, which fixes b = $\sum_{j \neq i^*} A_j (x_j - y_j)$ mod 2

  A(x-y) = 0 mod 2 if and only if A_{i^*}= b

  $\Pr_{A_{i^*}}[A_{i^*} = b] = \frac{1}{2^m} = \frac{1}{M}$

So h(x) = Ax mod 2 is universal

k-wise Independent Families

- Definition: A hash function family H is k-universal if for every set of k distinct keys $x_1, \ldots, x_k$ and every set of k values $v_1, \ldots, v_k \in \{0, 1, \ldots, M-1\}$,

$$\Pr[h(x_1) = v_1 \text{ AND } h(x_2) = v_2 \text{ AND } \ldots \text{ AND } h(x_k) = v_k] = \frac{1}{M^k}$$

- If H is 2-universal, then it is universal. Why?

- h(x) = Ax mod 2 for a random binary A is not 2-universal. Why?

- Exercise: Show Ax + b mod 2 is 2-universal, where A in $\{0,1\}^{m \times n}$ and b $\in$ $\{0,1\}^m$ are chosen independently and uniformly at random

More Efficient Universal Hashing

- Given a key x, suppose x = $[x_1, \ldots, x_k]$ where each $x_i \in \{0, 1, \ldots, M-1\}$
- Suppose M is prime
- Choose random $r_1, .., r_k \in \{0, 1, \ldots, M-1\}$ and define
$$h(x) = r_1 x_1 + r_2 x_2 + \ldots + r_k x_k \bmod M$$
- Uses less randomness than matrix product
- Claim: the family of such hash functions is universal, that is, $\Pr_h[h(x) = h(y)] \leq \frac{1}{M}$ for all x and y

More Efficient Universal Hashing

- Claim: the family of such hash functions is universal, that is, $\Pr_h[h(x) = h(y)] \le \frac{1}{M}$ for all x ≠ y
- Proof: Since $x \neq y$, there is an i^* for which $x_{i^*} \neq y_{i^*}$

  Let $h'(x) = \sum_{j \neq i^*} r_j x_j$, and h(x) = h'(x) + $r_{i^*} x_{i^*}$ mod M

  If h(x) = h(y), then h'(x) + $r_{i^*} x_{i^*}$= h'(y) + $r_{i^*} y_{i^*}$ mod M

  So $r_{i^*}(x_{i^*} - y_{i^*}) = h'(y) - h'(x)$ mod M, or $r_{i^*} = \frac{h'(y) - h'(x)}{x_{i^*} - y_{i^*}}$ mod M

  This happens with probability exactly 1/M

Perfect Hashing

- If we fix the dictionary S of size N, can we find a hash function h so that all query(x) operations take constant time?

- Claim: If H is universal and $\mathrm{M} = \mathrm{N}^2$, then $\Pr_{h \leftarrow H}[\text{ no collisions in S}] \geq \frac{1}{2}$

- Proof: How many pairs (x,y) in S are there?

  Answer: N(N-1)/2

  For each pair, the probability of a collision is at most 1/M

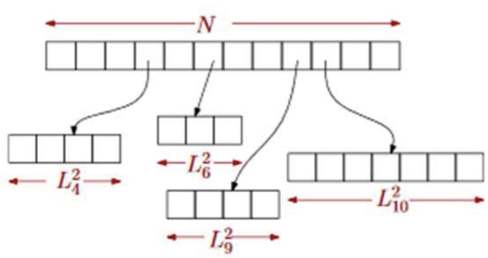
  Pr[exists a collision]$\leq$ (N(N-1)/2)/M $\leq \frac{1}{2}$

Just try a random h and check if there are any collisions

Problem: our hash table has $\mathrm{M} = \mathrm{N}^2$ space! How can we get O(N) space?

Perfect Hashing in O(N) Space – 2 Level Scheme

- Choose a hash function $\mathrm{h}: \mathrm{U} \rightarrow \{1, 2, \ldots, \mathrm{N}\}$ from a universal family
- Let $\mathrm{L_i}$ be the number of items x in S for which h(x) = i
- Choose N "second-level" hash functions $\mathrm{h_1}, \mathrm{h_2}, \ldots, \mathrm{h_N}$, where $\mathrm{h_i}: \mathrm{U} \rightarrow \{1, \ldots, \mathrm{L_i^2}\}$



By previous analysis, can choose hash functions $\mathrm{h_1}, \mathrm{h_2}, \ldots, \mathrm{h_N}$ so that there are no collisions, so O(1) time

Hash table size is $\sum_{\mathrm{i}=1,\ldots,\mathrm{n}} \mathrm{L_i^2}$

How big is that??

Perfect Hashing in O(N) Space – 2 Level Scheme

- Theorem: If we pick h from a universal family H, then

$$\Pr_{h \leftarrow H}[\sum_{i=1,\ldots,N} L_i^2 > 4N] \leq \frac{1}{2}$$

- Proof: It suffices to show $E[\sum_i L_i^2] < 2N$ and apply Markov's inequality

  Let $C_{x,y} = 1$ if h(x) = h(y). By counting collisions on both sides, $\sum_i L_i^2 = \sum_{x,y} C_{x,y}$

  If x = y, then $C_{x,y} = 1$. If $x \neq y$, then $E[C_{x,y}] = \Pr[C_{x,y} = 1] \leq \frac{1}{N}$

  $E[\sum_i L_i^2] = \sum_{x,y} E[C_{x,y}] = N + \sum_{x \neq y} E[C_{x,y}] = N + N(N-1)/(2N)$ < 2N

So just choose a random h in H, and check if $\sum_{i=1,\ldots,n} L_i^2 \leq 4N$, and if so, then choose $h_1, \ldots, h_N$