Lectures 24 and 25: Sketching

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Some slides from Jonathan Davis and Andrew McGregor

Outline

- Sketching model
  - Estimating the Euclidean norm of a vector
  - Finding a non-zero coordinate of a vector
- Graph sketching
  - Boruvka's spanning tree algorithm
  - Finding a spanning tree from a sketch

Sketching

- Random linear projection S: $\mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{k}}$ that preserves properties of any $\mathrm{x} \in \mathrm{R}^{\mathrm{n}}$ with high probability ,where $\mathrm{k} \ll \mathrm{n}$

$$\begin{pmatrix} & S & \end{pmatrix} \begin{pmatrix} \\ x \\ \\ \end{pmatrix} = \begin{pmatrix} Sx \end{pmatrix} \longrightarrow \text{answer}$$

- Matrix S does not depend on x, e.g., S could be a random matrix (require the entries of S be O(log n) bits)

Application: Streams with Deletions

- n-dimensional vector x initialized to 0^n
- Stream of updates $x_i \leftarrow x_i + \Delta_j$ for Δ_j in {-1,1}
- One pass over the stream, have small memory
- Given $S \cdot x$ and update $x_i \leftarrow x_i + \Delta_j$, replace $S \cdot x$ with $S \cdot x + S \cdot e_i \cdot \Delta_j$
- Memory to store $S \cdot x$ is (# of rows of S)$\cdot \log n$ bits
  - Also need to store S, which typically can be stored implicitly

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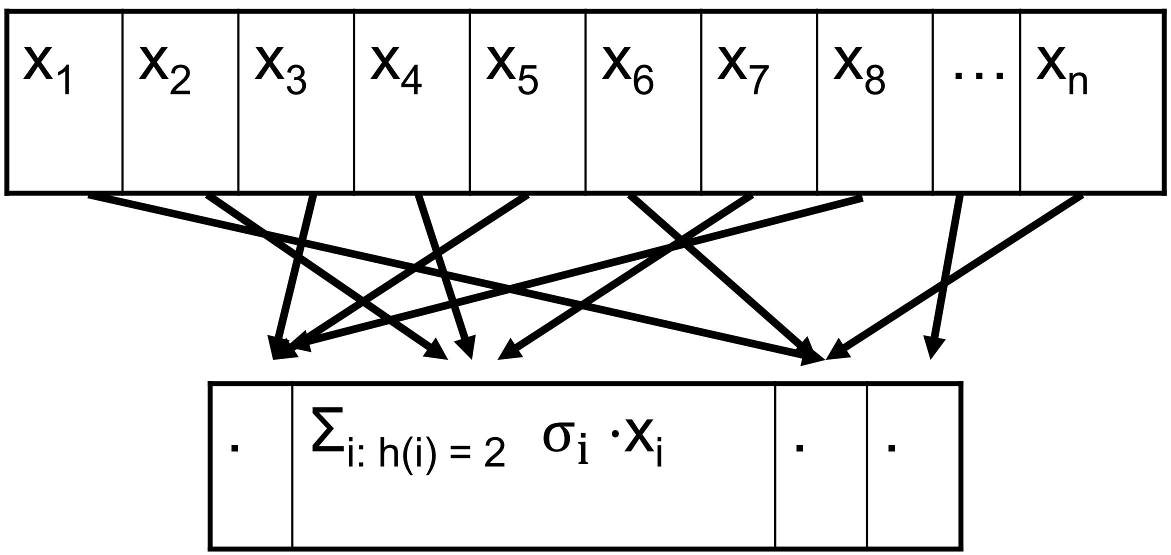
Example: Estimating the Norm of a Vector

- For $x \in R^n$, its squared Euclidean norm is $|x|^2 = \sum_i x_i^2$
- Find a number Z for which $(1-\epsilon)|x|^2 \leq Z \leq (1+\epsilon)|x|^2$
- Choose a 2-wise independent hash function h: [n] -> $[k]$
- Choose a 4-wise independent hash function $\sigma: [n] \rightarrow \{-1,1\}$



CountSketch

- CountSketch is a linear map S: $R^n \to R^k$
- A row i of S is a hash bucket, and $(Sx)_i$ is the value in the bucket
- Output $|Sx|^2$

- $E[|Sx|^2]$ $= E[\sum_j (\sum_i \delta(h(i) = j)\sigma(i)\, x_i)^2]$

$$= \sum_j \sum_{i,i'} x_i x_{i'} E[\delta(h(i) = j)\delta(h(i') = j)\sigma(i)\sigma(i')]$$

$$= \sum_j \sum_{i,i'} x_i x_{i'}\, E[\delta(h(i) = j)\delta(h(i') = j)] \cdot E[\sigma(i) \cdot \sigma(i')]$$

$$= \frac{\sum_j \sum_i x_i^2}{k} = |x|^2$$

CountSketch

- In recitation, will show $\mathrm{Var}[|Sx|^2] = O(|x|^4/k)$

- By Chebyshev's inequality,

$$\Pr\left[\left||Sx|^2 - |x|^2\right| > \epsilon|x|^2\right] \leq \frac{\mathrm{Var}\left[|Sx|^2\right]}{\epsilon^2|x|^4} \leq \frac{1}{10} \text{ provided } k = \Theta(\frac{1}{\epsilon^2})$$

- If S has $k = \Theta(\frac{1}{\epsilon^2})$ rows, can estimate $|x|^2$ from Sx up to a $(1+\epsilon)$-factor with probability at least 9/10

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A 1-Sparse Recovery Algorithm

- n-dimensional vector x initialized to 0^n
- Stream of updates $x_i \leftarrow x_i + \Delta_j$ for Δ_j in {-1,1}
  - Promised at all times, $-\mathrm{poly(n)} \leq \mathrm{x_i} \leq \mathrm{poly(n)}$
- Want a procedure which with probability 1-1/poly(n),
  - if x is 1-sparse, i.e., has exactly one non-zero entry $\mathrm{x_i}$, it returns i
  - otherwise outputs FAIL

A 1-Sparse Recovery Algorithm

- Let p = poly(n) be a random prime, and z a random integer mod p

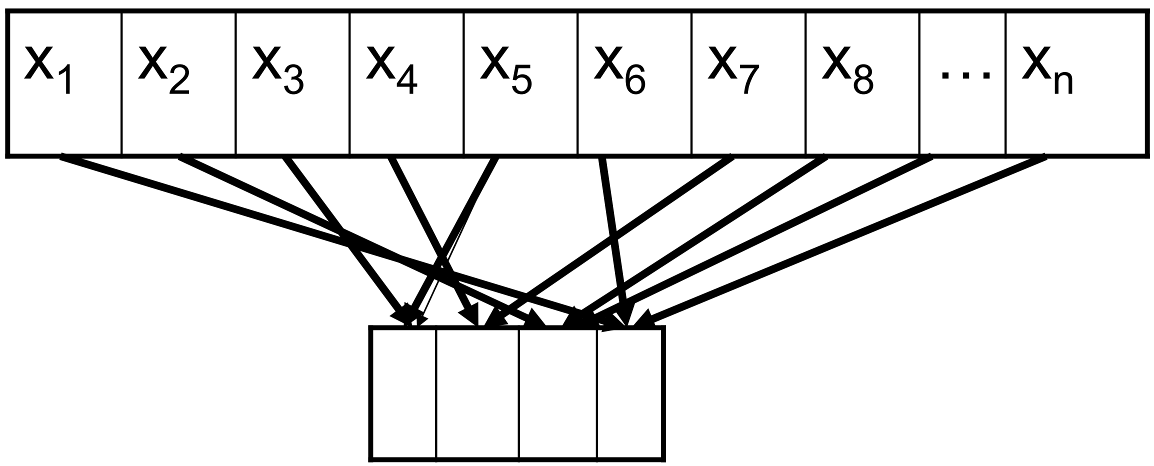
  Maintain A = $\sum_i x_i$, $B = \sum_i x_i \cdot i$, and $C = \sum_i x_i \cdot z^i$ mod p
- If B/A is not in {1, 2, …, n}, output FAIL
- Else if $C = A \cdot z^{B/A}$ mod p, output B/A. Otherwise output FAIL
- Claim: If x is 1-sparse, we succeed. Why?
- Claim: If x not 1-sparse, we output FAIL with probability 1-1/poly(n)
- Proof: If B/A is not in {1, 2, …, n}, we output FAIL
- Else, B/A is in {1, 2, …, n}, and let q(y) = $\sum_i x_i y^i - A \cdot y^{B/A}$ mod p
- q(y) is a degree at most n polynomial, and so has at most n roots. The chance that z is one of them is at most n/p = 1/poly(n)

Outputting a Non-Zero Coordinate of a Vector

- Maintain A = $\sum_i x_i$, $B = \sum_i x_i \cdot i$, and $C = \sum_i x_i \cdot z^i$ mod p
- O(log n) bits of space
- If x is 1-sparse, output single non-zero coordinate
- Otherwise, with probability 1-1/poly(n), output FAIL
- Call this algorithm 1-Sparse-Finder
- *Can we use 1-Sparse-Finder to find a non-zero item of x if x is not 1-sparse?*

Outputting a Non-Zero Coordinate of a k-Sparse Vector

- If x is k-sparse, i.e., has k non-zero entries, use hashing
- Let h be a 2-universal hash function from [n] to [10k]



- In the j-th hash bucket, run 1-Sparse Finder

k-Sparse Algorithm Analysis

- In each bucket, find a non-zero entry i or output FAIL, with probability 1-1/poly(n)
- *What if all non-zero items of x collide in a bucket?*
- Consider a non-zero entry i of x
- Since h is 2-universal, with probability at least 1-k/(10k) = 9/10,

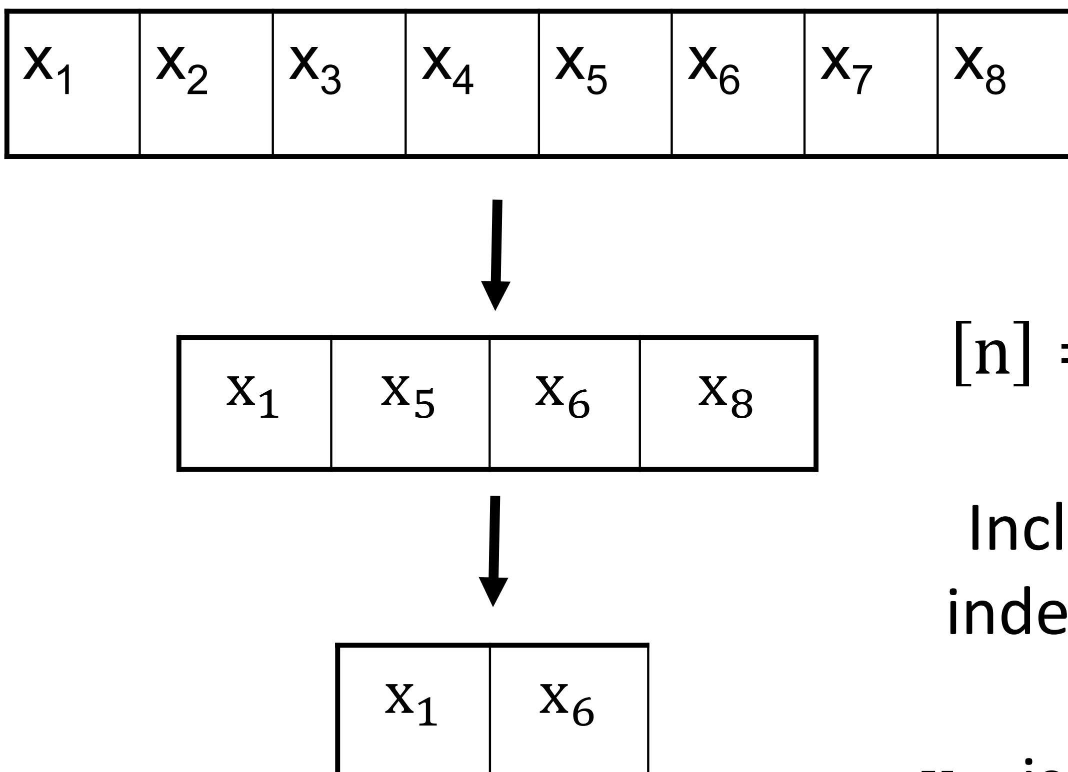
  $$h(i) \notin \{h(j) \mid j \neq i \text{ and } x_j \neq 0\}$$

- With 9/10 probability, we output a non-zero entry i of x
- We know when we fail to output a non-zero entry (except with probability 1/poly(n))

Reducing the Space

- Have an algorithm which, if x is k-sparse, outputs a non-zero item or says FAIL
- Outputs a non-zero item with probability at least 9/10
- Use O(k log n) bits of space
- Good if k is small, but how to reduce the space for large k?

Subsampling



Uniformly sample
the coordinates
as nested subsets

$$[n] = S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_{\log_2 n}$$

Include each item from S_{i-1} in S_i
independently with probability 1/2

x_{S_i} is x restricted to coordinates in S_i

Algorithm for Finding a Non-Zero Item

- *If x has k non-zero entries, what's the expected number of non-zero entries in* x_{S_i}*?*
  - For each non-zero entry j in x, let $Z_j = 1$ if $j \in S_i$, and $Z_j = 0$ otherwise
  - $Z = \sum_j Z_j$,
  - $E[Z] = k \cdot E\left[Z_j\right] = \frac{k}{2^i}$
  - $\mathrm{Var}[Z] = \sum_j \mathrm{Var}\left[Z_j\right] = k \cdot \mathrm{Var}[Z_1] = k \left(\frac{1}{2^i}\right)\left(1 - \frac{1}{2^i}\right) \leq \frac{k}{2^i}$
- If $i = \lfloor \log_2 k \rfloor - 5$, then $32 \leq E[Z] < 64$ and $\mathrm{Var}[Z] < 64$
- By Chebyshev's inequality, $\Pr[|Z - E[Z]| \geq 32] \leq \frac{\mathrm{Var}[Z]}{32^2} \leq \frac{1}{16}$
- If we run a k'-sparse algorithm with k' = 96 on x_{S_i}, we recover a non-zero item of x_{S_i} with probability at least 1-1/16 – 1/10 > 4/5, or output FAIL
- *But we don't know i?*

Algorithm for Finding a Non-Zero Item

- Run a k'=96-sparse vector algorithm on every x_{S_i} !

- For each x_{S_i}, our algorithm either returns a non-zero item of x_{S_i}, and hence of x, or outputs FAIL

- For $i = \lfloor \log_2 k \rfloor - 5$, with probability at least 4/5, we output a non-zero item of x_{S_i}, and hence of x

- Space is $(\log_2 n) \cdot O(k' \log n) = O(\log^2 n)$ bits!

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Sketching Graphs

Are there sketches for graphs? A_G is the n x n adjacency matrix of a graph G

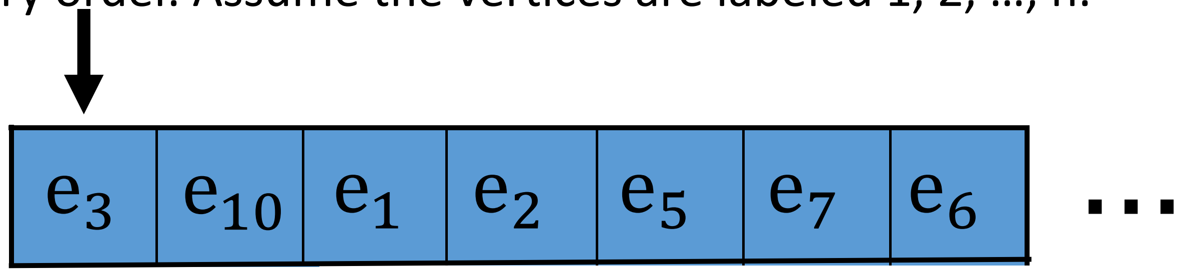
- $(A_G)_{i,j} = 1$ if {i,j} is an edge, and $(A_G)_{i,j} = 0$ otherwise

$$\begin{pmatrix} \quad S \quad \end{pmatrix}\begin{pmatrix} \\ \quad A_G \quad \\ \\ \end{pmatrix} = \begin{pmatrix} \quad SA_G \quad \end{pmatrix} \longrightarrow \text{answer}$$

- Is there a distribution on matrices S with a small number of rows so that you can output a spanning tree of G, given SA_G, with high probability?

Application: Graph Streams

- Want to process a graph stream, where we see the edges of a graph $e_1, \ \dots, e_m$ in an arbitrary order. Assume the vertices are labeled 1, 2, …, n.



- Make 1 pass over the stream
- Trivially store stream using $O(n^2)$ bits of memory.
- Want to use $n \cdot \mathrm{poly}(\log n)$ bits of memory

- How would you compute a spanning forest?

Computing a Spanning Forest

- For each edge e in the stream
  - If ______________________, store edge e
- ______________________ is “doesn’t form a cycle”
- Store at most n-1 edges, so O(n log n) bits of memory
- *But what if you are allowed to delete edges? This is called a dynamic stream*

Handling Deletions with Sketching

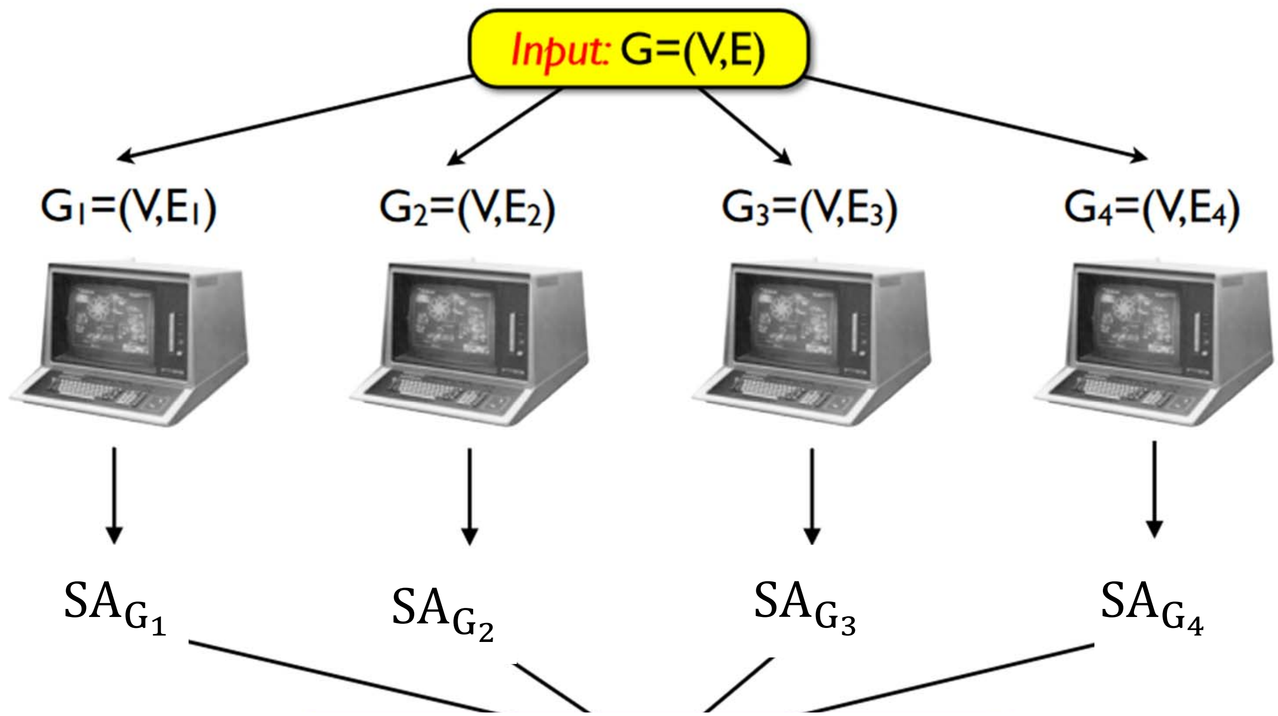
- Given $S \cdot A_G$, if e is deleted, replace it with $S \cdot A_G - S \cdot A_e = S \cdot A_{G-e}$

- Memory to store $S \cdot A_G$ is (# of rows of S)$\cdot$ $n \cdot \log n$ bits
  - Also need to store S, which is (# of rows of S)$\cdot$ $n \cdot \log n$ bits

- Goal: find a distribution on matrices S with a small # of rows so that given $S \cdot A_G$, can output a spanning tree of G with high probability

- Theorem: there is a distribution on S with $O(\log^2 n)$ rows!

Parallel Computing



$$SA_G = SA_{G_1} + SA_{G_2} + SA_{G_3} + SA_{G_4}$$

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Boruvka's Spanning Tree Algorithm (Modified)

- Assume input graph is connected

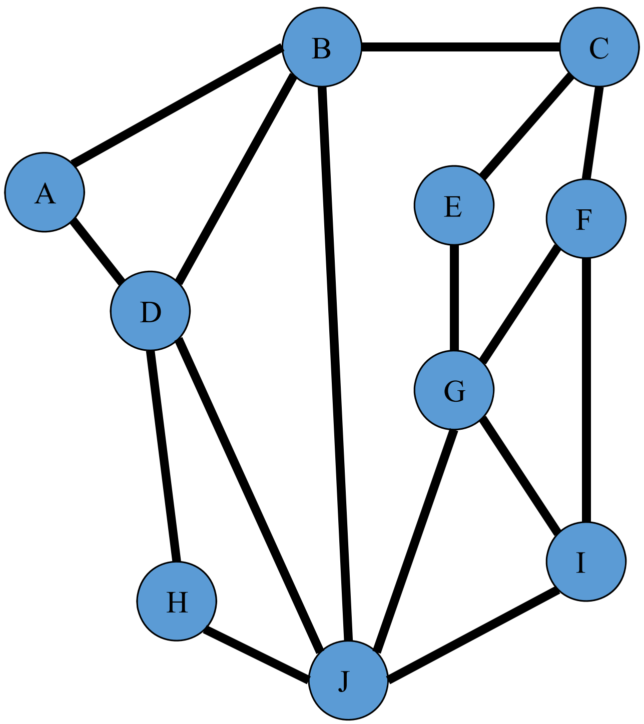
- Initialize edgeset E' to Ø

- Create a list of n groups of vertices, each initialized to a single vertex

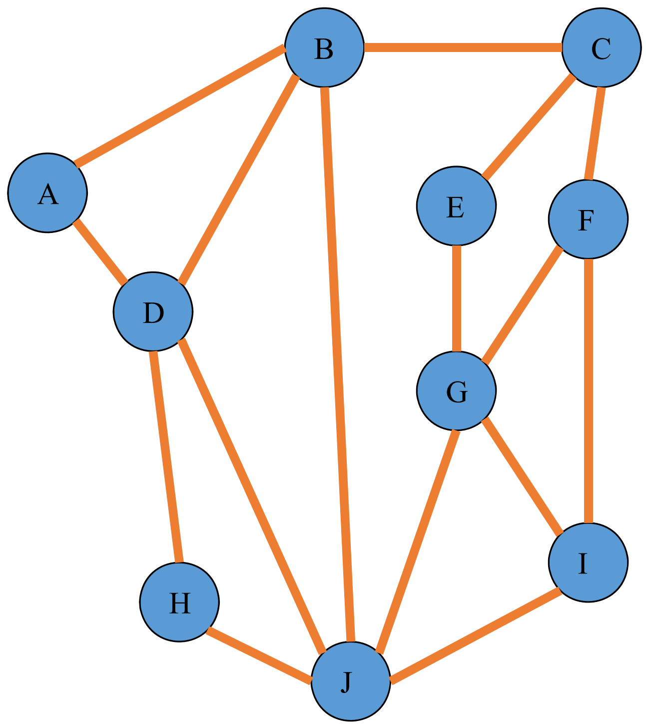
- While the list has more than one group
  - For each group G, include in E' an edge e from a vertex in G to a vertex not in G
  - Merge groups connected by an edge in the previous step

- Find a spanning tree among the edges in E'

Input Graph



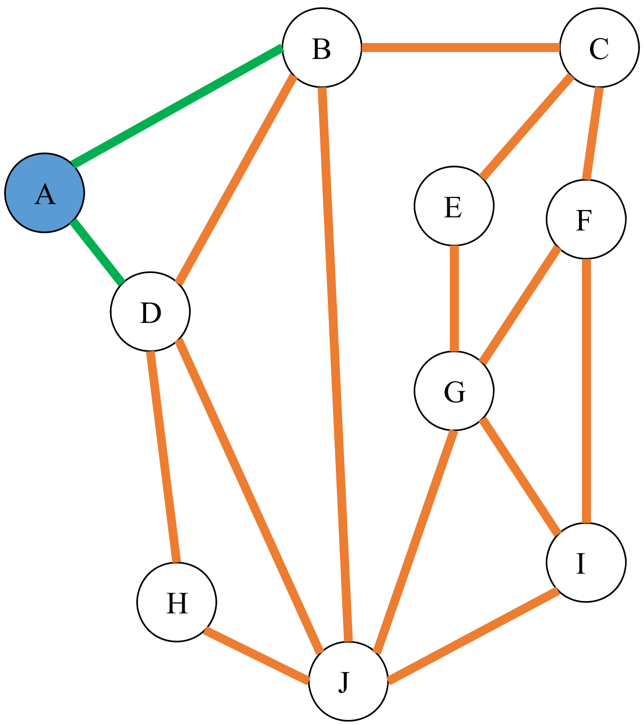
Groups at Beginning of Round 1



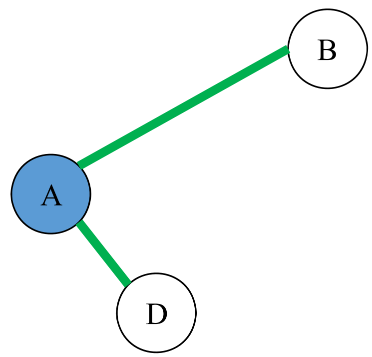
List of Groups

- A
- B
- C
- D
- E
- F
- G
- H
- I
- J

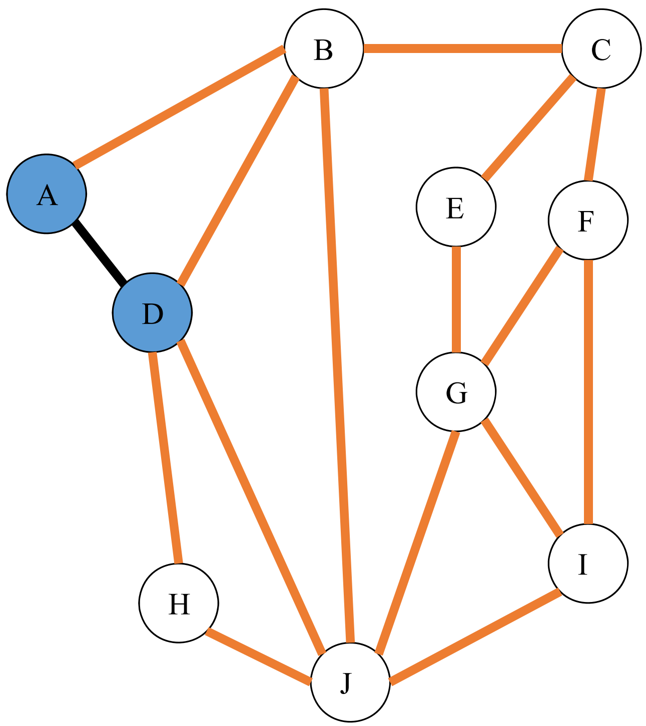
Round 1



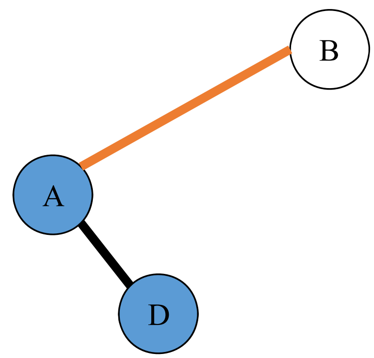
Group A



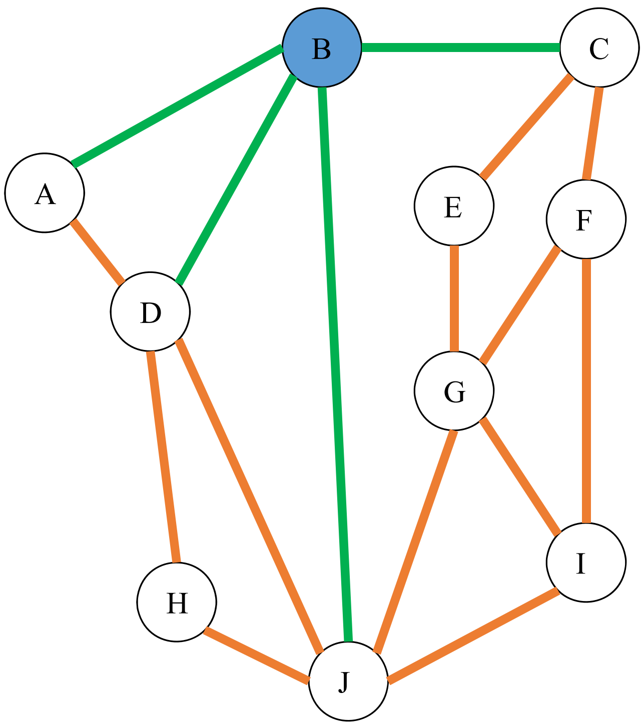
Round 1



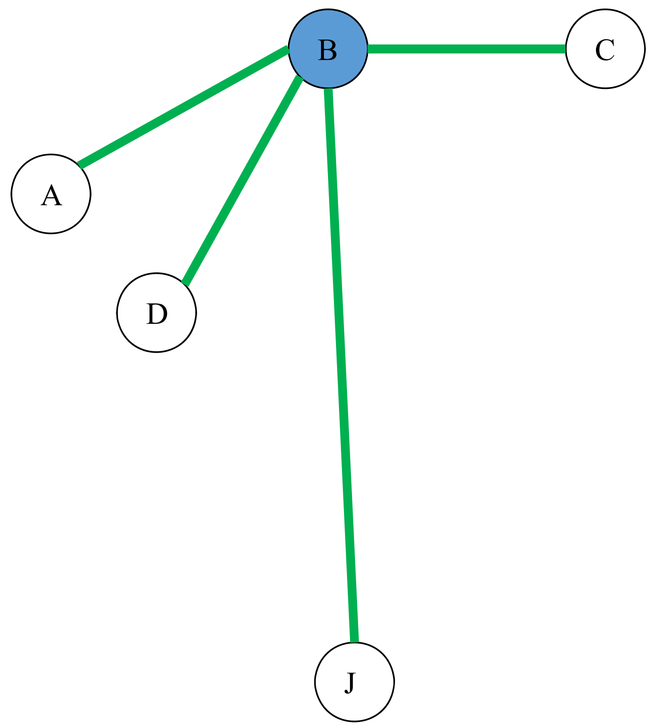
Edge A-D



Round 1



Group B



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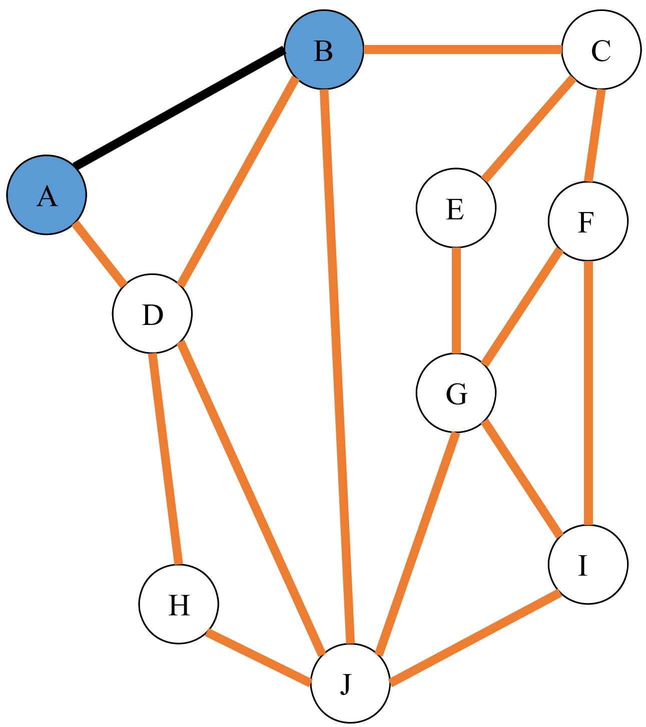
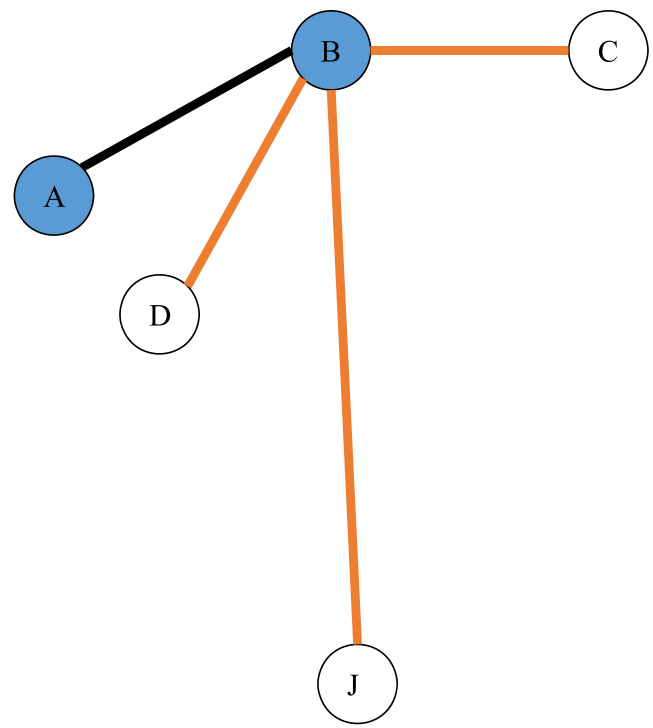
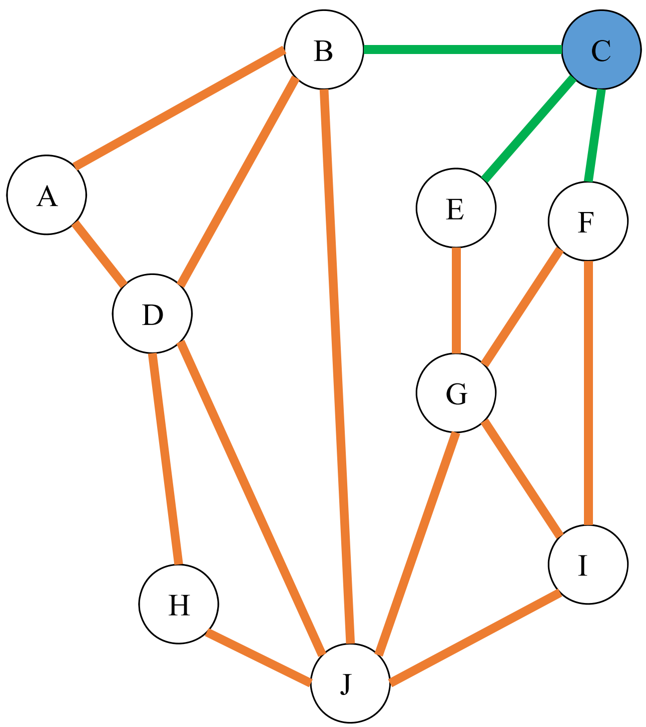


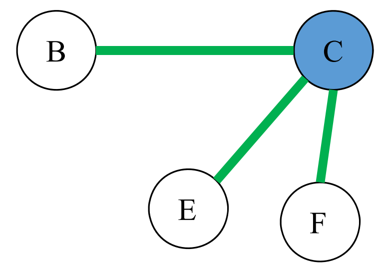
Edge B-A



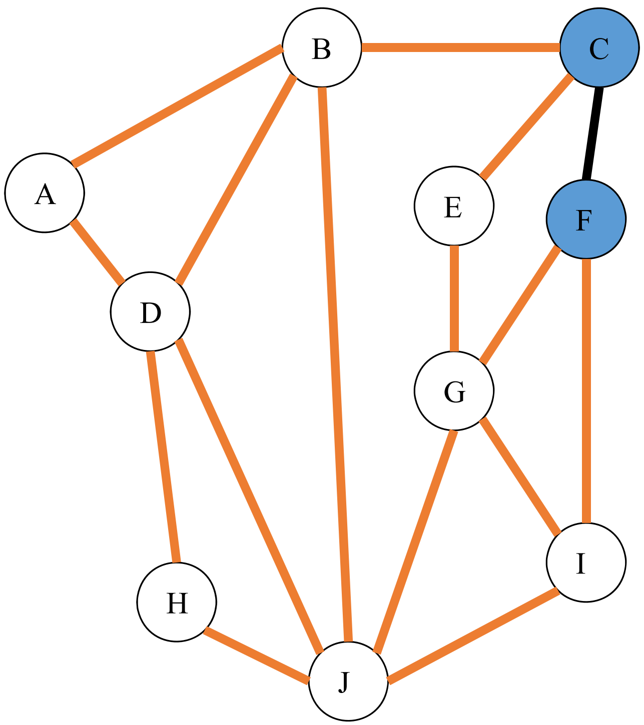
Round 1



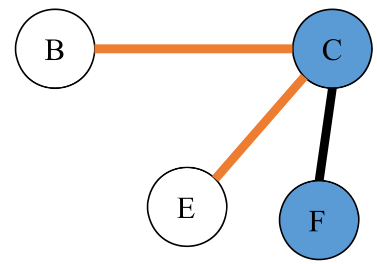
Group C



Round 1



Edge C-F



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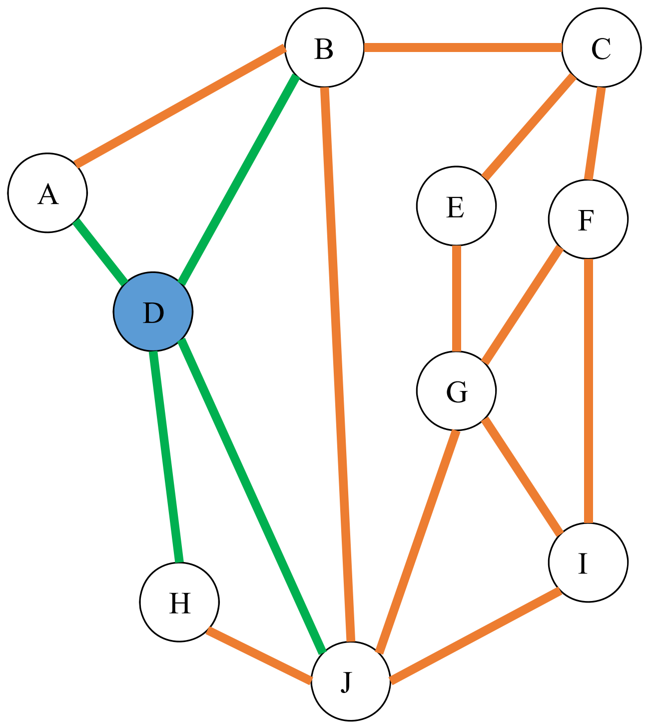
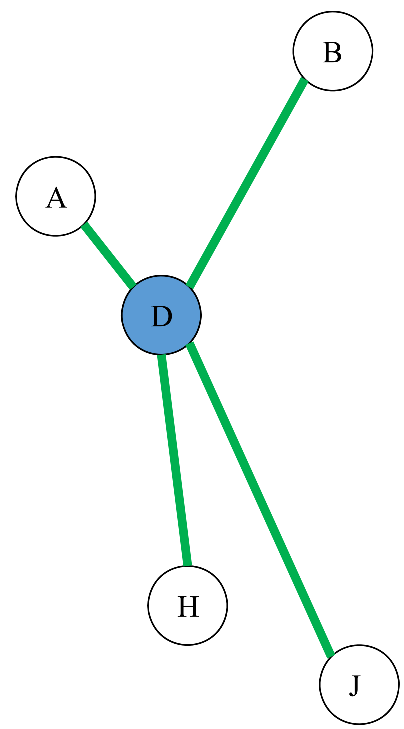


Group D



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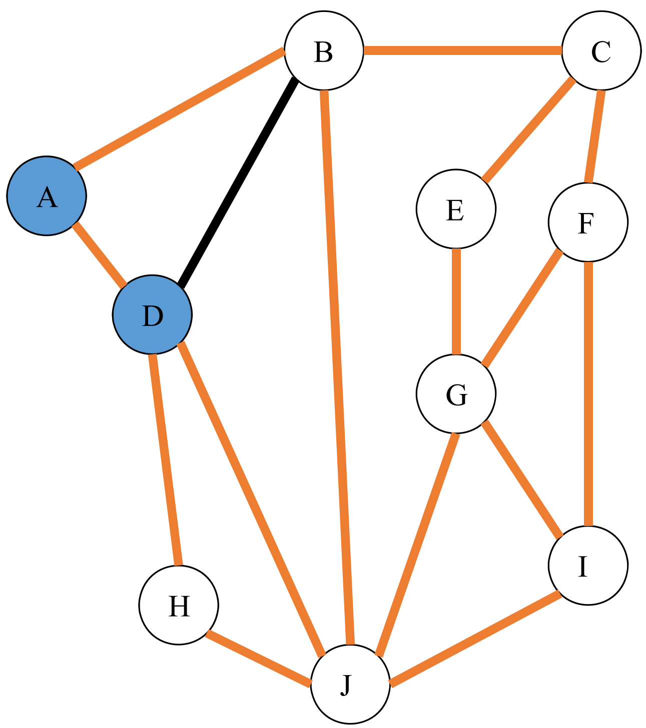
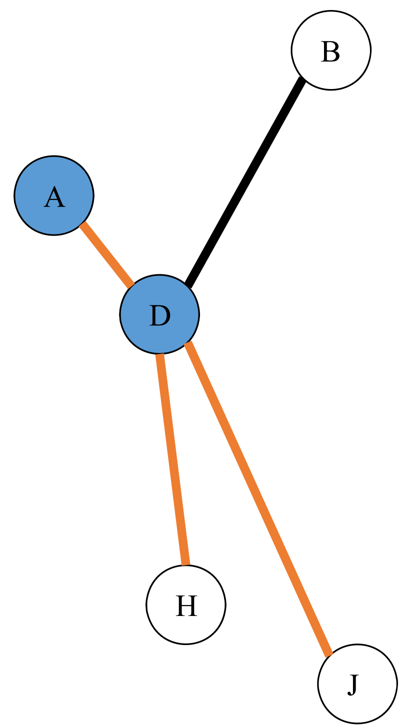


Edge D-A



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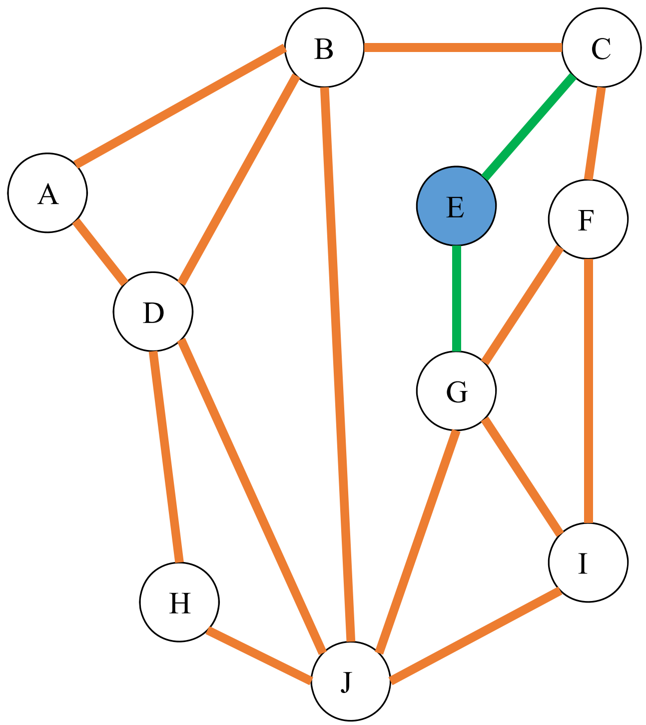
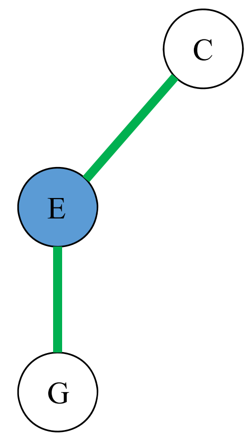


Group E



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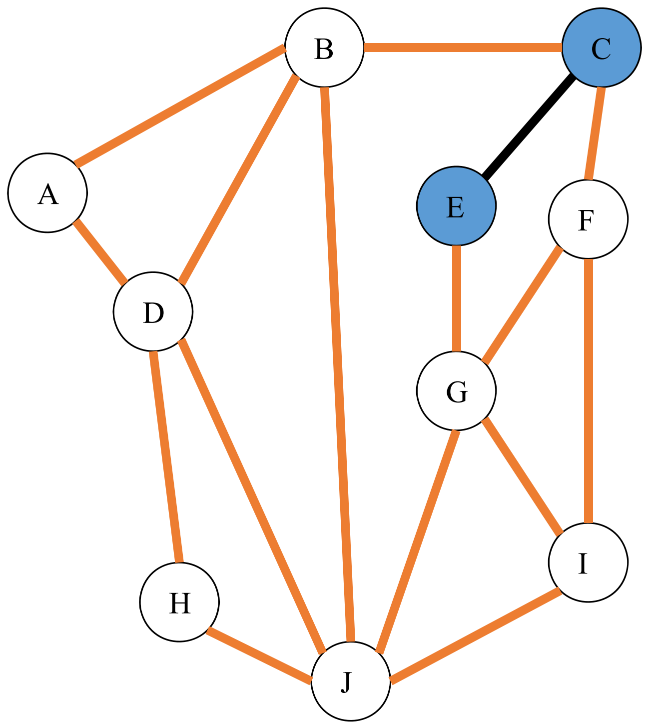
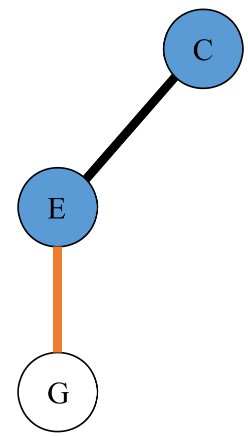


Edge E-C



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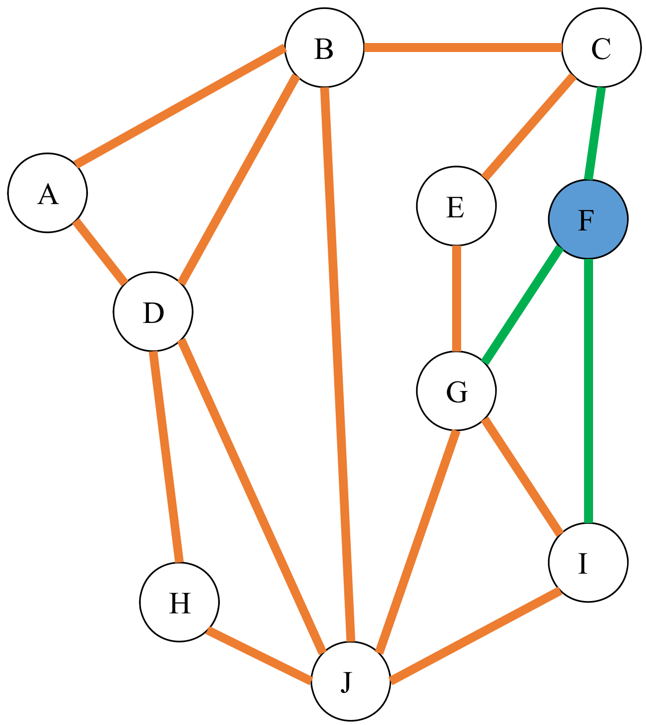
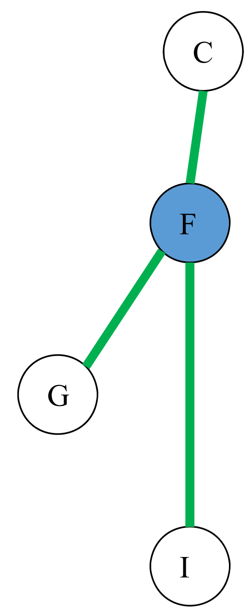


Group F



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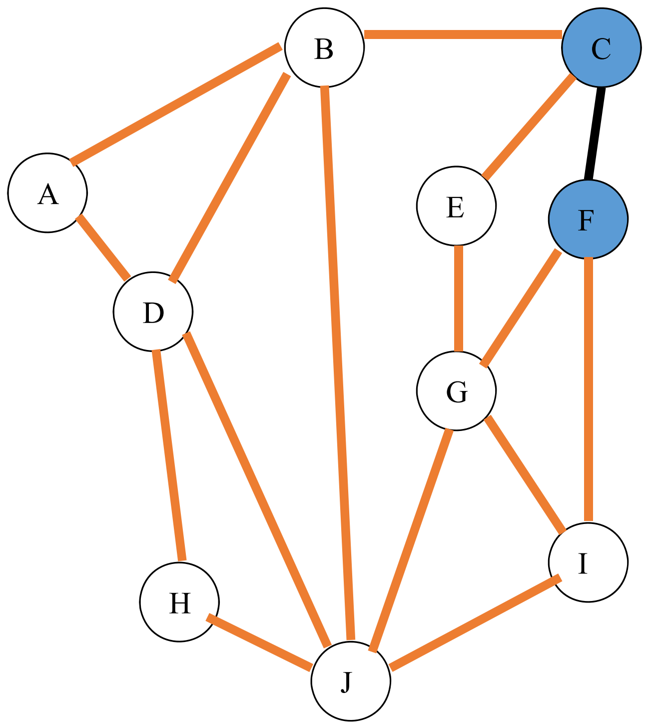
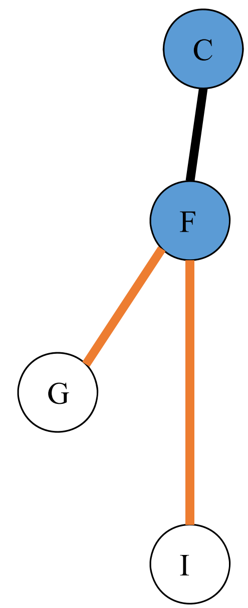


Edge F-C



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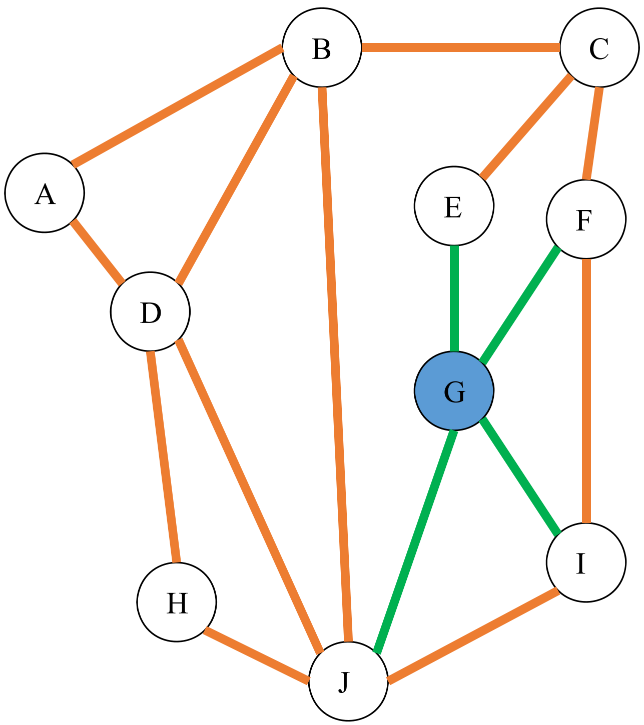
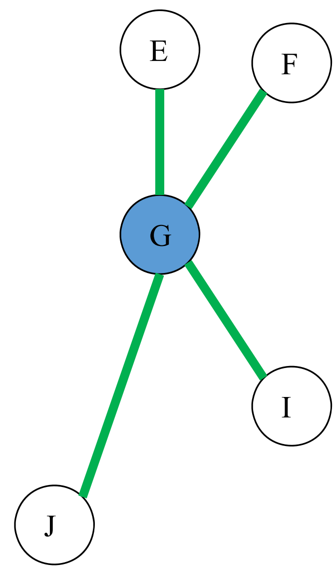


Group G



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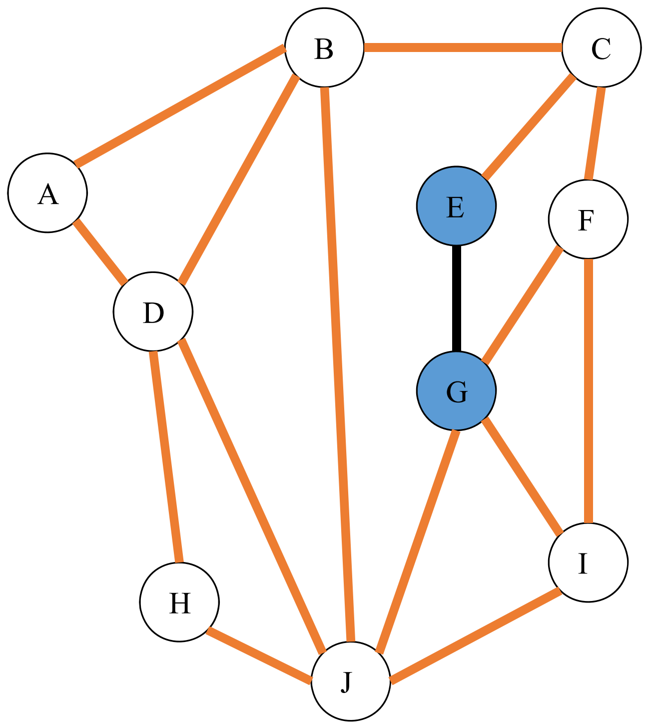
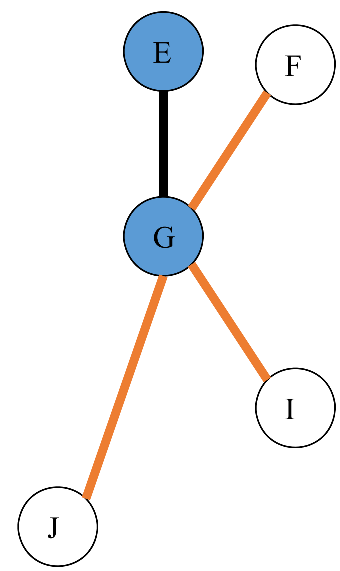


Edge G-E



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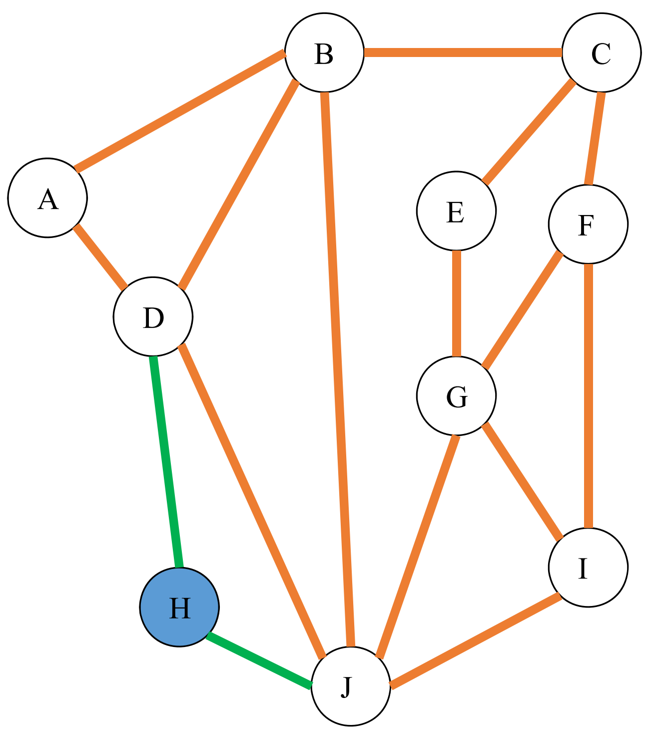
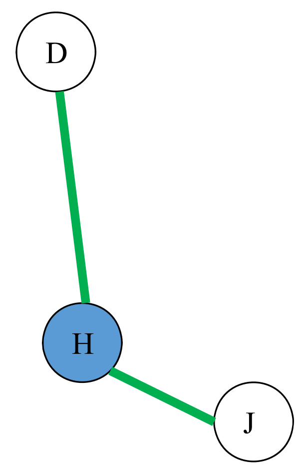


Group H



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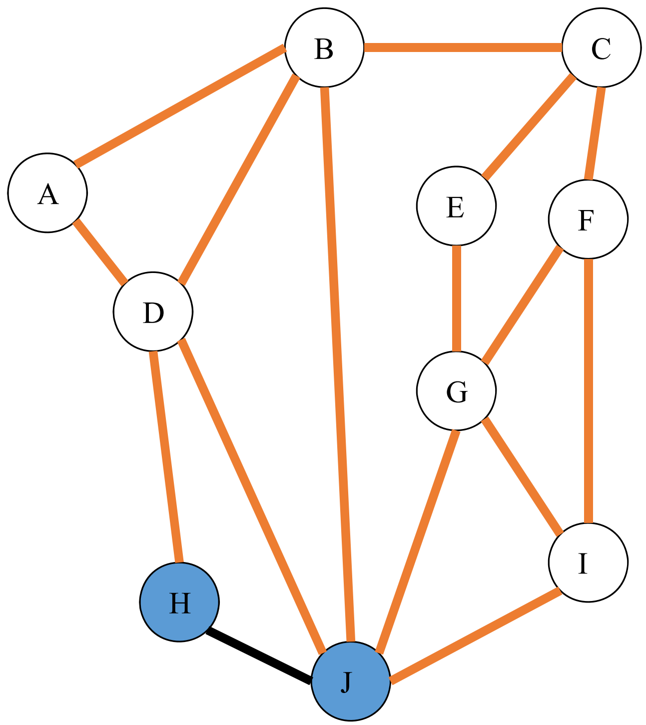
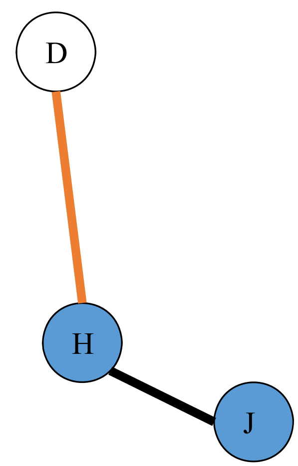


Edge H-J



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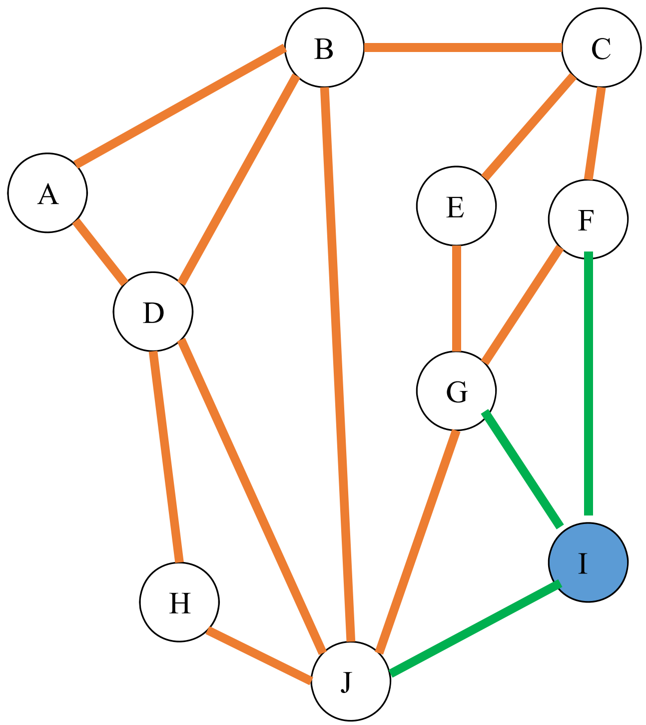
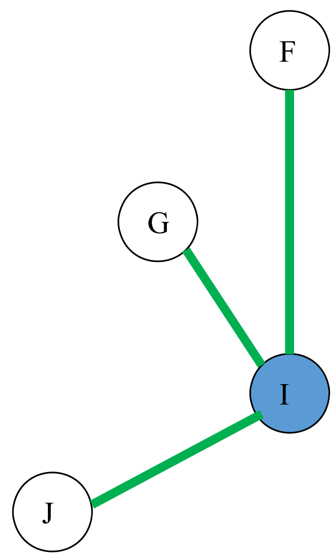


Group I



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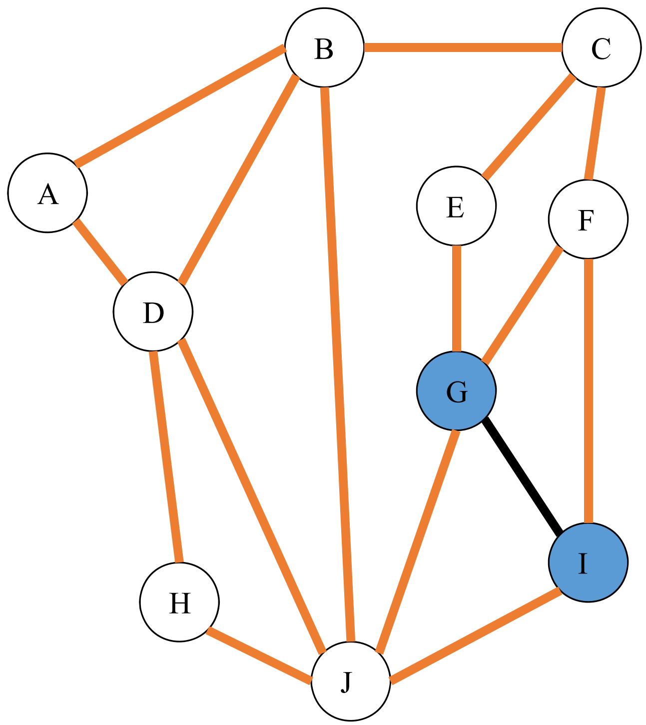
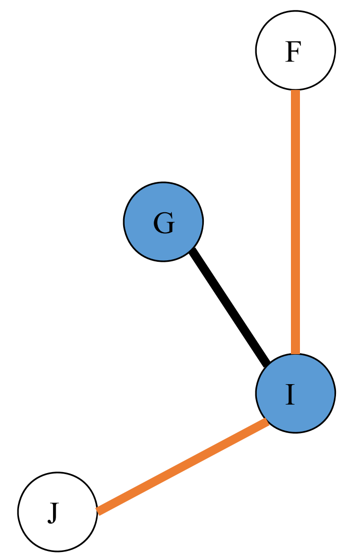
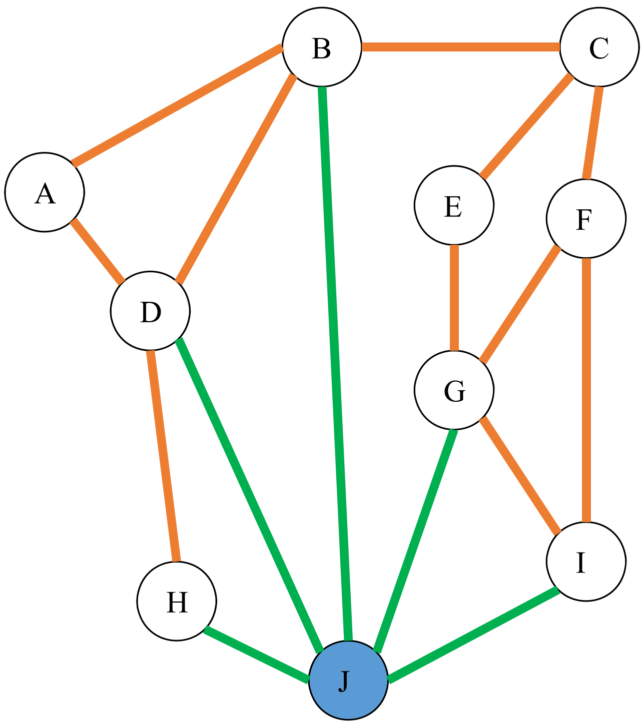


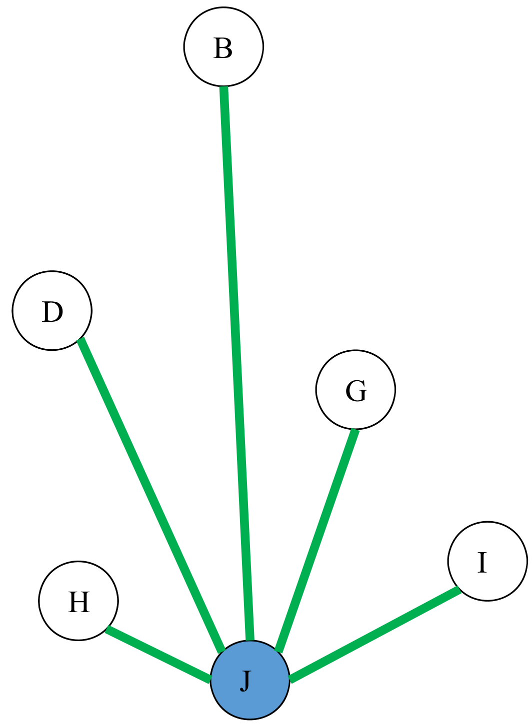
Edge I-G



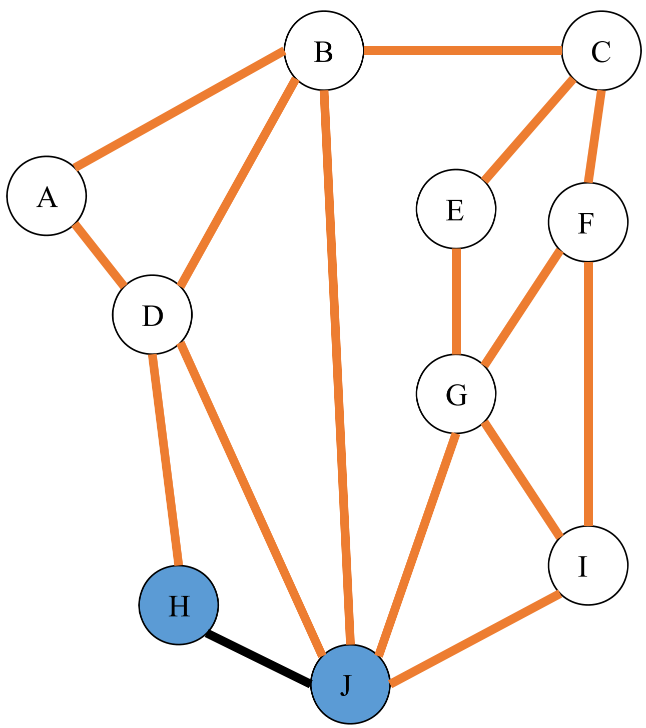
Round 1



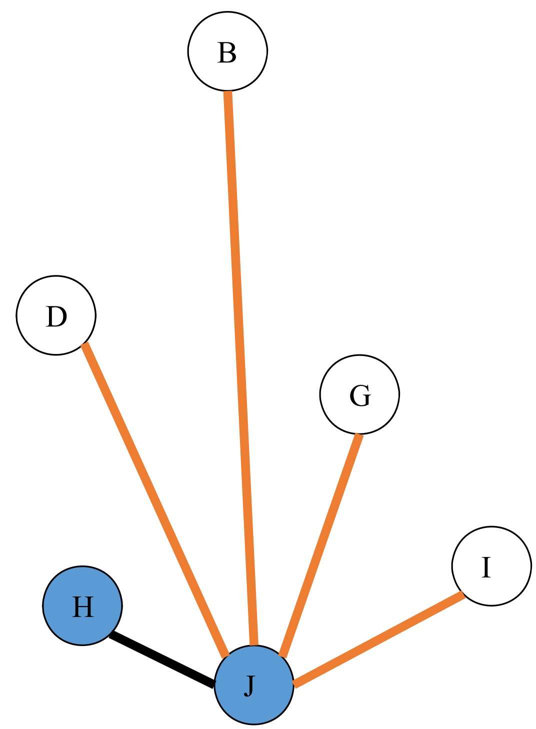
Group J



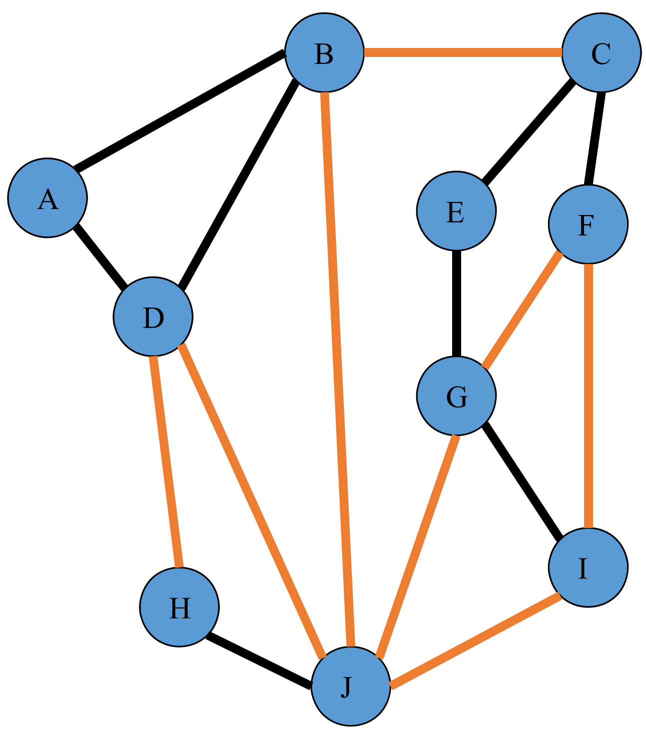
Round 1



Edge J-H



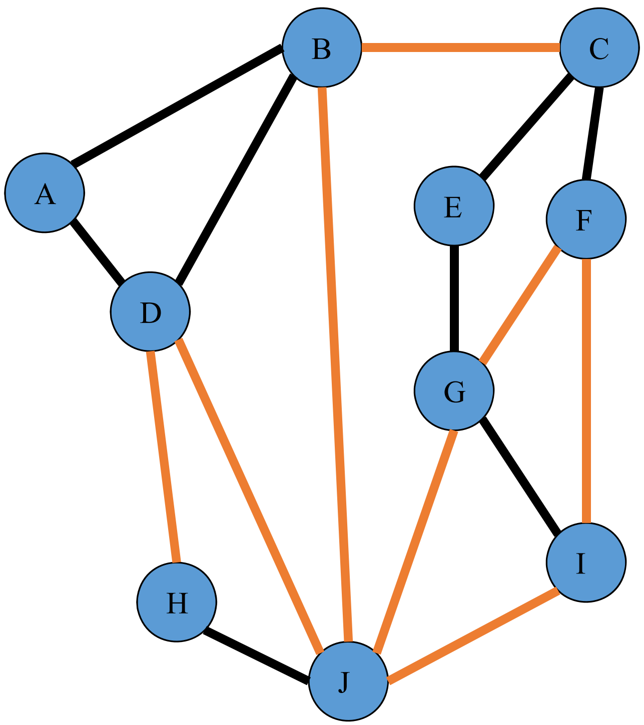
Round 1 Ends



List of Edges Added

- A-D
- B-A
- C-F
- D-B
- E-C
- F-C
- G-E
- H-J
- I-G
- J-H

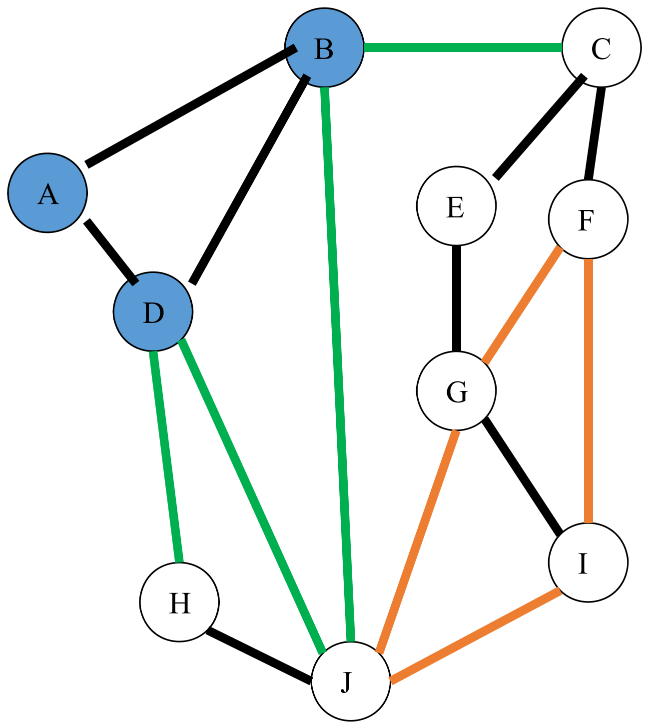
Groups at Beginning of Round 2



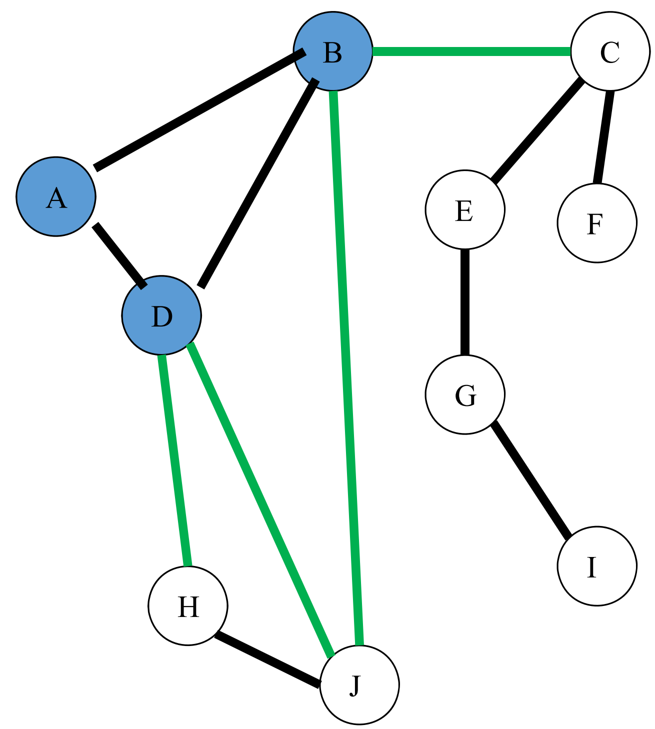
List of Groups

- D-A-B
- F-C-E-G-I
- H-J

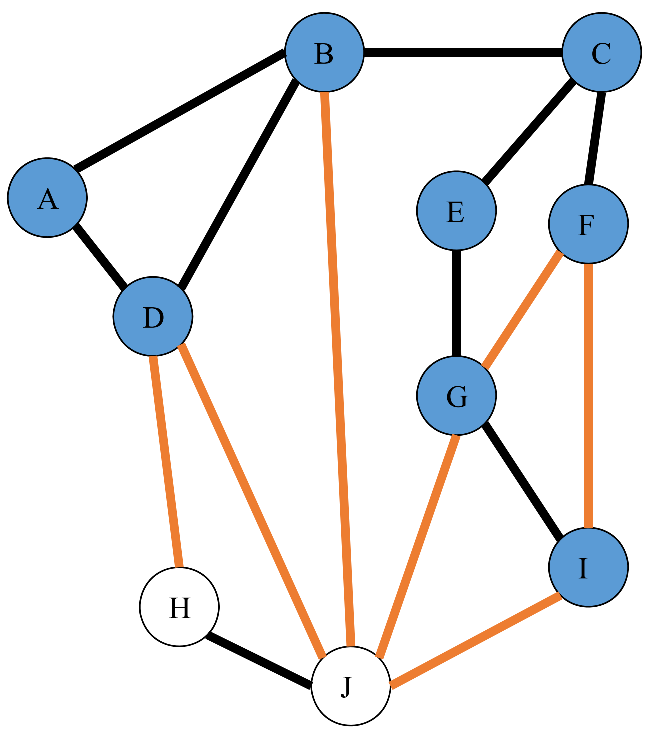
Round 2



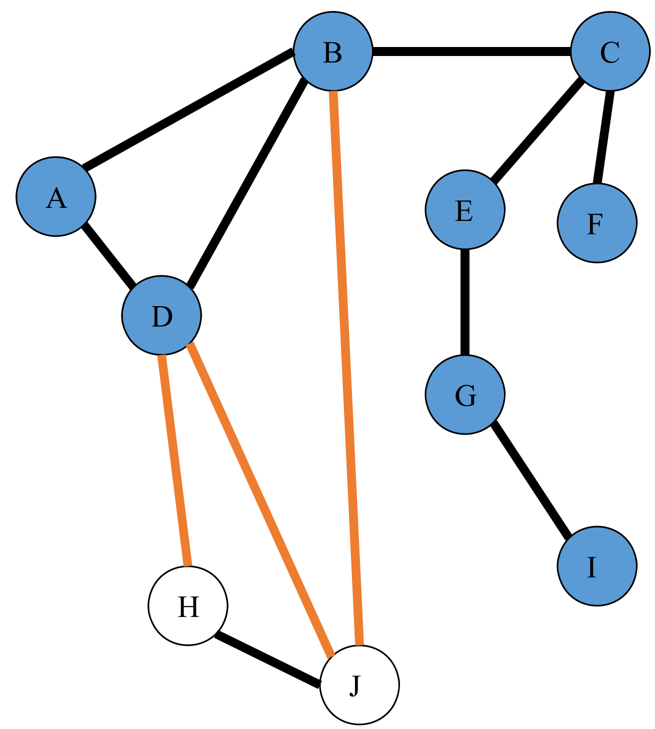
Group D-A-B



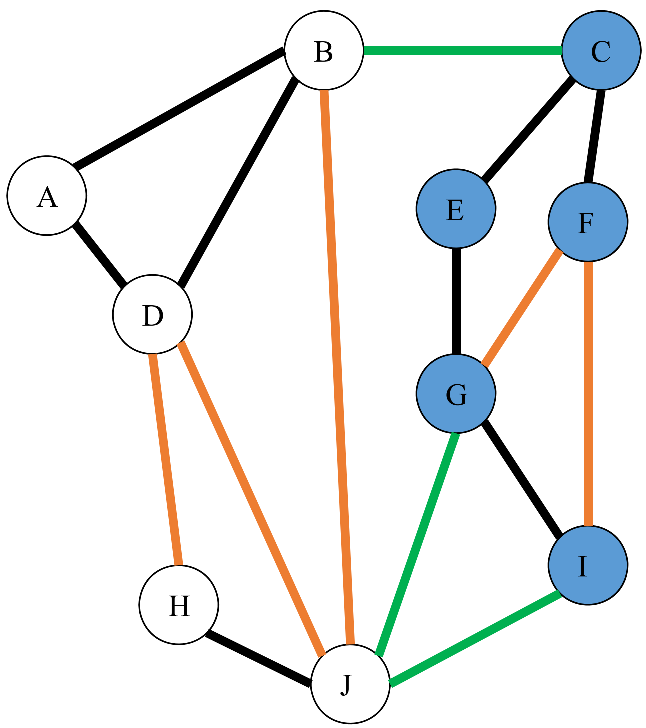
Round 2



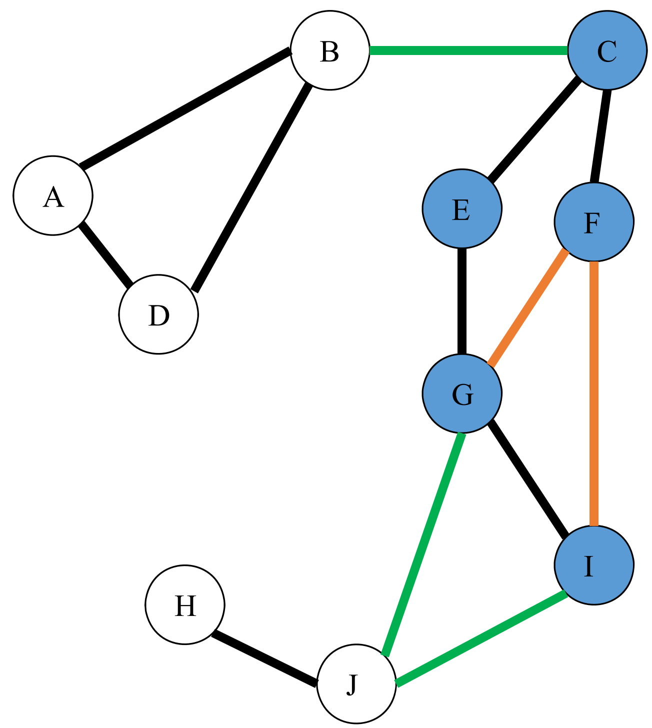
Edge B-C



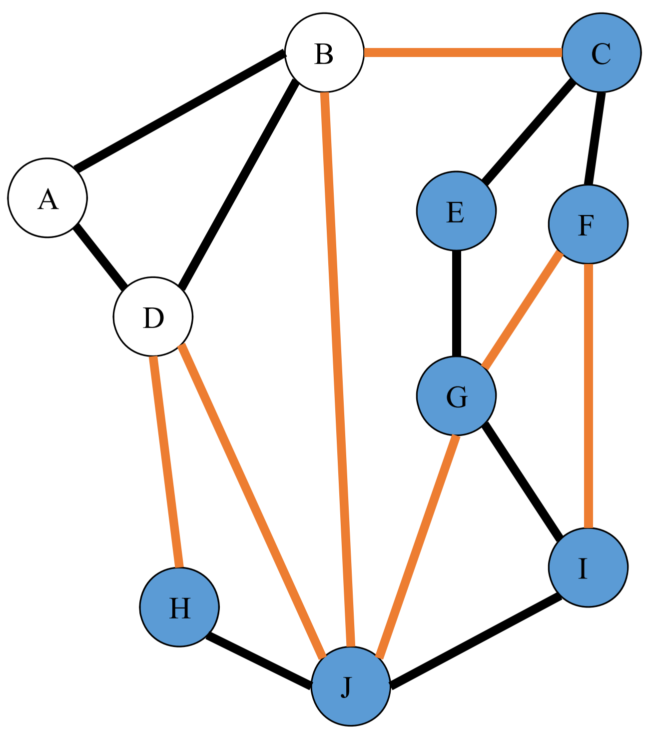
Round 2



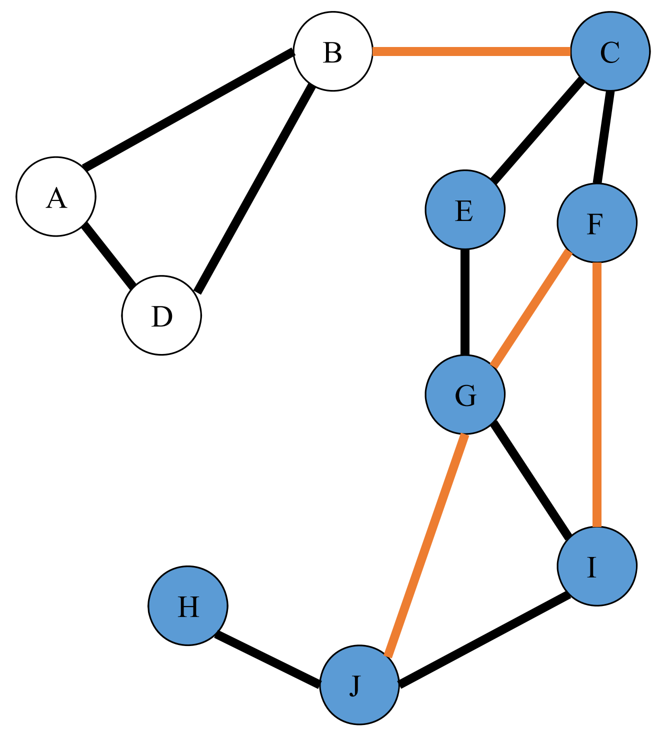
Group F-C-E-G-I



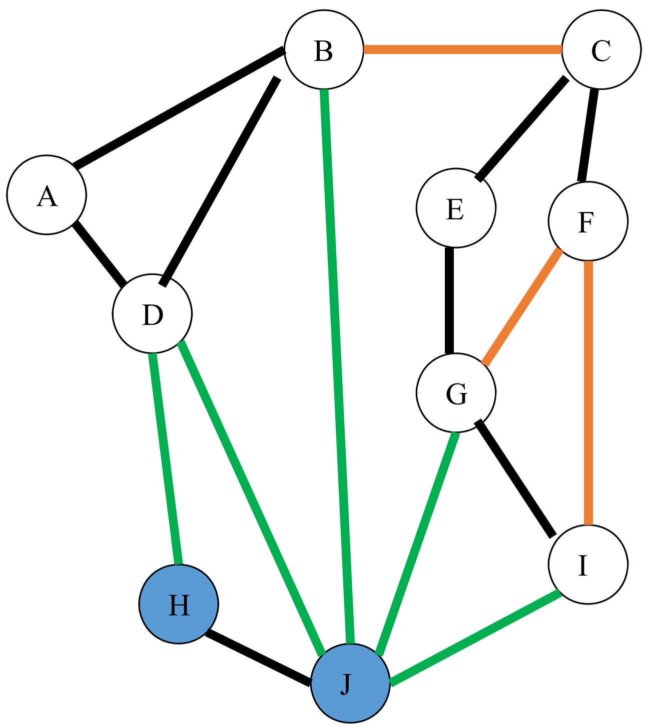
Round 2



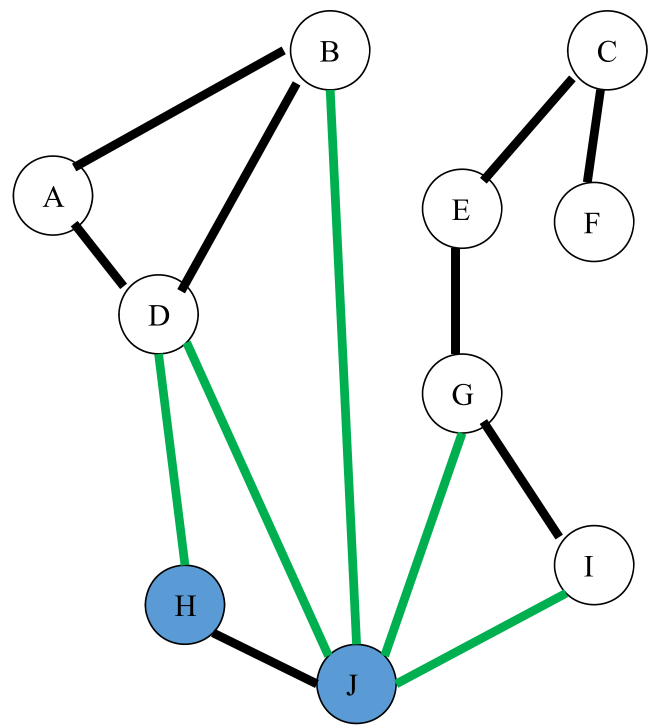
Edge I-J



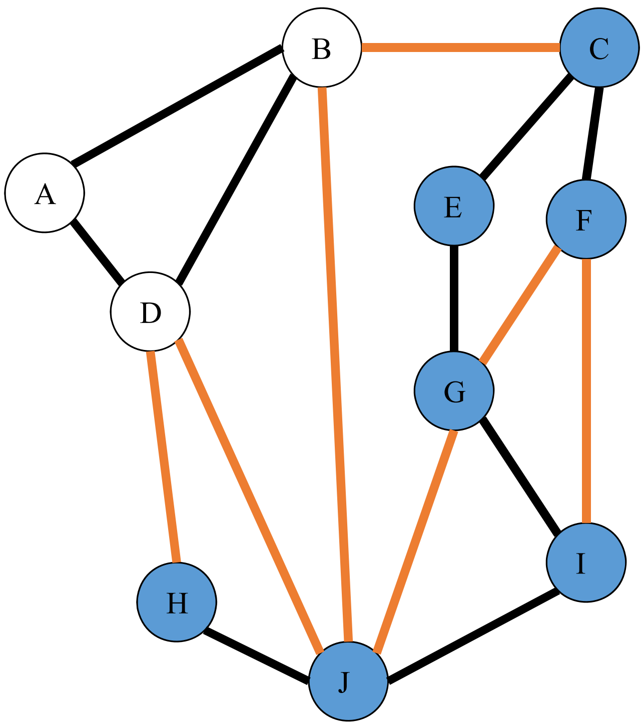
Round 2



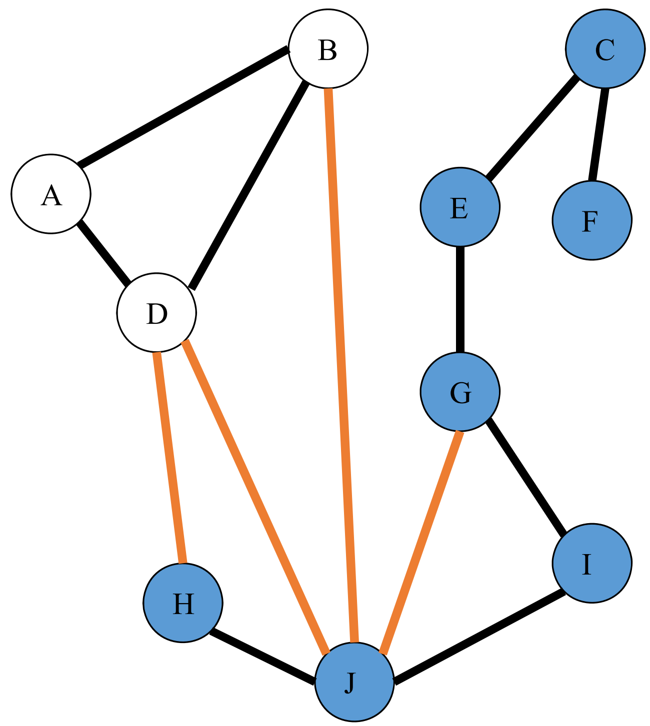
Group H-J



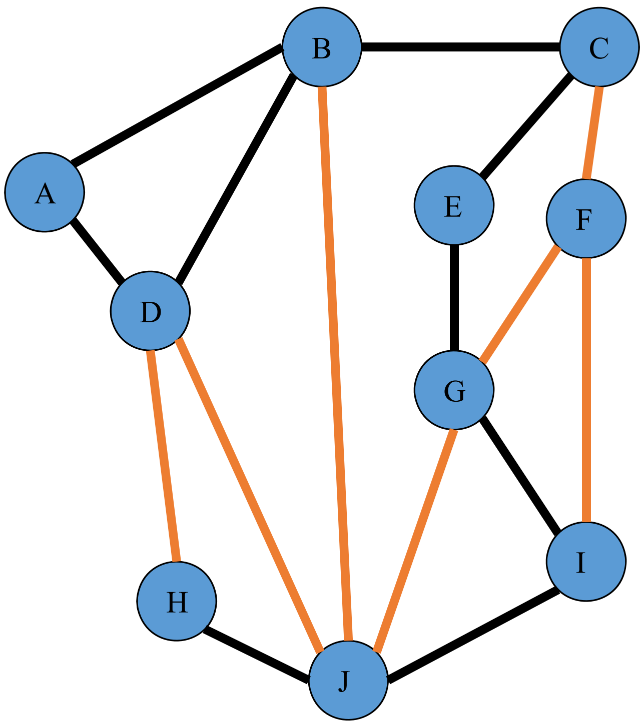
Round 2



Edge J-I



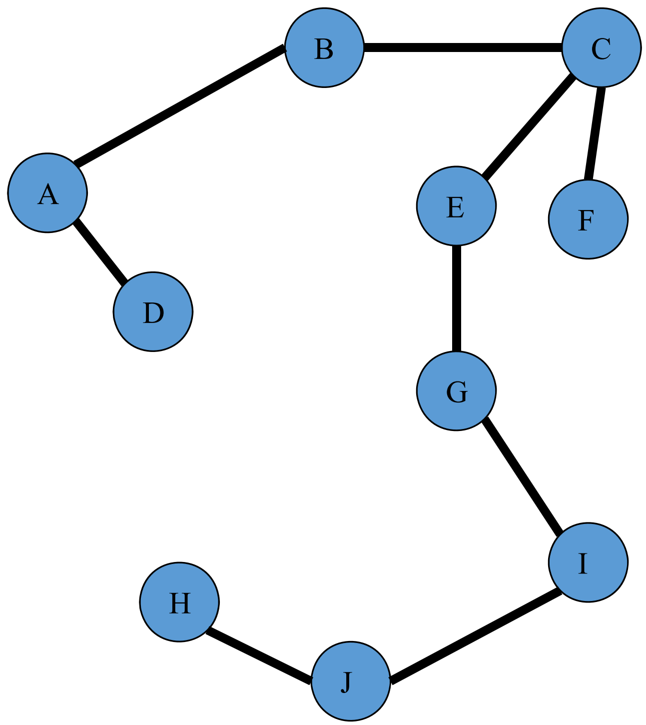
Round 2 Ends



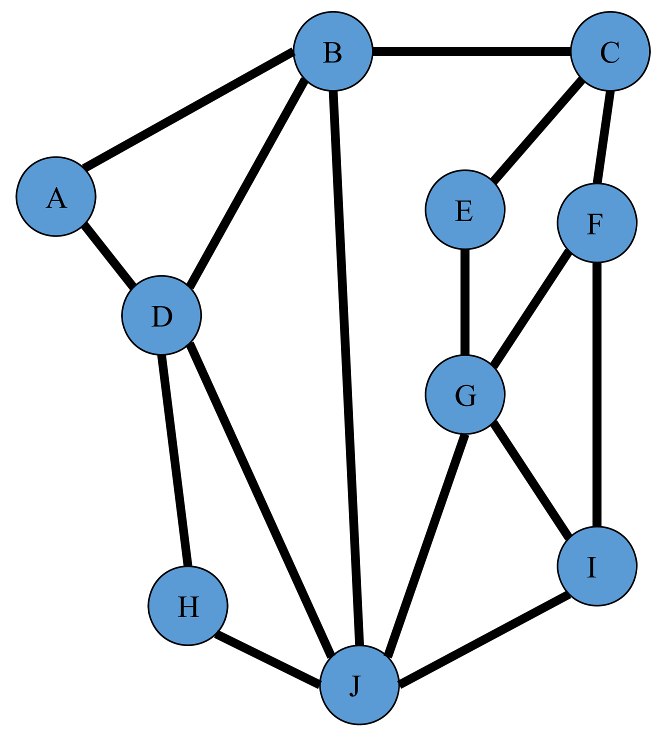
List of Edges Added

- B-C
- I-J
- J-I

Spanning Tree



Input Graph



Analysis

- If $G_1, G_2, \ldots, G_r$ are groups of vertices in an iteration, for each G_i, there is a G_j, $j \neq i$, and an edge {u,v} from a vertex $u \in G_i$ to a vertex $v \in G_j$
  - Else, graph is disconnected

- If t groups at start of an iteration, at most t/2 groups at end of iteration
  - Consider graph H with vertex set $G_1, G_2, \ldots, G_r$ and r edges, where edges correspond to the groups we connect
  - Number of groups now at most number of connected components in H. *Why?*

- After $\log_2 n$ iterations, one group left
  - At most n + n/2 + n/4 + … + 1 $\leq 2n$ edges in E’

- E’ contains a spanning tree
  - Invariant: the vertices in a group are connected

Outline

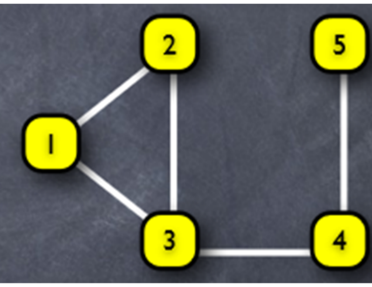
- Sketching Model
  - Estimating the Euclidean norm of a vector
  - Finding a non-zero coordinate of a vector

- Graph sketching
  - Boruvka’s spanning tree algorithm
  - Finding a spanning tree from a sketch

Representing a Graph

- For node i, let a_i be a vector indexed by node pairs
- If {i,j} is an edge, $a_i[i, j] = 1$ if j > i, and $a_i[i, j] = -1$ if j < i
- If {i,j} is not an edge, $a_i[i, j] = 0$

$$
\begin{array}{r|cccccccccc}
 & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} & \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \\
\mathbf{a}_1 = & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{a}_2 = & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$



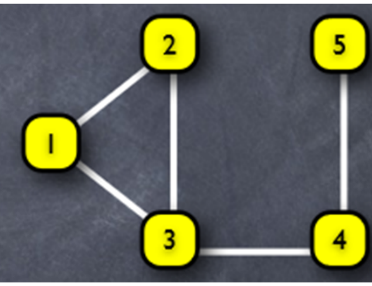
Representing a Graph

- Lemma: for a subset S of nodes,

$$\text{Support}\left(\sum_{i \in S} a_i\right) = E(S, V \backslash S)$$

- Proof: for edge {i,j}, if i, j $\in S$, the sum of entries on {i,j}-th column is 0

$$\begin{array}{rc} & \begin{array}{cccccccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} & \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \end{array} \\ \mathbf{a}_1 = & \left(\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \mathbf{a}_2 = & \left(\begin{array}{cccccccccc} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$



Spanning Tree Algorithm

- Compute $O(\log n)$ sketches $C_1 \cdot a_j, \ldots, C_{O(\log n)} \cdot a_j$ for each a_j
- Each $C_i \cdot a_j$ outputs a non-zero item of a_j with probability > 4/5, else returns FAIL
- Idea: Run Boruvka's algorithm on sketches
- For each node j, use $C_i \cdot a_j$ to get incident edge on j
- For i = 2, …, O(log n)
  - To get incident edge on group $G \subseteq V$, use

$$\sum_{j \in S} C_i \mathbf{a}_j = C_i \left(\sum_{j \in S} \mathbf{a}_j \right) \longrightarrow e \in \mathrm{support}\left(\sum_{j \in S} \mathbf{a}_j\right) = E(S, V \setminus S)$$

Spanning Tree Wrapup

- O(n log n) sketches $C_i \cdot a_j$, as i and j vary, so $O(n \log^3 n)$ space
- Note: a 1/5 fraction of sketches fail in each iteration in expectation, but on the remaining 4/5 fraction of vertices, the number of connected components halves
- Expected number of iterations is O(log n)
- Since sketches are linear, can maintain with insertions and deletions of edges
- Overall, $O(n \log^3 n)$ bits of space to output a spanning tree!