Algorithm Design and Analysis

Computational Geometry (Incremental Algorithms)

Goals for today

- Apply **randomized incremental algorithms** to geometry
- Give randomized incremental algorithms for two key problems:
  - The **closest pair** problem
  - The **smallest enclosing circle** problem

Closest Pair

The closest pair problem

Problem (closest pair): Given n points P, define $CP(P)$ to be the closest distance, i.e.

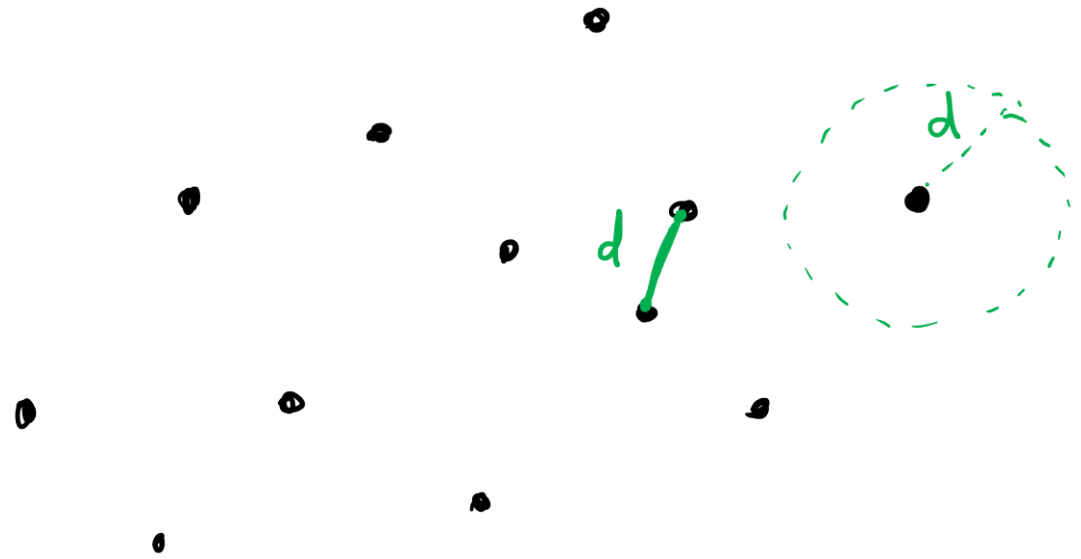
$$CP(P) = \min_{p,q \in P} \|p - q\|$$

Brute force solution: Try all pairs $\rightarrow$ $O(n^2)$

Improving brute force: incremental

- Brute force reuses no information whatsoever
- Geometry problems often have a lot of reusable information!

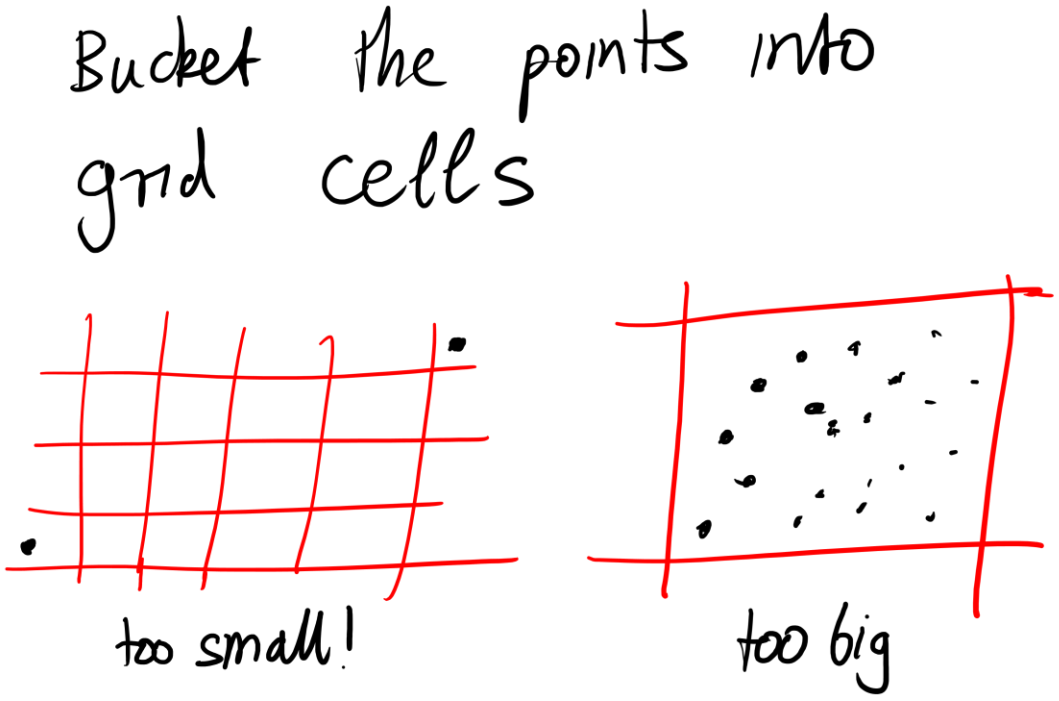
- Suppose I know the closest pair among the first i points…

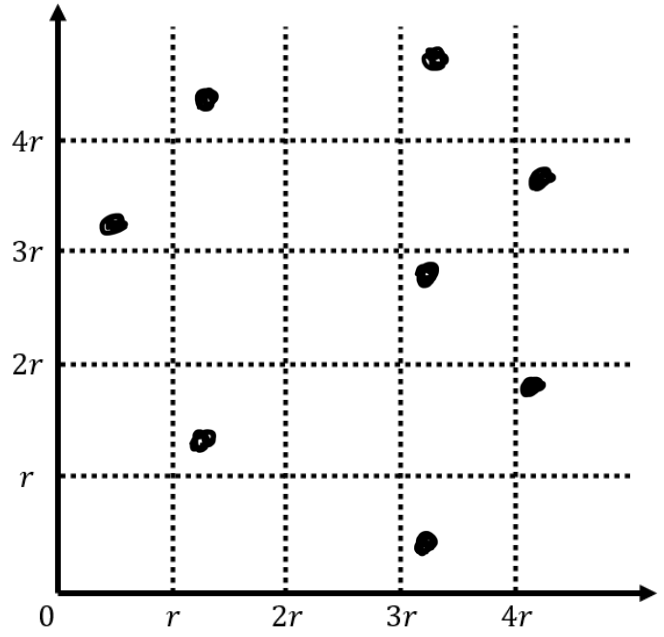


Only need to check within distance d

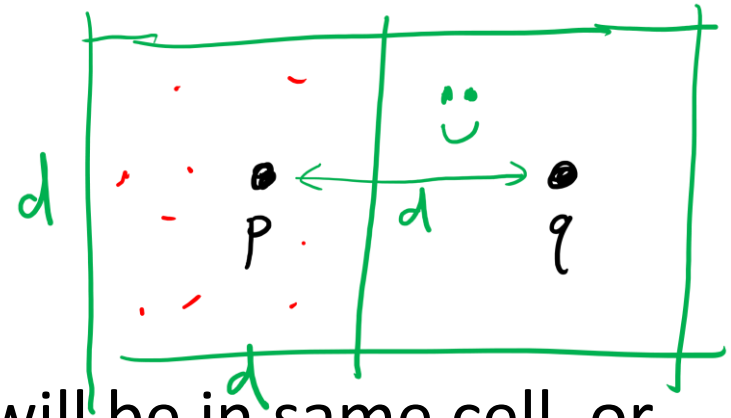
The problem

New Question: How do we find the set of points within distance d of the new point?





A grid data structure!



- If the grid size is sufficiently large, closest pair will be in same cell, or in neighboring cells
- If the grid size is too large, there will be too many points per cell…

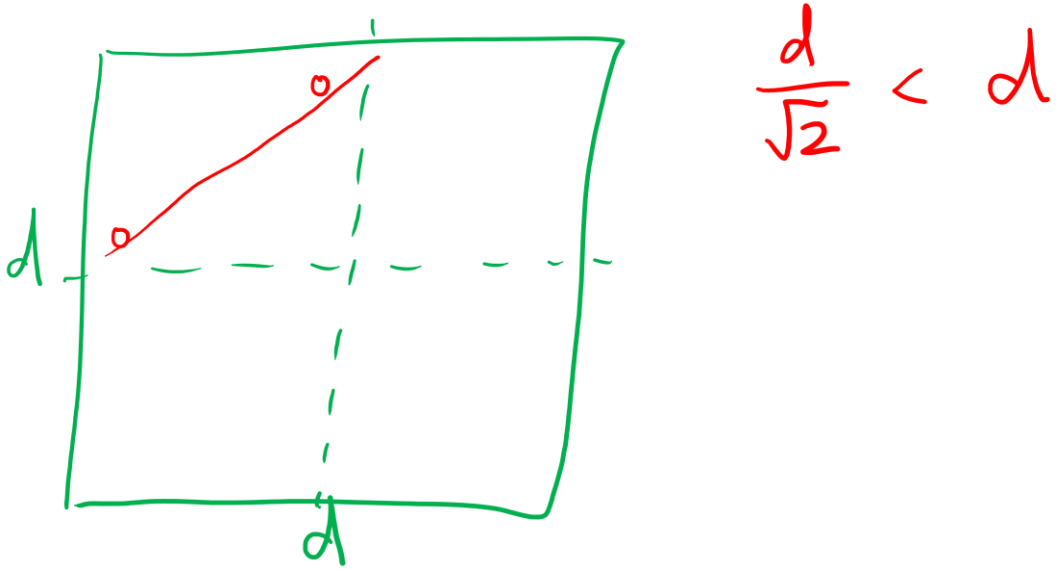
Goal: Choose the right grid size.

- Want few points per cell, so that looking in a cell is fast
- Want the closest pair to be in neighboring cells so we find them fast

The right grid size

Claim (the right grid size): Given a grid with points P and grid size $r = CP(P)$, no cell contains more than four points

Proof:



The incremental approach

Key idea (incremental): Add the points one at a time

- Check neighboring cells to see if there's a new closest pair
- If so, rebuild the grid with the new size
- Otherwise keep going

A grid data structure

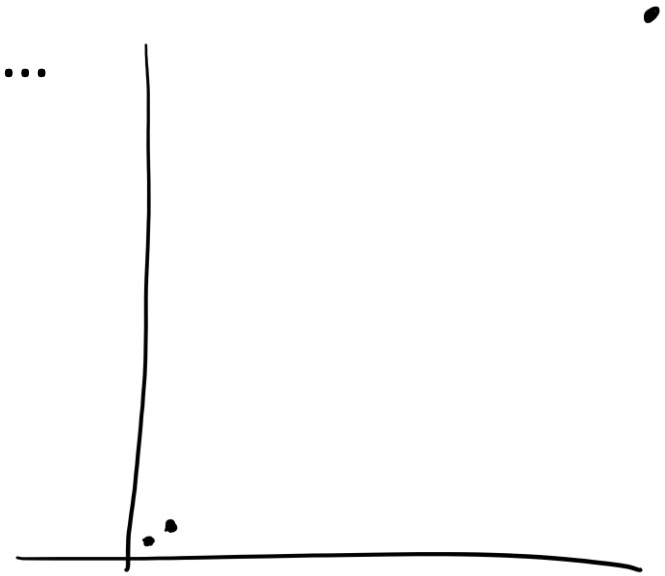
Invariant (grid size): Given a grid containing a set of points P, we want the grid size r to always equal $CP(P)$

- MakeGrid(p, q): Make a grid containing p and q, with $r = \|p - q\|$
- Lookup(G, p): Given a grid G and point p (not currently in the grid), we want to know whether p is part of a new closest pair
- Insert(G, p): Given a grid G and point p, inserts p and returns the grid size (which may have changed because of p)

Implementing the grid

Issue: The number of grid cells could be unbounded…

Use a hashtable



Implementing the grid

Implement MakeGrid(p, q):

$r = \|p - q\|$

Insert p & q into hashtable

$(x, y) \to \left(\left\lfloor \frac{x}{r} \right\rfloor, \left\lfloor \frac{y}{r} \right\rfloor\right) \xrightarrow{\text{dict}}$ list of points

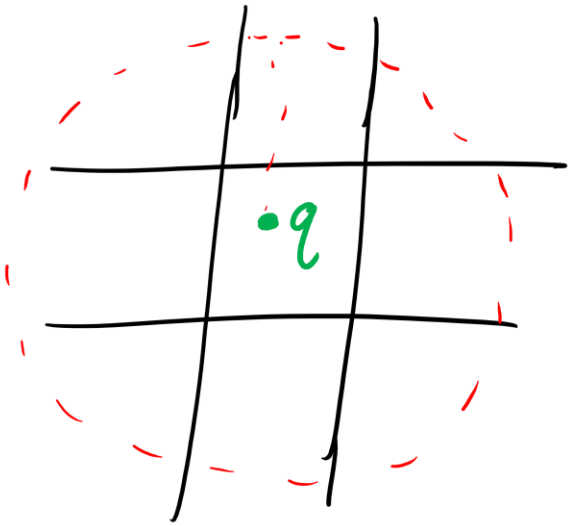
Implementing the grid

Implement Lookup(G, q):

Search neighbouring grid cells
$\leq$ 36 points (constant)

If $\exists p$ s.t $\|p-q\| < r$
return p, $\|p-q\|$

return None



Implementing the grid

Implement Insert(G, q):

If p, r = Lookup(G, q) is not None:

    Rebuild my grid with distance r (expensive)

else

    put q in the current grid (cheap)

Runtime

Claim (runtime): The worst-case runtime of the incremental grid algorithm is $O(n^2)$

Proof:

$$\text{Cost} = O\left(\sum_{i=1}^{n} i\right) = O(n^2)$$

Randomization to the rescue!!!

Randomized runtime

Claim (randomized incremental is fast): Randomly shuffle the points, then run the incremental algorithm, it takes $O(n)$ time in expectation

Proof:

$$P_i = \langle p_{\pi_1}, p_{\pi_2}, \cdots, p_{\pi_i} \rangle$$

$$X_i = \begin{cases} 1 & \text{if } CP(P_i) \neq CP(P_{i-1}) \\ 0 & \text{o.w.} \end{cases}$$ (answer changes)

$$T = \sum_{i=2}^{n} (1 + i \cdot X_i)$$

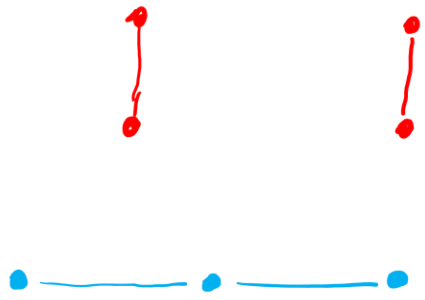
$$\mathbb{E}[T] = O(n) + \sum_{i=2}^{n} i \cdot \Pr[CP(P_i) \neq CP(P_{i-1})]$$

Randomized runtime (continued)

We need to bound $\Pr[X_i = 1]$... *(i.e.,* $\Pr[CP(P_i) \neq CP(P_{i-1})]$*)*

Call a point q "critical" if $CP(P_i \setminus \{q\}) \neq CP(P_i)$

≤ 2 critical points



$\Pr[CP(P_i) \neq CP(P_{i-1})]$

$= \Pr[p_{\pi_i} \text{ is critical}] \leq \frac{2}{i}$

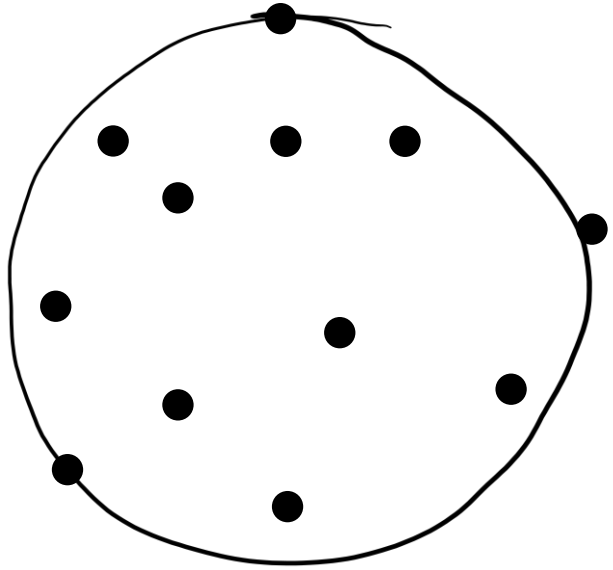
$$E[T] = O(n) + \sum_{i=2}^{n} i \cdot \frac{2}{i} = O(n)$$

□

Smallest enclosing circle

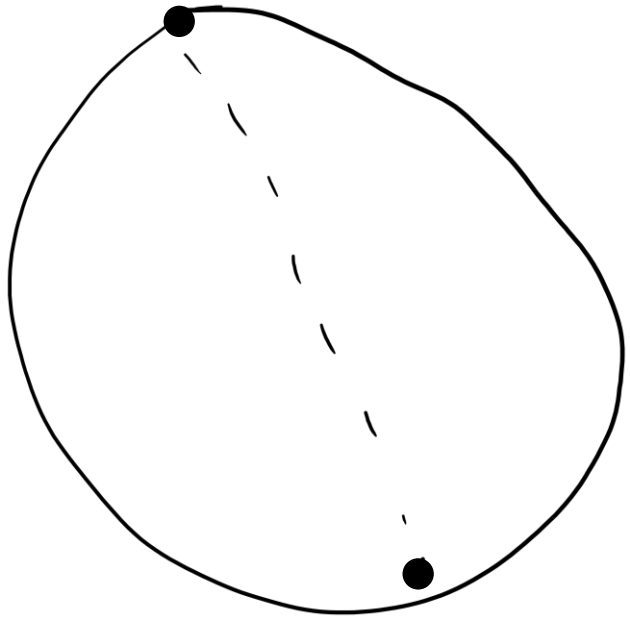
The smallest enclosing circle

Problem (Smallest enclosing circle): Given $n \geq 2$ points in two dimensions, find the smallest circle that contains all of them



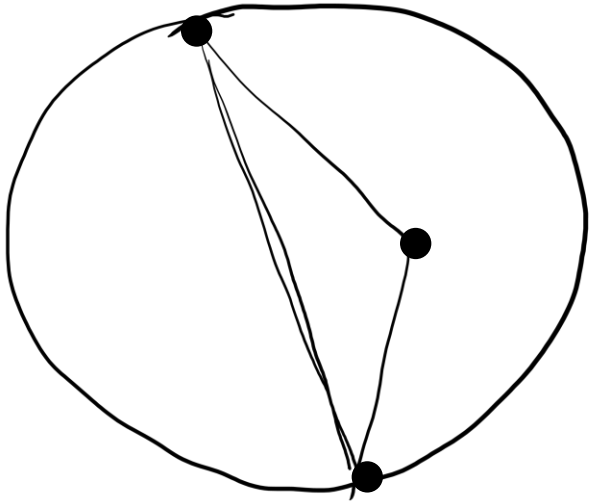
Base cases

Base case (two points):

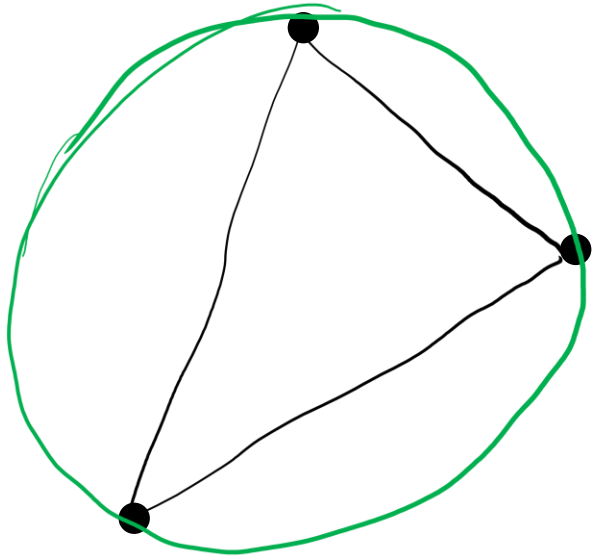


Base cases

Base case (three points):



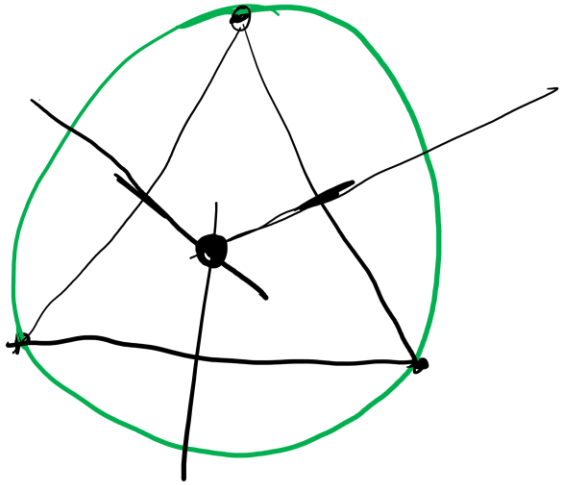
Case 1: Obtuse angle

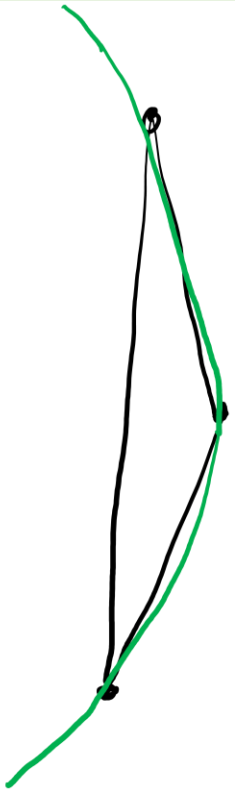


Case 2: Acute angle

Three points and a circle

Fact (unique circle): Given three non-colinear points, there is a unique circle that goes through them





The general case

Given $n > 3$ points, how many circles do we need to consider?

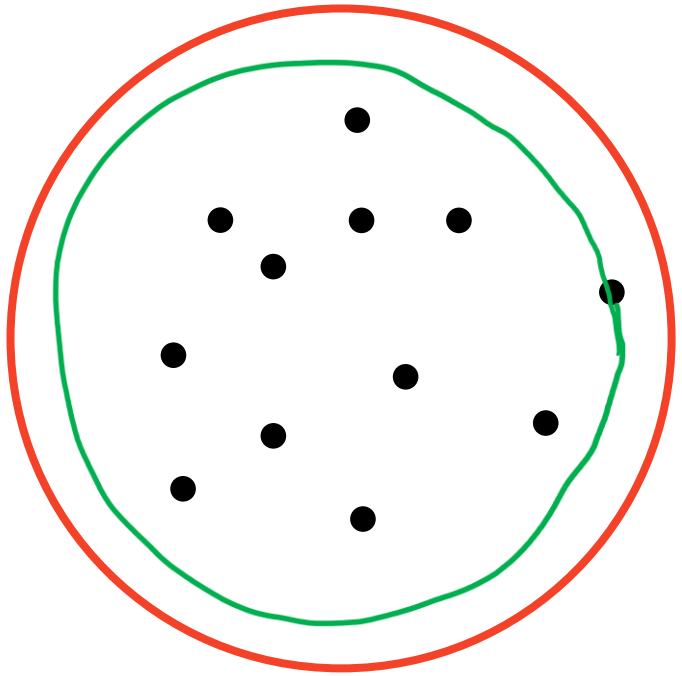
Theorem (three points is always enough): For any set of points, the smallest enclosing circle either touches two points p_i, p_j at a diameter, or touches three points p_i, p_j, p_j forming an ***acute*** triangle

In other words: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

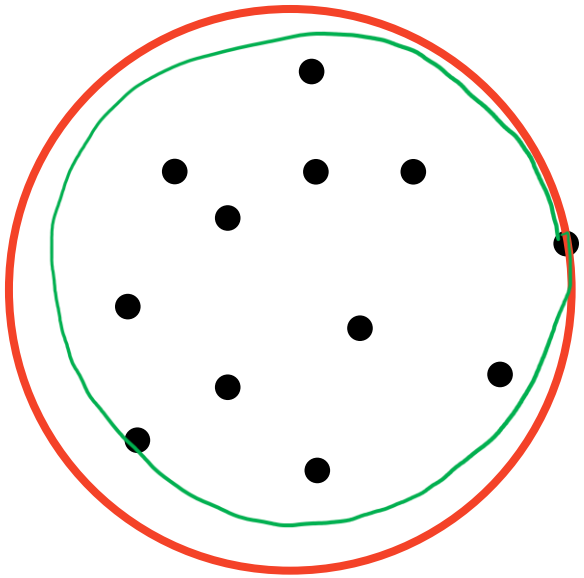
Proof of theorem

Case 1 (no points):



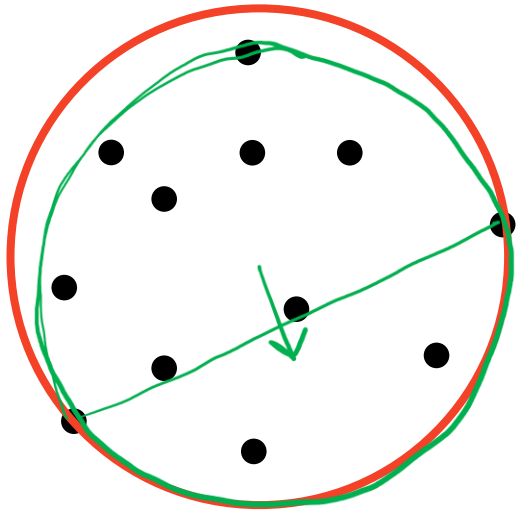
Proof of theorem

Case 2 (one point):



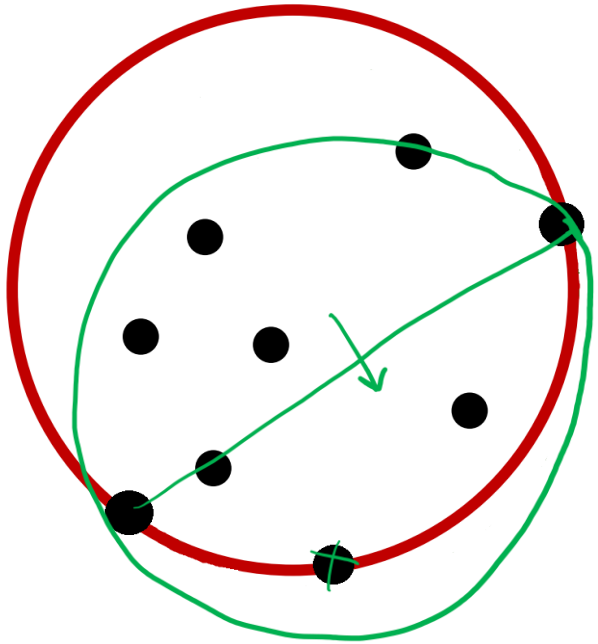
Proof of theorem

Case 3 (two points, not on a diameter):



Proof of theorem

Case 4 (three points, no acute angle):



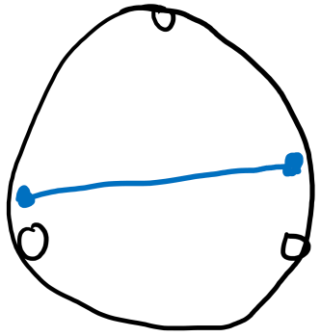
We just proved

Theorem: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

- Either two points at a diameter, or
- Three points forming an acute triangle

Brute force algorithms



Algorithm 1 (brute force): Try all triples of points and find their smallest enclosing circle. Check whether this circle contains every point. Returns the smallest such circle.

Algorithm 2 (better brute force): Try all triples of points and find their smallest enclosing circle. Return the **largest** such circle.

Beating brute force: incremental

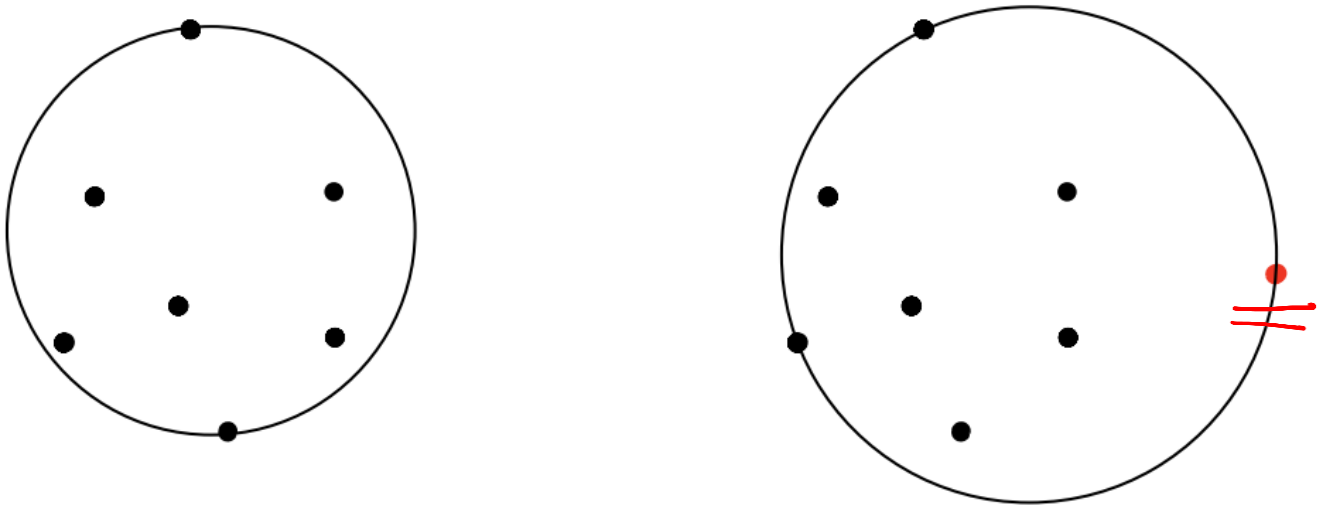
Incremental approach: Insert points one by one and maintain the smallest enclosing circle

When inserting p_i:

- ***Case 1:*** p_i is inside the current circle. Great, do nothing!
- ***Case 2:*** p_i is outside the current circle. Need to find the new one

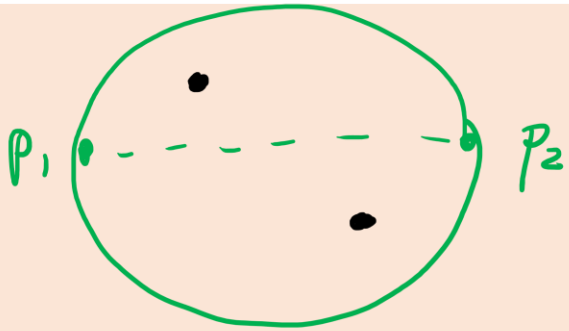
Making incremental fast

Observation: When we add p_i, if it is not in the current circle, then it is on the boundary of the new circle



Incremental algorithm

```
SEC([p_1, p_2, ..., p_n]) = {
  Let C = circle touching p_1 and p_2
  for i = 3 to n do {
    if p_i is not inside C then
      C = SEC1([p_1, p_2, ... p_{i-1}], p_i)
  }
  return C
}
```

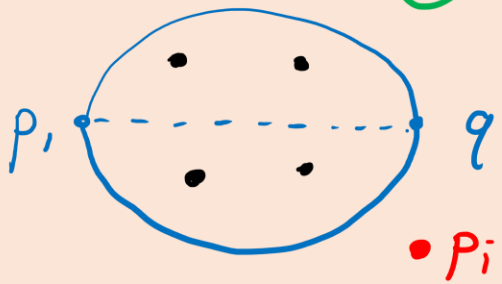


p_i is definitely on boundary

Incremental algorithm continued

```
SEC1([p_1, p_2, ..., p_k], q) = {
  Let C = circle touching p_1 and q
  for i = 2 to k do {
    if p_i is not inside C then
      C = SEC2([p_1, p_2, ..., p_{i-1}], p_i, q)
  }
  return C
}
```

q is definitely on boundary

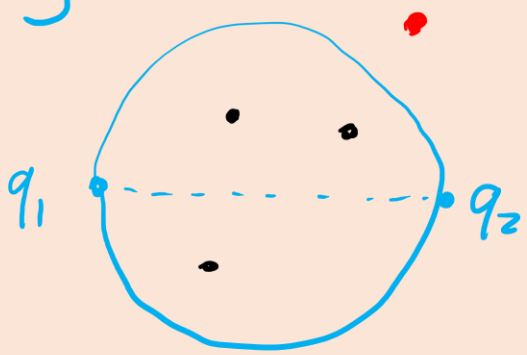


Incremental algorithm deeper again

```
SEC2([p_1, p_2, ..., p_k], q_1, q_2) = {
  Let C = circle touching q_1 and q_2
  for i = 1 to k do {
    if p_i is not inside C then
      C = SEC of p_i, q_1, and q_2
  }
  return C
}
```

on the boundary

(base case)



Runtime

Lemma (SEC2): SEC2 runs in $O(k)$ time

Lemma (SEC1): In the worst case, SEC1 runs in $O(k^2)$ time

Theorem (SEC): In the worst case, SEC runs in $O(n^3)$ time

Randomization to the rescue!!!

Claim (randomized SEC is fast): If we ***randomly shuffle*** the points in SEC and SEC1, then SEC1 runs in $O(k)$ expected time and SEC runs in $O(n)$ expected time

Theorem $\Rightarrow$ ≤ 3 "critical points"

$\Pr[$ a point is critical $] \leq 3/i$

$\mathbb{E}[T] = O(n + \frac{3}{i} \cdot i) = O(n)$

Putting it all together

```
SEC([p_1, p_2, …, p_n]) = {
    random_shuffle(p)
    C = circle touching p_1 and p_2
    for i = 3 to n do
        if p_i is not inside C then
            C = circle touching p_1 and p_i
            for j = 2 to i-1 do
                if p_j is not inside C then
                    C = circle touching p_i and p_j
                    for k = 1 to j-1 do
                        if p_k is not inside C then
                            C = circle touching p_i, p_j, p_k
    return C
```

SEC: $O(n)$
in expectation!!

Pr[entering loop**]** $\leq 3/i$

Pr[entering loop**]** $\leq 3/j$

SEC1: $O(1)$
in expectation

SEC2: $O(j)$
in expectation

Summary

- **Randomized incremental algorithms** are pretty great. We can turn slow brute force algorithms into expected linear-time algorithms!
- We got $O(n)$ expected time for the **closest pair** and **smallest enclosing circle** problems