Algorithm Design and Analysis

Hashing: Universal and Perfect Hashing

Roadmap for today

- Review the ***dictionary problem*** and motivate hashing
- See ***universal hashing*** and how to prove that a family is universal
- See an algorithm for ***static perfect hashing***

Formal model of computation

Model (word-RAM):

- We have unlimited constant-time addressable memory (“registers”)
- Each register can store a w-bit integer (a “word”)
- Reading/writing, arithmetic, logic, bitwise operations on a constant number of words takes constant time
- With input size n, we need $w \geq \log n$.

Dictionaries & Hashing

The dictionary problem

The dictionary data type stores ***items*** that have associated unique ***keys***

unique key →

STUDENT	
id:	*integer*
name:	*string*
grade:	*character*

Dictionary Interface	***Python equivalent***
insert(*item*): Insert the given item (associated with its key)	*d[key] = item*
lookup(*key*): Return the item with the given key if it exists	*item = d[key] (throws **KeyError** if not present)*
delete(*key*): Delete the item with the given key if it exists	*d.pop(key) (throws **KeyError** if not present)*

Formal setup for hashing/hash tables

- The keys come from $U = [0 \ldots u - 1]$ (the ***universe*** of keys)
- We want to store items in a table A of size m. Assume $u \gg m$, so we can not just store key x at $A[x]$

Key idea (Hashing): Define a function $h : U \rightarrow \{0,1, \ldots, m - 1\}$. Try to store item with key x at $A[h(x)]$

Handling collisions

Approach #1 (Open addressing): When a collision occurs, cleverly find a **different location in the table** for the new item

- Very hard to analyze, bad performance if not implemented well
- Amazing performance if done well! **All state-of-the-art hashtables do this**

Approach #2 (Chaining): Instead of storing a single item in each slot, store a **list of items**. Add all items that hash to that slot to the list

- Simple to analyze and implement
- Decent performance in practice, used by the C++ standard library
- Much easier to parallelize

Prehashing non-integer keys

Idea (prehashing): For non-integer keys, we want to convert them into some representative integer.

Example (strings): Strings can be interpreted as integers by interpreting each character as a digit, in base alphabet size (e.g., base-128 for ASCII)

B A C Z

66 65 67 90

$$= \mathbf{66 \cdot 128^3 + 65 \cdot 128^2 + 67 \cdot 128 + 90}$$

$$= \mathbf{142{,}340{,}002}$$

Choosing a hash function h

Main goal: We want it to be unlikely that $h(x) = h(y)$ for $x \neq y$

- We want $m = O(n)$, where n is the number of keys in the table
  - We could just pick $m = u$ then there are no collisions!!
  - But this is an unacceptable amount of memory if $u \gg n$
- We also want $h(x)$ to be fast to compute. Ideally $O(1)$ time
- How long does a hashtable operation take using chaining?

O(len chain)

So which hash function do we pick?

- For any hash function you choose, I can find a set of n items that hash to the same location…
- There's no such thing as a hash function that works for every input.

Big idea (randomization): We need to employ **randomization** to build a hash function that doesn't have a horrible worst-case behaviour

- **Specifically, we want to choose a random hash function from some big set of possible hash functions**

Random hash families

Definition (totally random hash): A set $\mathcal{H}$ of hash functions is ***totally random*** if for all $x \in U, t \in \{0, \dots m-1\}$, independent of all $y \in U$

$$\Pr_{h \in \mathcal{H}}[h(x) = t] = \frac{1}{m}$$

- Essentially equivalent to "*Simple uniform hashing*" (if you know it)
- Totally random hashing has all nice properties, but it's not possible to do practically…

Less random, but still random

Goal: We need a hash function that is still "pretty random", but not totally random, since that's too expensive

Definition (Universal Hashing): A set $\mathcal{H}$ of hash functions $h : U \to \{0, \ldots, m-1\}$ is called ***universal*** if for all $x \neq y$

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{m}$$

Can compute probability by counting:

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{|h(x) = h(y)|_{h \in \mathcal{H}}}{|\mathcal{H}|}$$

Examples: Universal or not?

$$|U| = 2, \qquad m = 2$$

	a	b
h_1	0	0
h_2	0	1

✓

	a	b
h_1	0	0
h_2	1	1

✗

	a	b
h_1	0	1
h_2	1	0

✓

	a	b
h_1	0	1
h_2	1	0
h_3	0	1

✓

More examples

$|U| = 3, \quad m = 2$

	a	b	c
h_1	0	0	1
h_2	1	1	0
h_3	1	0	1

✗

$|U| = 3, \quad m = 3$

	a	b	c
h_1	0	0	0
h_2	0	1	2
h_3	1	2	0
h_4	2	0	1

✓

Analysis of Universal Hashing

Theorem: If $\mathcal{H}$ is a universal family, then for any set $S \subseteq U$ with $|S| = n$, for any $x \in S$, if h is chosen at random from $\mathcal{H}$, then the **expected** number of collisions between x and other elements is at most n/m.

$C_{xy} = \{1 \text{ if } h(x) = h(y) \text{ else } 0\}$

$C_x = \sum_y C_{xy}$

$\mathbb{E}[C_{xy}] = \Pr[h(x) = h(y)] \leq \frac{1}{m}$

$\mathbb{E}[C_x] = \mathbb{E}[\sum_y C_{xy}] = \sum \mathbb{E}[C_{xy}] \leq \frac{n}{m}$

Corollary

Definition (Load Factor): The quantity n/m is called the ***load factor***

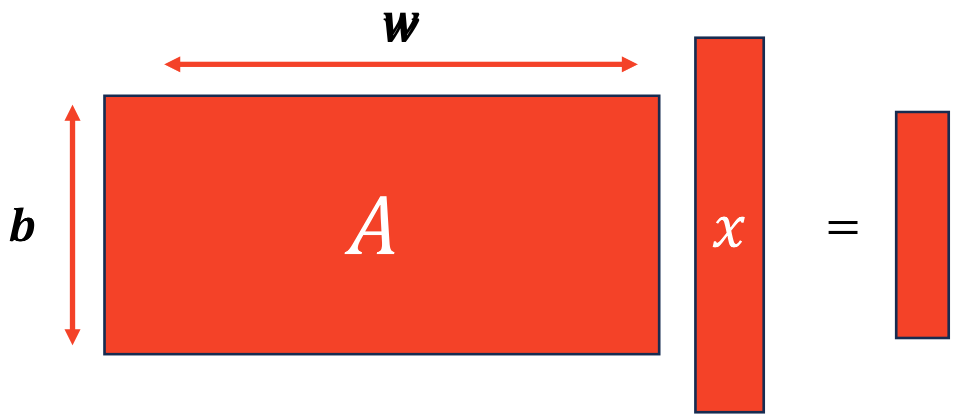
Corollary: Using separate chaining, given a universal family $\mathcal{H}$, the expected cost of each operation is $O(1 + n/m)$

- Therefore, if $m = \Theta(n)$, the expected cost of each operation is $O(1)$
- If you don't know n in advance, resize the table whenever the load factor exceeds some constant threshold

Assumes h can be computed in $O(1)$ time

Okay… how do we construct one?

Construction (Random binary matrix): Assume $|U| = 2^w, m = 2^b$
- Let A be a random $w \times b$ matrix of zeros and ones
- Interpret $x \in U$ as a w length vector of its bits
- Let $h(x) = Ax \bmod 2$, again interpreting $h(x)$ as a b length vector of bits



Analysis of random binary matrix

Theorem: The family produced by the random binary matrix method is universal, i.e., for $x \neq y$, $\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{m}$

Let $x \neq y$. $\Rightarrow \exists i^* \; x_{i^*} \neq y_{i^*}$ WLOG $x_{i^*} = 0$ $y_{i^*} = 1$

$h(x) = \sum_{i \neq i^*} A_i x_i \qquad h(y) = \sum_{i \neq i^*} A_i y_i + A_{i^*}$

independent

If $\Pr(h(x) = h(y)) \rightarrow h(x) - h(y) = 0 \qquad \Pr\left(\sum_{i \neq i^*} A_i (x_i - y_i) = A_{i^*}\right)$

$= \left(\frac{1}{2}\right)^b$

$= \frac{1}{m}$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \updownarrow b$$

Wait, that's not constant time!

- How efficient is computing $h(x)$?

$$O(b) = O(\log m)$$

- Thankfully, there exists universal families whose hash functions can be computed in constant time (but they are harder to analyze).

Example (The multiplication method): Suppose $|U| = 2^w$ and choose a power of two table size $m = 2^r$ and a **random odd integer $\boldsymbol{a}$**

$$h(x) = [(ax) \bmod 2^w] \gg (w - r)$$

Even more randomness!

- Can we make a hash family that is "more random" than universal, but still less than totally random? Yes!

Definition (pairwise independent)*:** A hash family $\mathcal{H}$ is called ***pairwise independent if for every pair $x_1 \neq x_2$ of distinct keys and every pair of values $v_1, v_2 \in \{0, \ldots, m-1\}$ (not necessarily distinct),

$$\Pr_{h \in \mathcal{H}}[h(x_1) = v_1 \text{ and } h(x_2) = v_2] = \frac{1}{m^2}$$

Intuitively, for every pair of distinct keys (x_1, x_2), all pairs of values (v_1, v_2) are equally likely to occur (there are m^2 possible pairs of values).

Even more randomness!

Definition (k-wise independent): A hash family $\mathcal{H}$ is called k-wise independent if for every set of k distinct keys $x_1, \ldots, x_k$ and k values $v_1, \ldots, v_k$ (not necessarily distinct) we have

$$\Pr_{h \in \mathcal{H}}[h(x_1) = v_1 \text{ and } \ldots \text{ and } h(x_k) = v_k] = \frac{1}{m^k}$$

- The $k = 1$ case is usually called *uniform* (since "1-wise independent" sounds funny)
- The $k = 2$ case is pairwise independence from the previous slide

Static perfect hashing (Optional content)

Static perfect hashing

Problem: Suppose we ***know the n keys in advance*** want deterministic constant query time in the worst case? Is this possible?

Idea: Reduce collision probability by making the table really really big!

Theorem: Given a universal family $\mathcal{H}$, taking $m = n^2$ gives us

$$\Pr_{h \in \mathcal{H}}[\text{no collisions}] \geq \frac{1}{2}$$

Some analysis

Theorem: Given a universal family $\mathcal{H}$, taking $m = n^2$ gives us

$$\Pr_{h \in \mathcal{H}}[\text{no collisions}] \geq \frac{1}{2}$$

That's a bit too much

- Okay, no collisions is nice, but n^2 space is way too much.
- Can we achieve the same with only $O(n)$ space?

Idea: Put hashtables inside a hashtable! The number of collisions per element is usually small, so squaring *those numbers* might not be too big

FKS Hashing

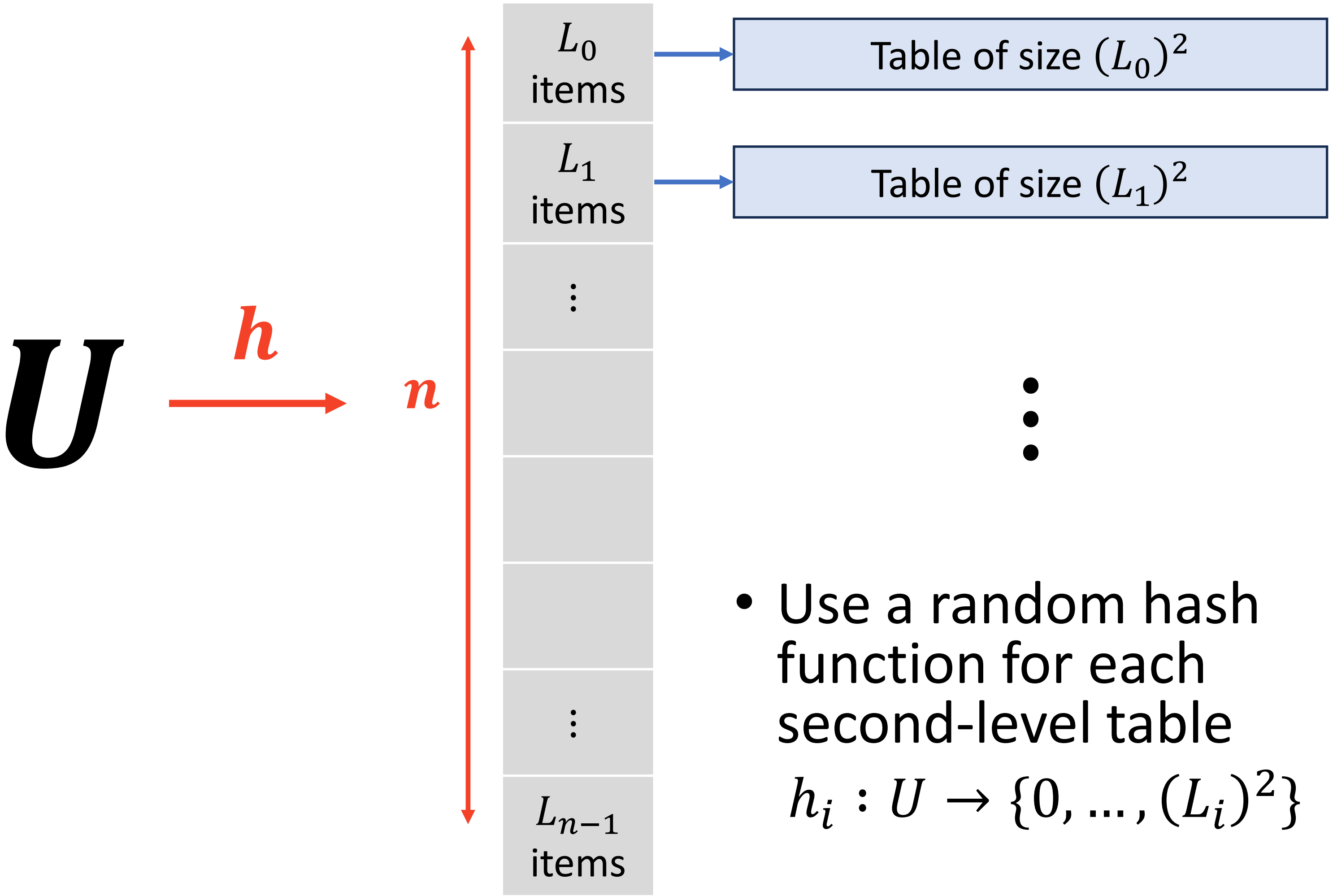
- Choose a hash function $h \in \mathcal{H}$ (universal)

  $h: U \to \{0, \ldots, n-1\}$

- Let L_i be the number of keys x such that

  $$h(x) = i$$

- Store the L_i items at position i in a second-level table of size $(L_i)^2$



- Use a random hash function for each second-level table

  $h_i : U \to \{0, \ldots, (L_i)^2\}$

Analysis of second-level tables

- We know that for each second-level table, we have a $\geq 1/2$ probability that there are no collisions
- There are n such tables, so there are bound to be ***some*** with collisions

Solution: If there are collisions in a second-level table, just pick another random hash from the family until there isn't.

Analysis of top level

Theorem: If h is chosen from a universal family $\mathcal{H}$, then

$$\Pr_{h \in \mathcal{H}}\left[\sum L_i^2 > 4n\right] \leq \frac{1}{2}$$

Analysis continued…

Lemma: Define $C_{xy} = 1$ if $h(x) = h(y)$, else $C_{xy} = 0$

$$\sum (L_i)^2 = \sum \sum C_{xy}$$

Analysis continued continued…

Lemma: If h is chosen from a universal family $\mathcal{H}$, then

$$\mathbb{E}\left[\sum (L_i)^2\right] < 2N$$

Completing the analysis

Theorem: If h is chosen from a universal family $\mathcal{H}$, then

$$\Pr_{h \in \mathcal{H}}\left[\sum L_i^2 > 4n\right] \leq \frac{1}{2}$$

Summary of today

- ***Universal hashing*** gives us "enough" randomness to get nice results
  - Operations on a hash table with separate chaining run in $O(1 + n/m)$ time.
  - Static FKS hashing gives deterministic lookup in constant worst-case time.

- ***Proving that a hash family is universal / k-wise independent*** can be quite tricky, but is very important

- For "more randomness", we can employ pairwise independent, or k-wise independent hashing.