Algorithm Design and Analysis

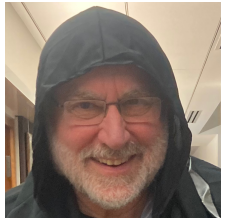
Introduction, Algorithm Analysis, and Selection

Logistics

Who is 15-451/15-651?

Instructors

Daniel



Danny

Tutor

Emily

Teaching Assistants

Tanisha

Thomas

Eileen

Jolyne

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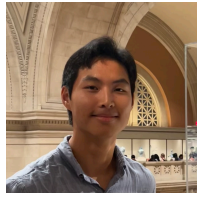
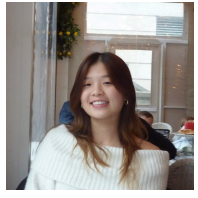
Ethan

Lillian

Jerick

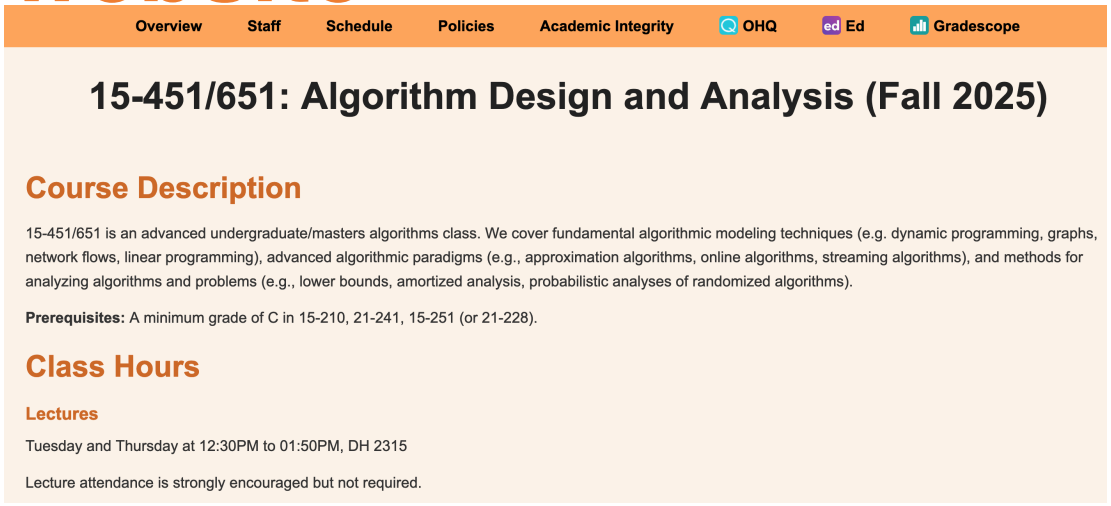


Michael



Brenda

Course website



https://www.cs.cmu.edu/~15451-f25/

- Contains course calendar with office hour schedule
- Contains all lecture notes, homework handouts, policies, etc.
- Links to other important platforms (OHQ, Ed, Gradescope)

Homework

- **6 Written Homework:** Each has 2-3 problems. Solutions must be typeset, not handwritten! Submitted to Gradescope.
- **3 Oral Homework:** Collaborate in groups of three and present your solutions to a TA
- **4 Programming Problems:** Submitted via Gradescope.
  - Officially recommended/supported languages are **C++** and **Python**
  - We will accept C, Java, OCaml, Rust when possible (not guaranteed for every assignment, and TA/instructor help will be limited.)

Recitation

- Please review the lecture notes and read the problems beforehand
- Only 50-minutes long so please show up on time!
- 5% of your grade from attendance
- ***Two styles of recitation!***
  - **Review-style (R):** More review and goes at a slower pace
  - **Problem-heavy (P):** Faster pace and cover more problems

Midterm exams

- Evening midterms!
- Put these in your calendar right now!

Midterm one (Week 5)
Tuesday September 23rd
7:00pm – 9:30pm

Midterm two (Week 10)
Tuesday November 4th
7:00pm – 9:30pm

Tutors

- One-on-one tutoring is available
- ***Highly recommended*** if you struggled with any of the prerequisite courses (15-210, 15-251, 15-122, concepts)
- Open to anyone, space permitting

Academic Integrity

- Healthy collaboration within the rules set out by the assignments is great for learning, but cheating is not.

Definition (Cheating): Includes, but is not limited to:

- Reading/copying answers/code from online sources, books, etc.
- Reading/copying answers of other people, current or former students or otherwise
- Reading/copying answers obtained from generative AI tools (e.g., ChatGPT, CoPilot)
- Accessing problems or solutions from past semesters (e.g., from a friend who took the course in a previous semester, or from a website, etc.)

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Detection: We are extremely accurate at catching cheating.

- In Spring 2024, we caught ***50-60 cases of cheating*** (out of approx. 180 students)
- In Fall 2024, we caught ***30-40 cases of cheating*** (out of approx. 130 students)

AIV Alternatives

- 451 is a hard course. We know that a lot of AIVs happen out of desperation, not a premeditated plan to cheat.
- If you are struggling with the course, please **sign up for tutoring**, attend **instructor office hours**, and reach out for additional help
- If you are behind and low on time, **request an extension** (see the **extension request form** on the course website)
- Try to start your homework early, but if you are stuck at the last minute, we don't close after business hours. **If you reach out for homework help at the last minute, we will almost always try to be available to help.**

Now onto the class!

Formal analysis of algorithms

- We want **provable guarantees** about the properties of algorithms
  - E.g., **prove** that it runs in a certain amount of **time**
  - E.g., **prove** that it outputs the correct **answer**
- **Important question:** How exactly do we measure time?
  - **Answer**: It depends :)
  - Lots more discussion about this in the coming lectures

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- **Important question:** How exactly do we measure time?
  - **Answer**: It depends :)
  - Lots more discussion about this in the coming lectures
- We need a ***model of computation!***
  - Specifies exactly what operations are permitted
  - How much each operation costs (sometimes called the *cost model*)

Today's model

The Comparison Model

- Input to the algorithm consists of an array of n items in some order
- The algorithm may perform comparisons (is $a_i < a_j$?) at a **cost of 1**
- Copying/moving items is *free*
- The items are of an arbitrary type. **We are not allowed to assume a type**
  - E.g., the items **can not** be assumed to be numbers
  - This means we can not add, multiply, XOR the items
  - We also can not use hashing, or use elements as array indices, etc.

Quicksort: A journey of algorithm design and analysis

- As seen in 15-122 and 15-210 (and possibly elsewhere!)
- One of the most well-known algorithms in all of computer science

function quicksort($a[0 \ldots n-1]$ **: list) {**
  select a pivot element p = a_0
  let LESS = [a_j such that $a_j < p$]
  let GREATER = [a_j such that $a_j > p$]
  return quicksort(LESS) + [p] + **quicksort**(GREATER)
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}

***Question*: What is the *complexity* of Quicksort?**

Which measure of complexity?

Definition (Worst-case complexity): The worst-case complexity of an algorithm is the *largest cost* it can incur over *any possible input* (usually as a function of input size n)

Theorem (15-122): The *worst-case* cost of QuickSort on an input of length n *is* $\Theta(n^2)$

***Note*:** By default, if not specified, we will pretty much always consider worst-case complexity.

Which measure of complexity?

Definition (Average-case complexity): The average-case complexity of an algorithm is the *average of the costs* of the algorithm over *every possible input.*

Note: Mathematically, this is equivalent to the *expected value of the cost* of the algorithm over an *input chosen uniformly randomly*.

***Note*: We will rarely, if ever, consider average-case complexity in 15-451/651.**

Which measure of complexity?

Definition (Average-case complexity): The average-case complexity of an algorithm is the *average of the costs* of the algorithm over *every possible input.*

Note: Mathematically, this is equivalent to the *expected value of the cost* of the algorithm over an *input chosen uniformly randomly*.

Theorem (15-210): The *average* cost of QuickSort on an input of length n *is* $\Theta(n \log n)$

***Note*:** We will rarely, if ever, consider average-case complexity in 15-451/651.

Making it better

- The average-case performance of QuickSort is great
- But its only reliable if the input is random! An evil adversary can always feed our code a worst-case input and ruin our day :(
- Most real-life data **is not random**. Hoping that data is random is **not** a good way to design your algorithms.

Making it better

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- Most real-life data **is not random**. Hoping that data is random is **not** a good way to design your algorithms.

Important idea: Instead of hoping that the input is random…
put the randomness *into the algorithm*!

Making it better: *Randomized* Quicksort

function random_quicksort($a[0 \ldots n-1]$ **: list) {**
  select a ***random*** pivot element p = a_i
  let LESS = [a_j such that $a_j < p$]
  let GREATER = [a_j such that $a_j > p$]
  return random_quicksort(LESS) + [p] + **random_quicksort**(GREATER)
}

Analyzing randomized algorithms

Theorem (15-210): The *expected* number of comparisons performed by randomized Quicksort on any input of size n is $\Theta(n \log n)$

IMPORTANT NOTES:

Note: When analyzing randomized algorithms, we are usually interested in the ***expected value*** *over the random choices* to process a ***worst-case user input***

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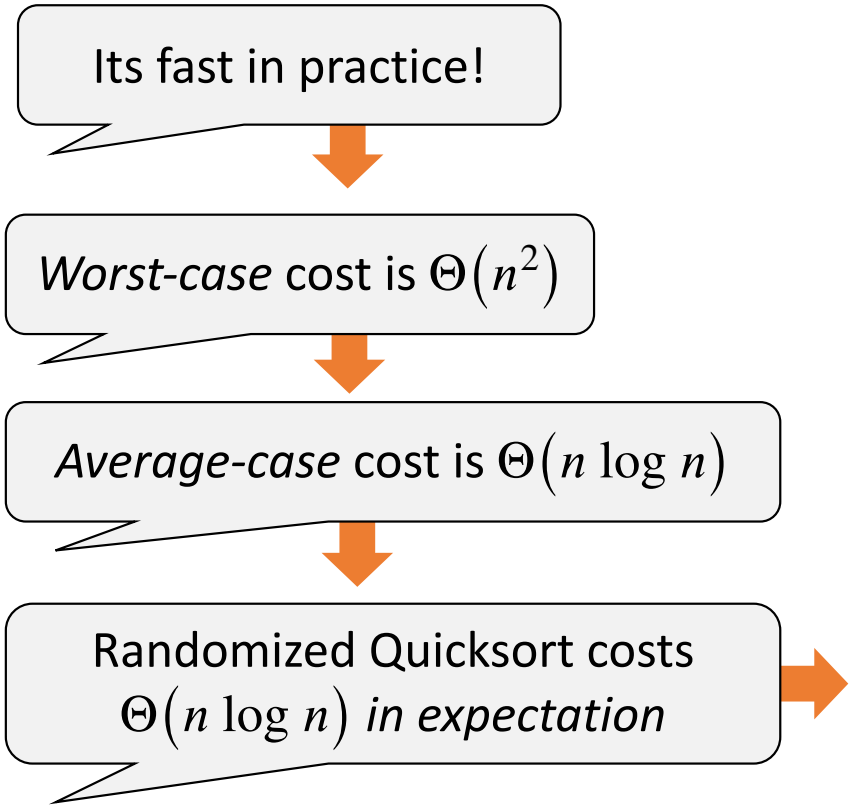
IMPORTANT NOTES:

Note: When analyzing randomized algorithms, we are usually interested in the ***expected value*** *over the random choices* to process a ***worst-case user input***

- we are **not** assuming that our random-number generator gives us the worst possible random numbers,
- we are **not** analyzing the algorithm for a randomly chosen input (that's average-case complexity!)

The Quicksort journey so far



The Quicksort journey so far

Its fast in practice!

Worst-case cost is $\Theta(n^2)$

Average-case cost is $\Theta(n \log n)$

Randomized Quicksort costs $\Theta(n \log n)$ *in expectation*

What if we could efficiently find the median element and use that as the pivot?

$T(n) = n - 1 + 2T(\frac{n}{2})$

$O(n \log n)$

The Quicksort journey so far

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Deterministic Quicksort in worst-case $\Theta(n \log n)$ cost??

New Problem: Median finding

New problem: Median / k^{th} smallest

Problem (Median) Given a range of distinct elements $a_1, a_2, \ldots, a_n$, output the median.

Definition (Median): The median is the $\left\lfloor \frac{n-1}{2} \right\rfloor$th smallest element

- More generally, we can try to solve the "k^{th} smallest" problem. Given a range of distinct elements and an integer k, we want to find the element such that there are exactly k smaller elements
- k is zero-indexed, so the minimum element is the 0th smallest element

Algorithm design strategy

Algorithm design idea: Start with a simple but inefficient algorithm, then optimize and remove unnecessary steps.

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Simple algorithm (k^{th} smallest): Sort the array and output element k

- **Redundancy**: We are finding the k^{th} *smallest* for **every** k

Take inspiration from Quicksort?

```
function quicksort(a[0...n − 1]) {
   select a pivot element p = a_i
   let LESS = [a_j such that a_j < p]
   let GREATER = [a_j such that a_j > p]
   return quicksort(LESS) + [p] + quicksort(GREATER)
}
```

***Question*: If we only want the k^{th} number, what is wasteful here?**

Take inspiration from Quicksort?

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```

***Question*: If we only want the k^{th} number, what is wasteful here?**

return quicksort(LESS) + [p] + **quicksort**(GREATER)

The answer is either in here *Or the answer is in here*

The result: Randomized Quickselect

function quickselect($a[0 \ldots n-1],\ k$**) {**

select a **random** pivot element p = a_i for a random i

let LESS = [a_j such that $a_j < p$]

let GREATER = [a_j such that $a_j > p$]

if |LESS| < k **then return** quickselect(LESS, k)

else if |LESS| > k **then return** quickselect(GREATER, k - |LESS| - 1)

else return p

}

Now the analysis

Theorem: The *expected* number of comparisons performed by Quickselect on an input of size n *is at most* $8n$

Warning: The proof is subtle because it uses probability. We must be careful to not make false assumptions about how probability and randomness work…

Let $T(n) =$ the ***expected*** number of comparisons performed by Quickselect on a ***worst-case input*** of size n

$$T(n) = n-1 + E[T(X)]$$

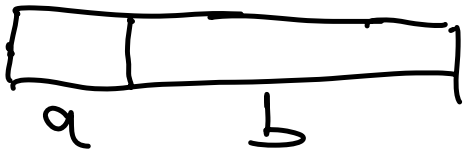
$X =$ Size of recurn. call

First attempt: Almost-correct analysis

Note: This proof is nearly, but not quite correct. It does, however, provide some useful insight that gets us closer to a correct proof.

Question: What is a (good) upper bound on the expected size of the recursive subproblem?

$E[X] \leq \frac{3}{4}n$



$a + b = n$

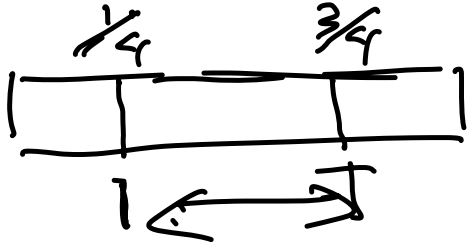
So, we might try the recurrence…

$T(n) \leq n - 1 + T(\frac{3}{4}n)$ ← in general

$O(n)$ $\quad T(E[X]) \neq E(T(X))$

A better proof

Question: Let's be more precise. How often is the recursive subproblem size at most $3n/4$?



$p = \frac{1}{2}$

So, a better recurrence relation is

$$T(n) \le n-1 + \frac{1}{2}T\left(\frac{3}{4}n\right) + \frac{1}{2}T(n)$$

$$\frac{1}{2}T(n) \le n-1 + \frac{1}{2}T\left(\frac{3}{4}n\right)$$

$$T(n) \le 2(n-1) + T\left(\frac{3}{4}n\right)$$

Validating the recurrence relation

$$T(n) \leq 2(n-1) + T\left(\frac{3n}{4}\right)$$

$$\leq 2n + \frac{3}{4}2n + \left(\frac{3}{4}\right)^2 2n + \cdots$$

$$\leq 2n\left(1 + \frac{3}{4} + \cdots\right) = 2n \frac{1}{1-\frac{1}{4}} = 8n$$

Summary of randomized Quickselect

function quickselect($a[0 \ldots n-1],\ k$**) {**
  select a random pivot element p = a_i for a random i
  let LESS = [a_j such that $a_j < p$]
  let GREATER = [a_j such that $a_j > p$]

  if $|\text{LESS}| \geq k$ **then return quickselect**(LESS, k)
  else if $|\text{LESS}| = k$ **then return** p
  else return quickselect(RIGHT, $k - |\text{LESS}| - 1$)
}

- Runs in $O(n)$ **expected** time in the **comparison model**.
- More tightly, uses at most $8n$ comparisons in expectation.
- An as exercise, the analysis can be improved to $4n$ comparisons.

Have we achieved our goal?

- Use Quickselect to select the pivot for Quicksort
- Guaranteed best-case recursion for Quicksort
- Problem?

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- Guaranteed best-case recursion for Quicksort
- Problem?

- Randomized Quickselect is still… randomized.
- So, Quicksort would still be $O(n \log n)$ randomized, not deterministic

We want a deterministic algorithm!!

- Where was the randomness in Randomized QuickSelect? How can we get rid of it?

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- What if we could deterministically find the optimal pivot? What would that be? The median! Oh…

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What we need: In $O(n)$ comparisons, we need to find a "good" pivot. A good pivot would leave us with cn elements in the recursive call, for some fraction $c < 1$, e.g., $\frac{3n}{4}$ elements is good.

Picking a good pivot

- Picking the median as the pivot is too much to ask for, so we want some kind of “approximate median”

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Idea (doesn’t quite work, but very close): Pick the median of a smaller subset of the input (faster to find) then hope that it is a good approximation to the true median.

Picking a good pivot

- Picking the median as the pivot is too much to ask for, so we want some kind of "approximate median"

Idea (doesn't quite work, but very close): Pick the median of a smaller subset of the input (faster to find) then hope that it is a good approximation to the true median.

Question: What if we find the median of half of the elements? How good of a pivot is this element?

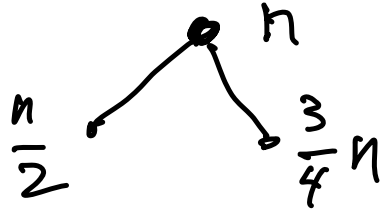
$T(\frac{n}{2})$ to find a pivot

which guarantees both parts are $\leq \frac{3}{4}n$

Median of half

If we pivot on the median of half of the elements, the number of comparisons will be

$$T(n) \leq n-1 + T\left(\frac{n}{2}\right) + T\left(\frac{3}{4}n\right)$$



n

$\frac{5}{4}n$

⋮

this solves to n^{α} where α satisfies

$\left(\frac{1}{2}\right)^{\alpha} + \left(\frac{3}{4}\right)^{\alpha} = 1$

ie $\alpha = 1.50712\cdots$

Median of half

If we pivot on the median of half of the elements, the number of comparisons will be

$$T(n) \leq$$

Exercise: Show that picking any constant-fraction sized subset (e.g., a quarter, one tenth) and taking the median doesn't work.

We need to go deeper!

Note: This idea is extremely subtle. It took four Turing Award winners to figure it out. We don't expect that you would produce this algorithm on your own.

- Finding the median of a smaller set **almost** worked, but it was just a bit too much work since the "approximate median" wasn't good enough.

We need to go deeper!

Note: This idea is extremely subtle. It took four Turing Award winners to figure it out. We don't expect that you would produce this algorithm on your own.

- Finding the median of a smaller set **almost** worked, but it was just a bit too much work since the "approximate median" wasn't good enough.

Huge idea (median of medians): Find the medians of several small subsets of the input, then find the **median of those medians**.

Median of medians algorithm

function DeterministicSelect($a[0 \ldots n-1],\ \ k$**) {**

group the array into $n/5$ groups of size 5, find the median of each group

recursively find the median of these medians, call it p

// Below is the same as Randomized Quickselect

let LESS = [a_j such that $a_j < p$]

let GREATER = [a_j such that $a_j > p$]

if $|\text{LESS}| \geq k$ **then return DeterministicSelect**(LESS, k)

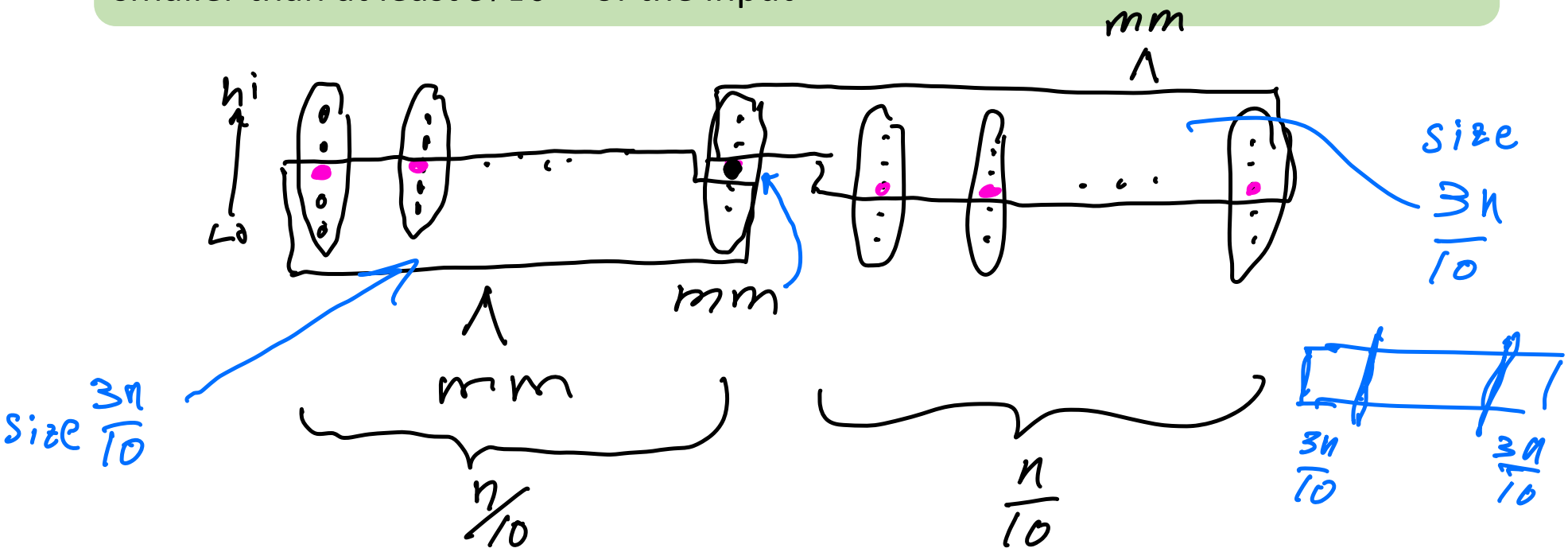
else if $|\text{LESS}| = k$ **then return** p

else return DeterministicSelect(RIGHT, $k - |\text{LESS}| - 1$)

}

How good is the median of medians?

Theorem: The median of medians is larger than at least $3/10^{\text{ths}}$ of the input, and smaller than at least $3/10^{\text{ths}}$ of the input



Analysis of DeterministicSelect

Theorem: The number of comparisons performed by DeterministicSelect on an input of size n *is* $O(n)$

1. Find the median of $n/5$ groups of size 5
2. Recursively find the median of medians
3. Split the input into LESS and GREATER
4. Recurse on the appropriate piece

$O(n)$ + $T(\frac{n}{5})$ + $n-1$ + $T(\frac{7n}{10})$

$$T(n) \leq c \cdot n + T(\tfrac{n}{5}) + T(\tfrac{7n}{10})$$

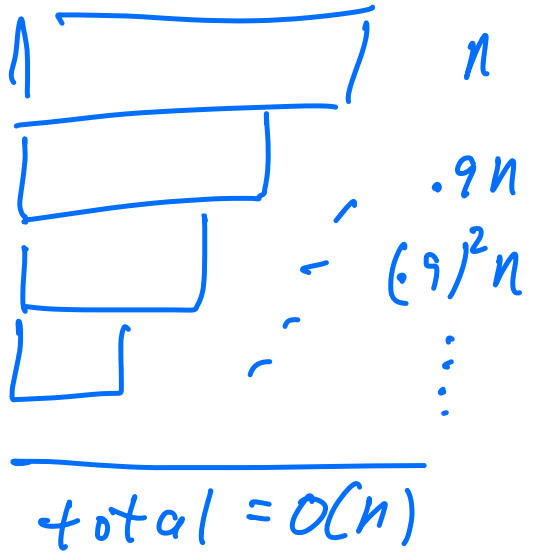
Solving the recurrence

$$\frac{1}{5} + \frac{7}{10} = \frac{9}{10} < 1$$

This is why it's linear.

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

This solves to $O(n)$. Each level contributes at most .9 times the one above it.



So, the total running time is…

$$T(n) \leq cn(1 + (9/10) + (9/10)^2 + (9/10)^3 + \ldots)$$

Summary of DeterministicSelect

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  group the array into $n/5$ groups of size 5,
    find the median of each group
  recursively find the median of these medians, call it p

  // Below is the same as Randomized Quickselect
  let LESS = [a_j such that $a_j < p$]
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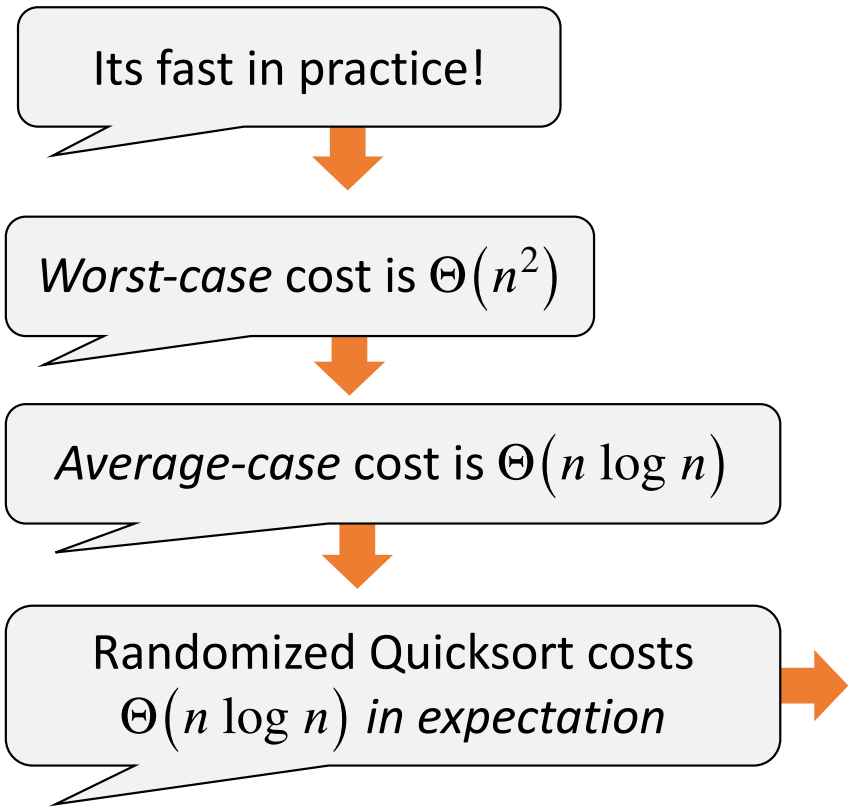
}

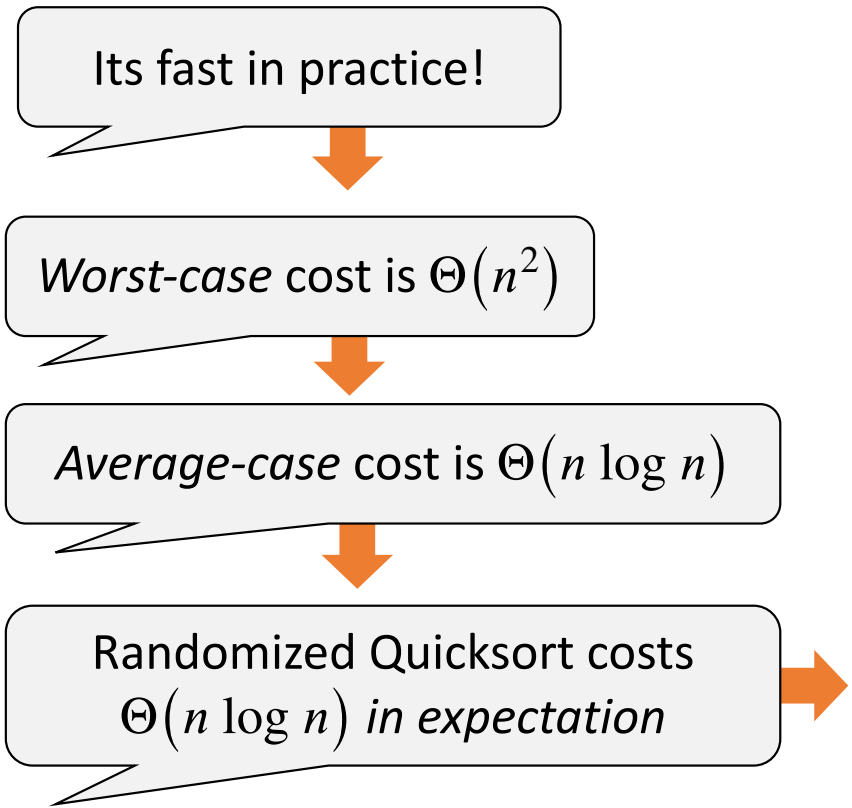
- The **median of medians** is the key ingredient for getting a deterministic algorithm
- To analyze the recurrence, we used the "stack of bricks" method.
- We could also prove it by induction, but this requires us to know the runtime already

The Quicksort journey



1. *Use the median-of-medians algorithm to find the median in* ***deterministic*** $\Theta(n)$ *cost*
2. *Use the median as the pivot for Quicksort*

The Quicksort journey



Deterministic Quicksort in worst-case $\Theta(n \log n)$ cost!!

1. *Use the median-of-medians algorithm to find the median in* ***deterministic*** $\Theta(n)$ *cost*
2. *Use the median as the pivot for Quicksort*

Take-home messages for today

- There's more to Quicksort than you think!
- Recursion is powerful, randomization is powerful.
- Analyzing ***randomized recursive algorithms*** is tricky. Be careful with expected values!!
- Analyzing runtime via ***recurrence relations*** is very useful.