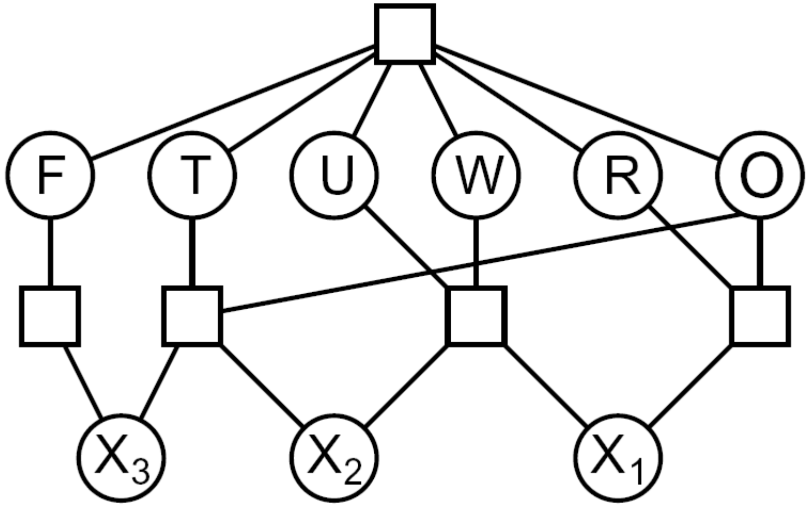
Warm-up: Cryptarithmetic

How would we formulate this as a linear program?

$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$



Announcements

Assignments:

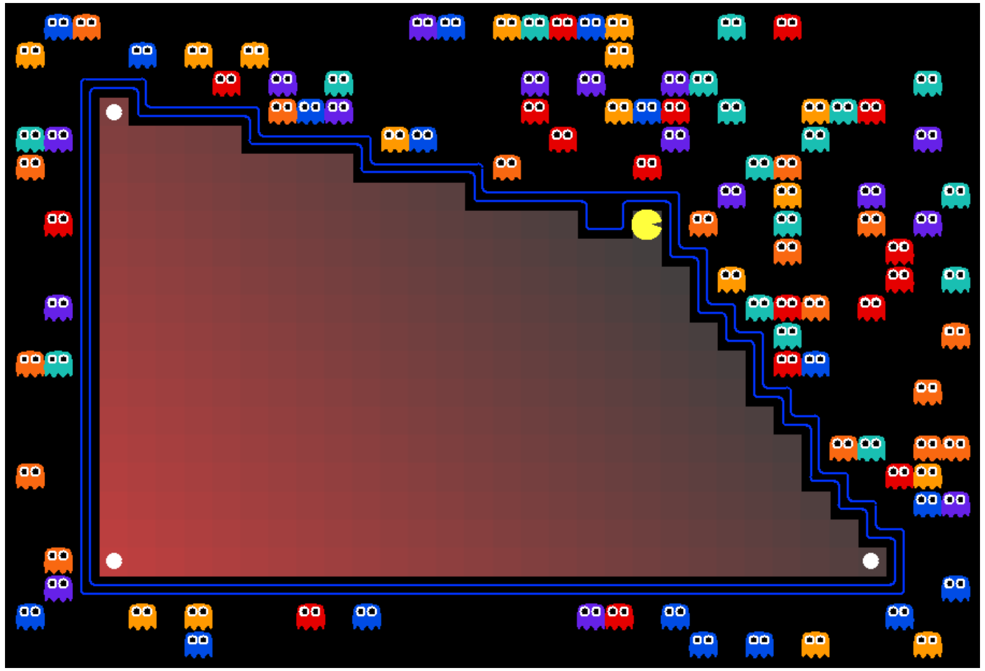
- HW4 (written)
  - Due Tue 2/12, 10 pm
- P2: Optimization
  - Released
  - Due Thu 2/21, 10 pm

Midterm 1 Exam

- Mon 2/18, in class
- Recitation Fri is a review session
- Practice midterm coming soon!

AI: Representation and Problem Solving

Integer Programming



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, http://ai.berkeley.edu

Linear Programming: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

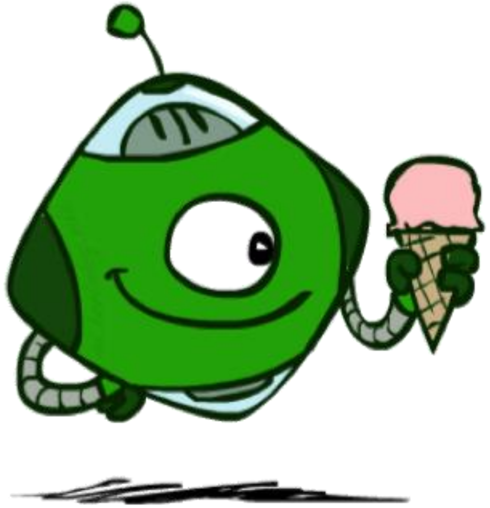
What is the cheapest way to stay “healthy” with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Optimization Formulation

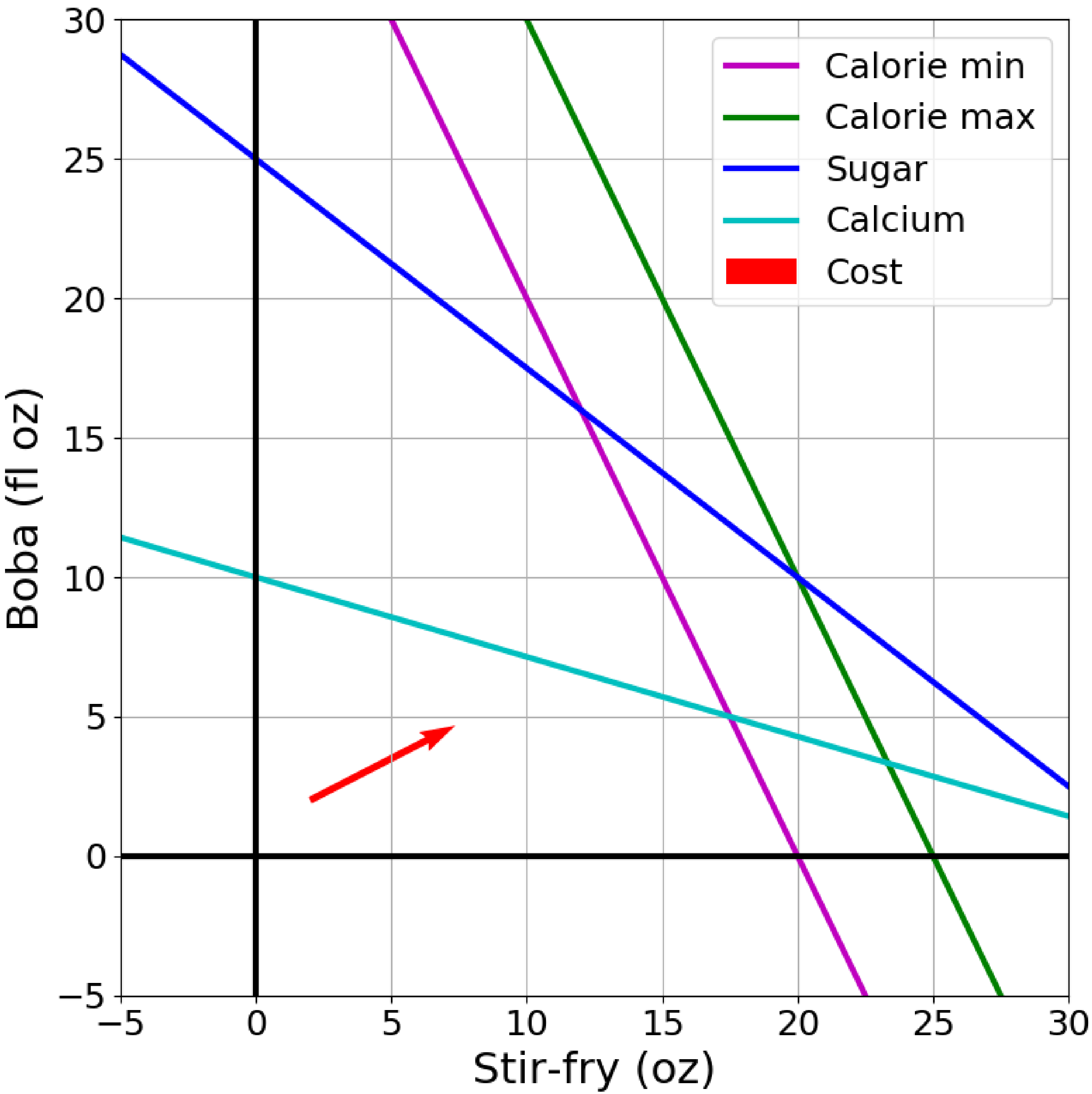
Diet Problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$



$$\boldsymbol{c} = \begin{array}{c} \text{Cost} \\ \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \end{array}$$

$$A = \begin{array}{c} \begin{array}{cc} \text{Stir-fry} & \text{Boba} \end{array} \\ \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \end{array} \quad \boldsymbol{b} = \begin{array}{cl} \text{Limit} & \\ -2000 & \text{Calorie min} \\ 2500 & \text{Calorie max} \\ 100 & \text{Sugar} \\ -700 & \text{Calcium} \end{array}$$

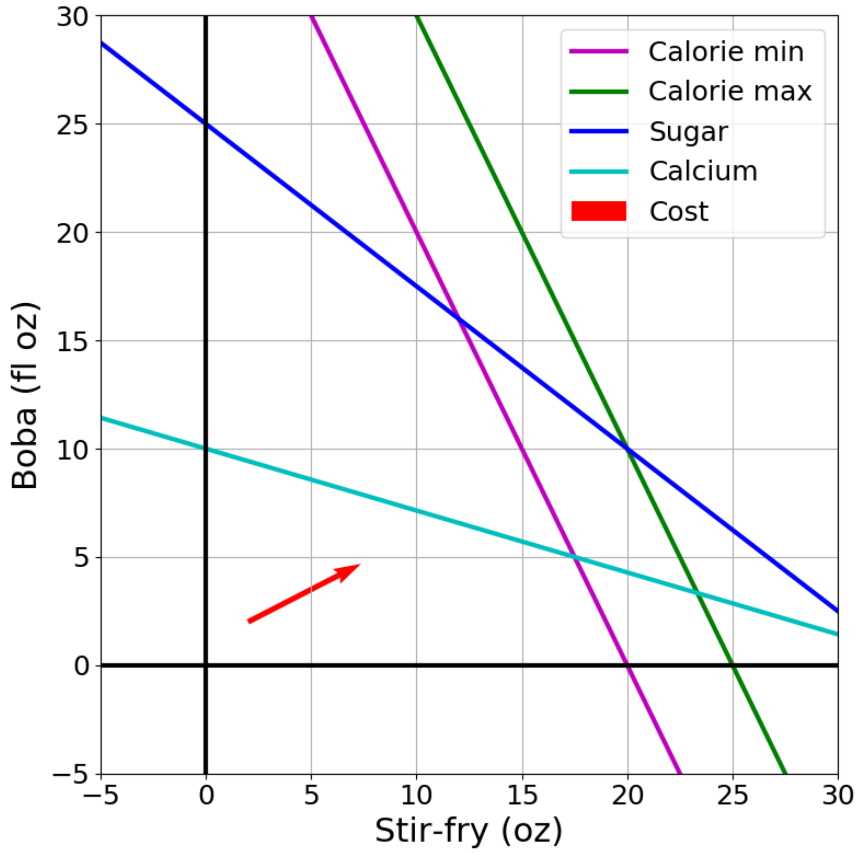


30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30
Boba (fl oz)
Stir-fry (oz)
Calorie min
Calorie max
Sugar
Calcium
Cost

Representation & Problem Solving

Problem Description

Graphical Representation



Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will $\boldsymbol{c}^T\boldsymbol{x} = 0$?

Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -2 ?

Piazza Poll 1

As the magnitude of $\boldsymbol{c}$ increases, the distance between the contours lines of the objective $\boldsymbol{c}^T\boldsymbol{x}$:

A) Increases

B) Decreases

Solving a Linear Program

Inequality form, with no constraints

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$

Solving a Linear Program

Inequality form, with no constraints

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

Piazza Poll 2

True or False: An minimizing LP with exactly on constraint, will always have a minimum objective at $-\infty$.

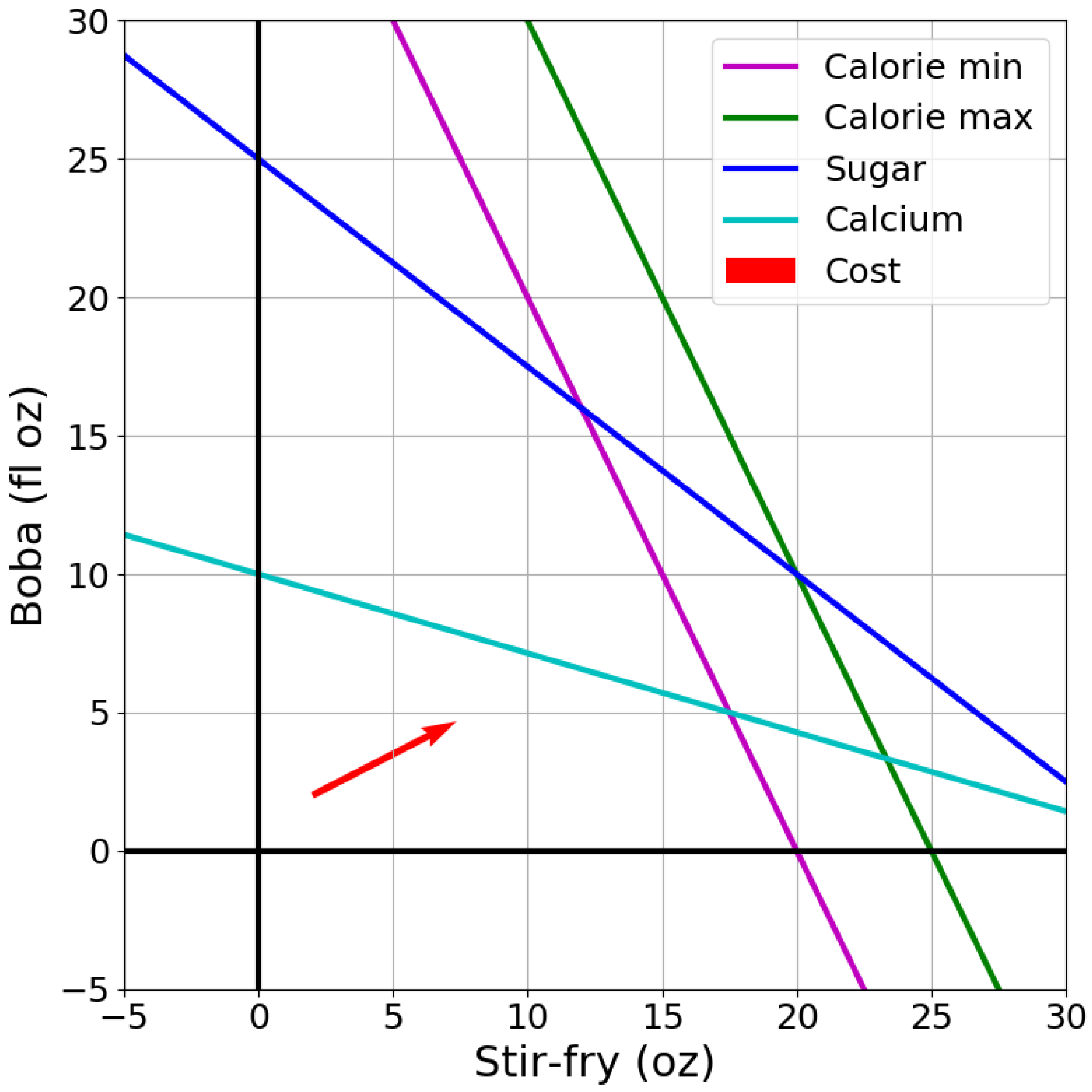
$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$

Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections



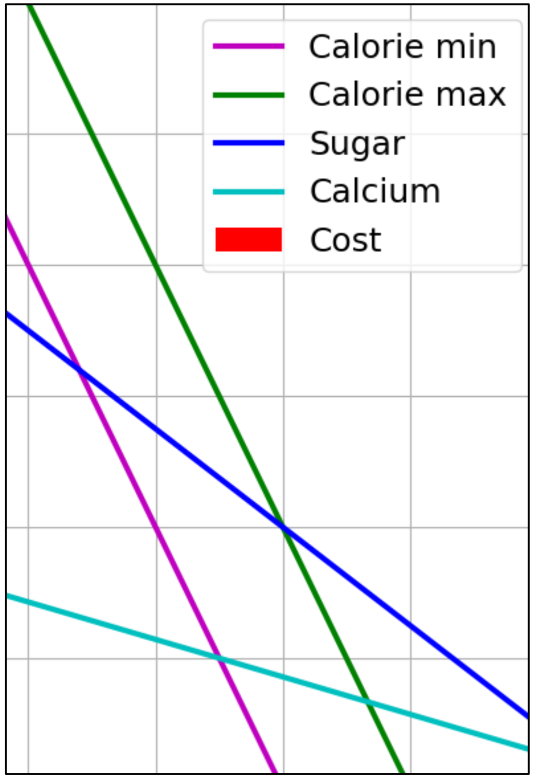
Solving an LP

But, how do we find the intersection between boundaries?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

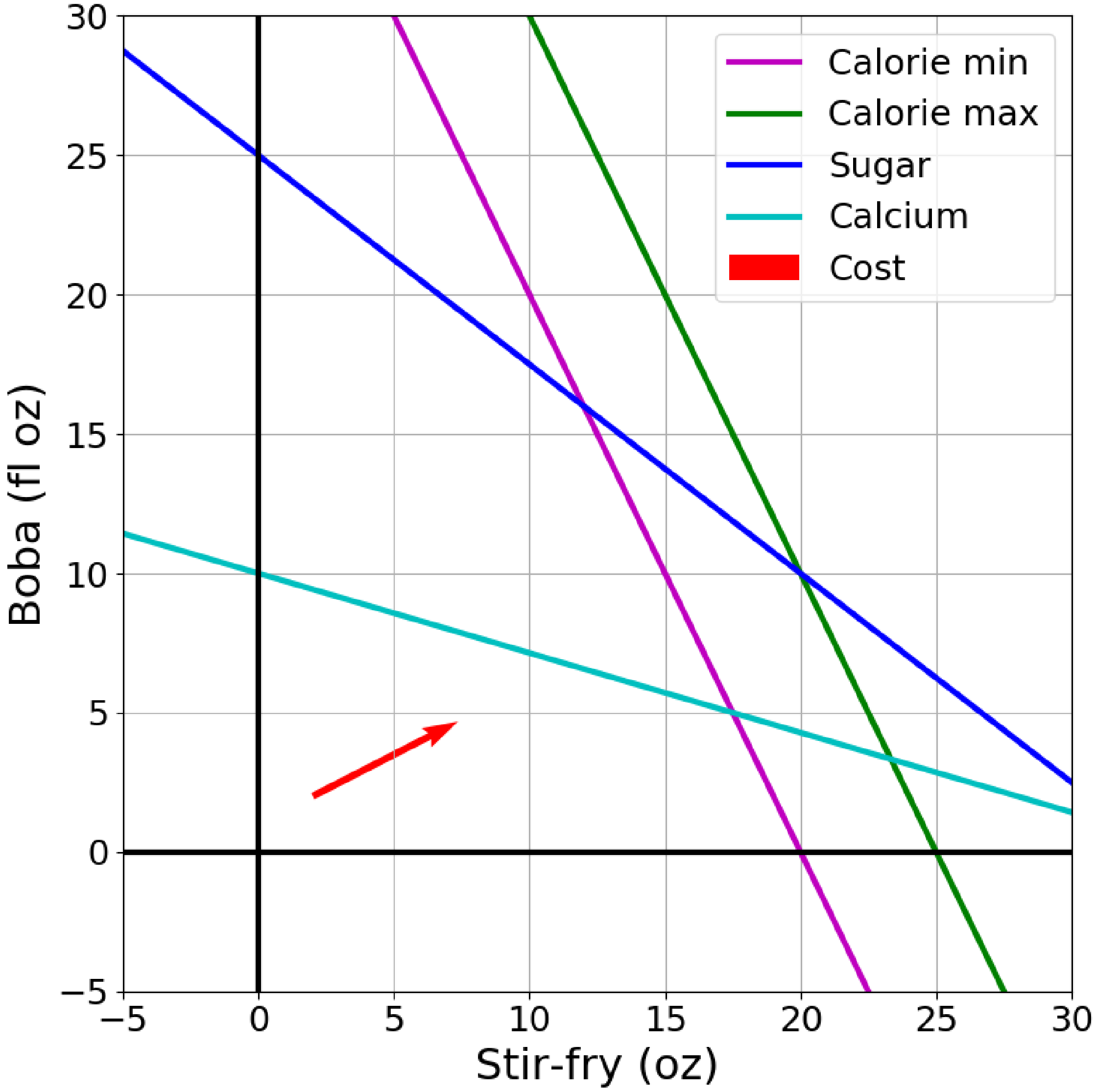


Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex



Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior Point

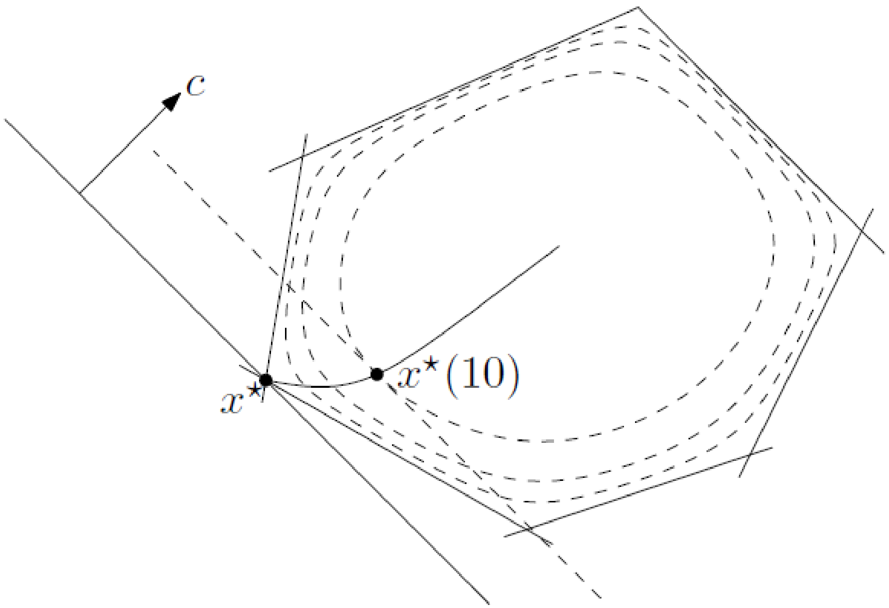


Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

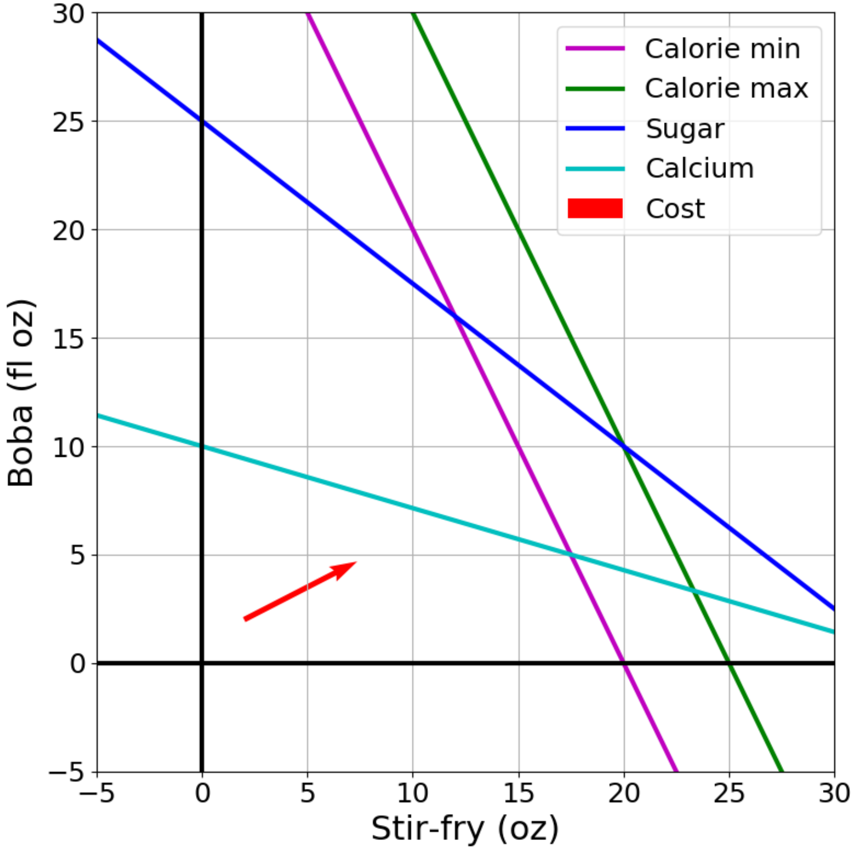
What about higher dimensions?

Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



“Marty, your not thinking fourth-dimensionally”

Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$$a_1\,x_1 + a_2\,x_2 = b_1$$

$$a_1\,x_1 + a_2\,x_2 \leq b_1$$

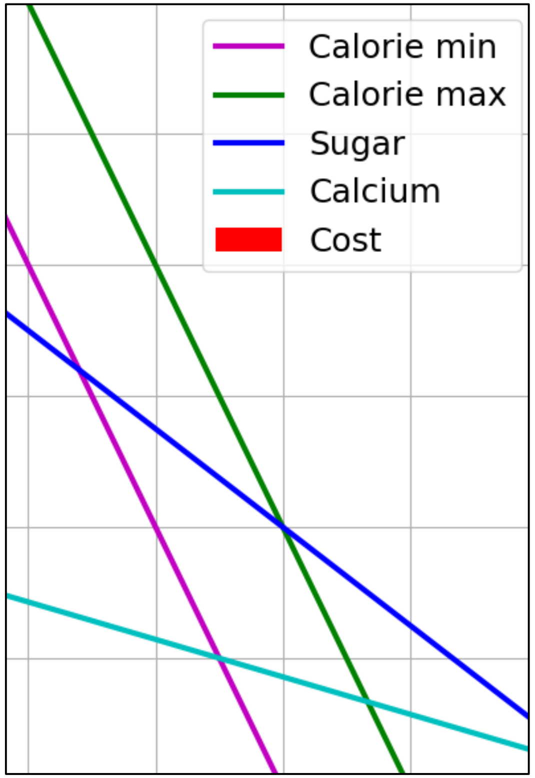
$$\begin{aligned}
a_{1,1}\,x_1 + a_{1,2}\,x_2 &\leq b_1\\
a_{2,1}\,x_1 + a_{2,2}\,x_2 &\leq b_2\\
a_{3,1}\,x_1 + a_{3,2}\,x_2 &\leq b_3\\
a_{4,1}\,x_1 + a_{4,2}\,x_2 &\leq b_4
\end{aligned}$$

What are intersections in higher dimensions?

How do these linear shapes extend to 3-D, N-D?

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \preceq \boldsymbol{b} \end{array}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$

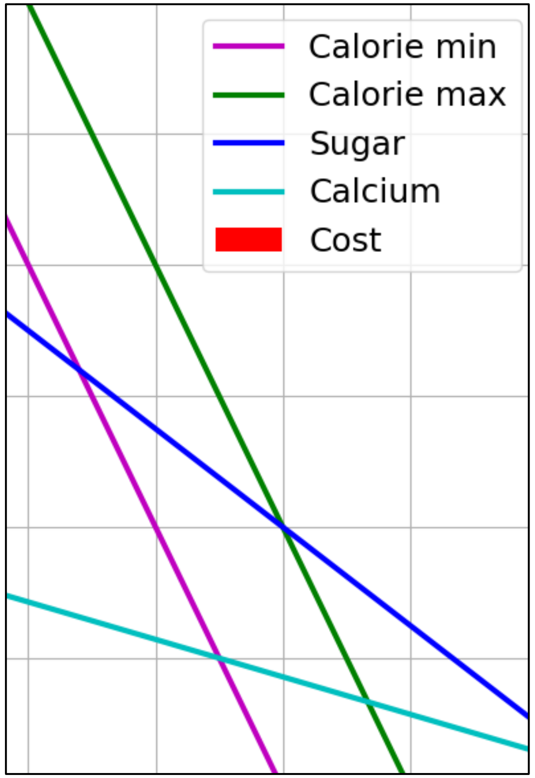


How do we find intersections in higher dimensions?

Still looking at subsets of A matrix

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{array}{cl} \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} & \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array} \end{array}$$



Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

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What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per bowl)	1	100	3	20
Boba (per glass)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned} \min_{\boldsymbol{x}}.\quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.}\quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned} \qquad\qquad \begin{aligned} \min_{\boldsymbol{x}}.\quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.}\quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

Notation Alert!

Integer Programming: Graphical Representation

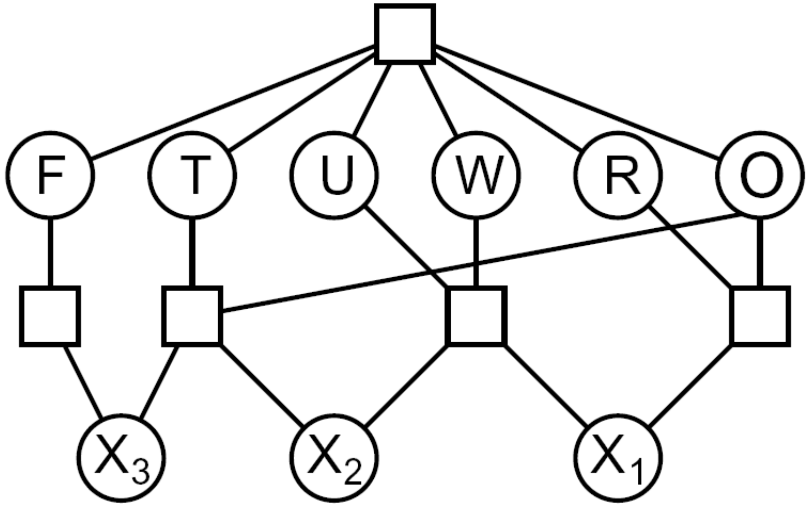
Just add a grid of integer points onto our LP representation

$$\begin{aligned} \underset{\boldsymbol{x}}{\min.} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

Integer Programming: Cryptarithmetic

How would we formulate this as a integer program?

$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$



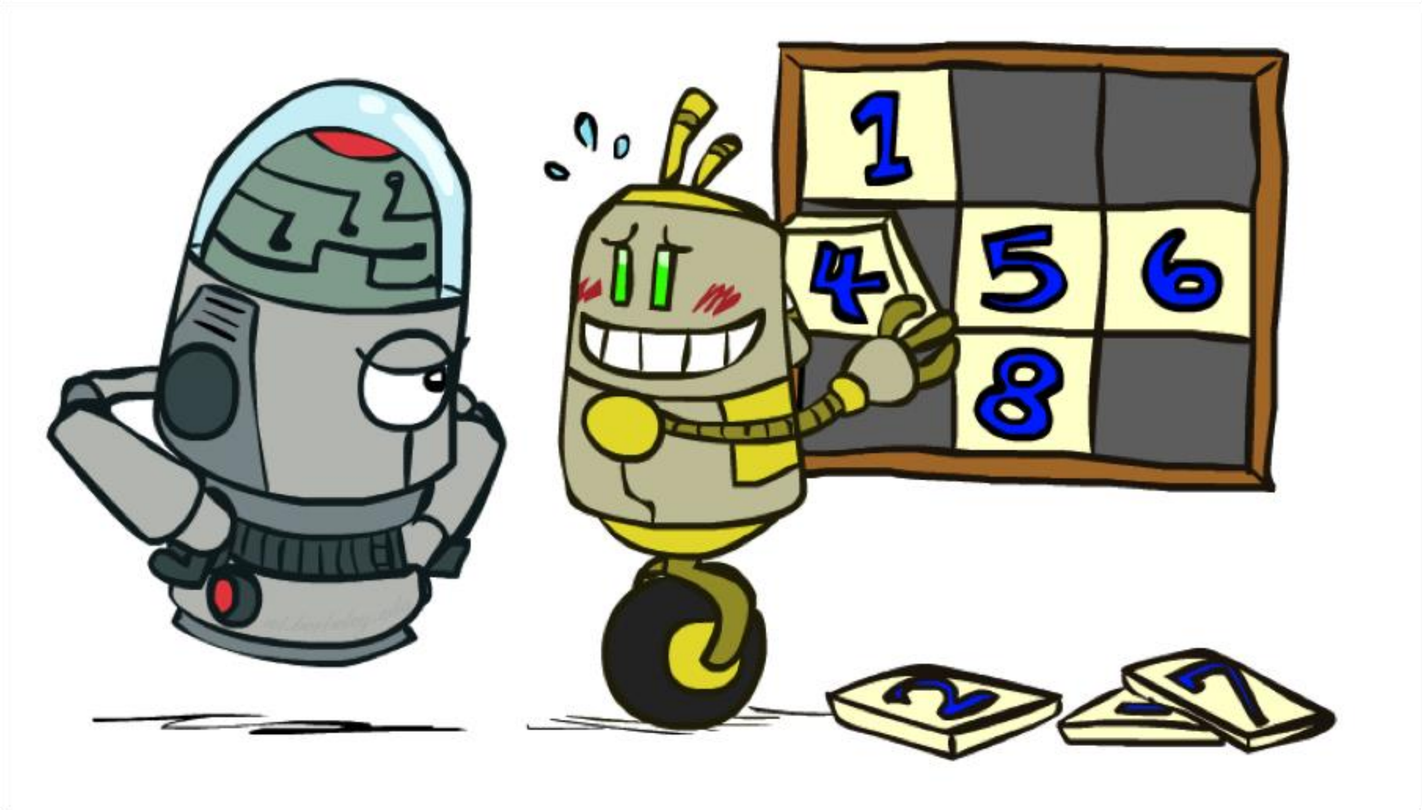
How would we could we solve it?

Relaxation

Relax IP to LP by dropping integer constraints

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \cancel{\boldsymbol{x} \in \mathbb{Z}^N} \end{aligned}$$

Remember heuristics?



Piazza Poll 3:

Let y_{IP}^* be the optimal objective of an integer program P.

Let $\boldsymbol{x}_{IP}^*$ be an optimal point of an integer program P.

Let y_{LP}^* be the optimal objective of the LP-relaxed version of P.

Let $\boldsymbol{x}_{LP}^*$ be an optimal point of the LP-relaxed version of P.

Assume that P is a minimization problem.

$$\begin{aligned} y_{IP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

$$\begin{aligned} y_{LP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Which of the following are true?

A) $\boldsymbol{x}_{IP}^* = \boldsymbol{x}_{LP}^*$

B) $y_{IP}^* \leq y_{LP}^*$

C) $y_{IP}^* \geq y_{LP}^*$

Piazza Poll 4:

True/False: It is sufficient to consider the integer points around the corresponding LP solution?

Solving an IP

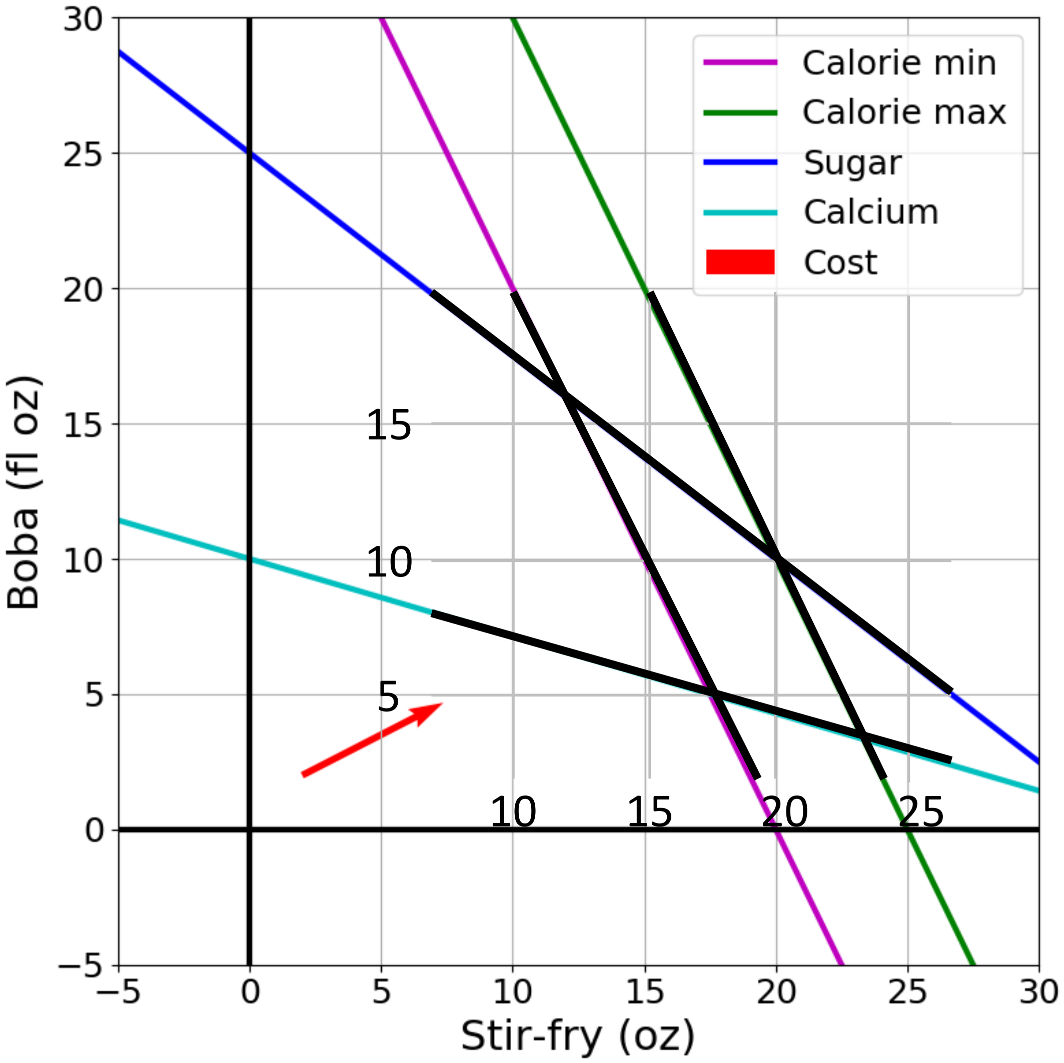
Branch and Bound algorithm

- Start with LP-relaxed version of IP
- If solution $\mathbf{x}_{LP}^{*}$ has non-integer value at x_i,
  Consider two branches with two different slightly more constrained LP problems:

  Left branch: Add constraint $x_i \leq floor(x_i)$

  Right branch: Add constraint $x_i \geq ceil(x_i)$
- Recursion. Stop going deeper:
  - When the LP returns a worse objective than the best **feasible IP objective** you have seen before (remember pruning!)
  - When you hit an integer result from the LP
  - When LP is infeasible

Branch and Bound Example



Branch and Bound Example