Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

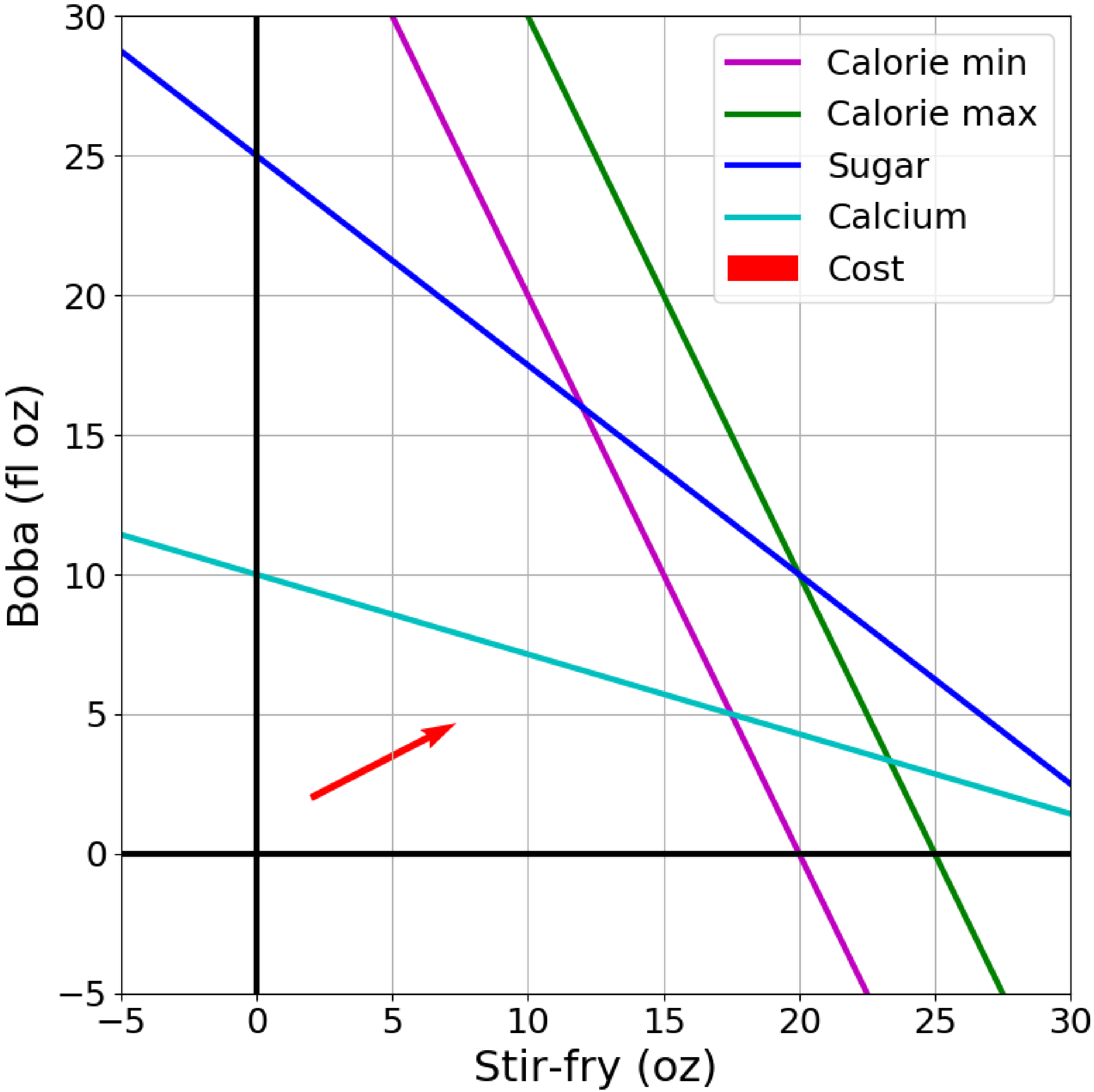
Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?



Calorie min
Calorie max
Sugar
Calcium
Cost
Boba (fl oz)
Stir-fry (oz)
30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30

Announcements

Assignments:

- HW3 (online)
  - Due Wed 2/6, 10 pm
- P1: Search & Games
  - Due Thu 2/7, 10 pm
- HW4 (written)
  - Released Wed 2/6
  - Due Tue 2/12, 10 pm
- P2: Optimization
  - Released later this week
  - Due Thu 2/21, 10 pm

Announcements

Midterm 1 Exam

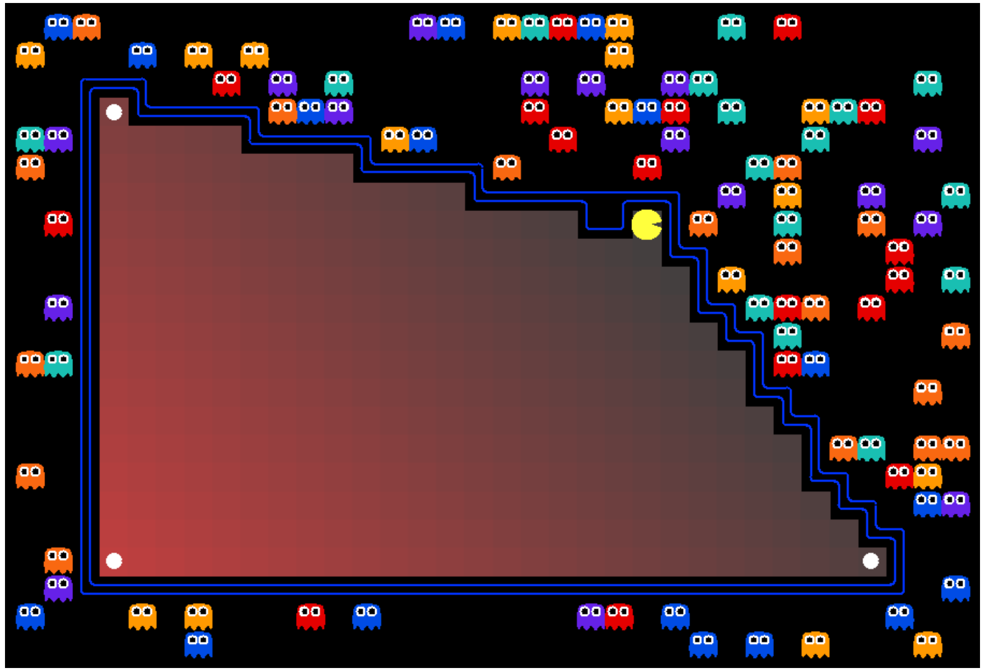
- Mon 2/18, in class

AAAI Conference

Honolulu, HI

AI: Representation and Problem Solving

Linear Programming



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, http://ai.berkeley.edu

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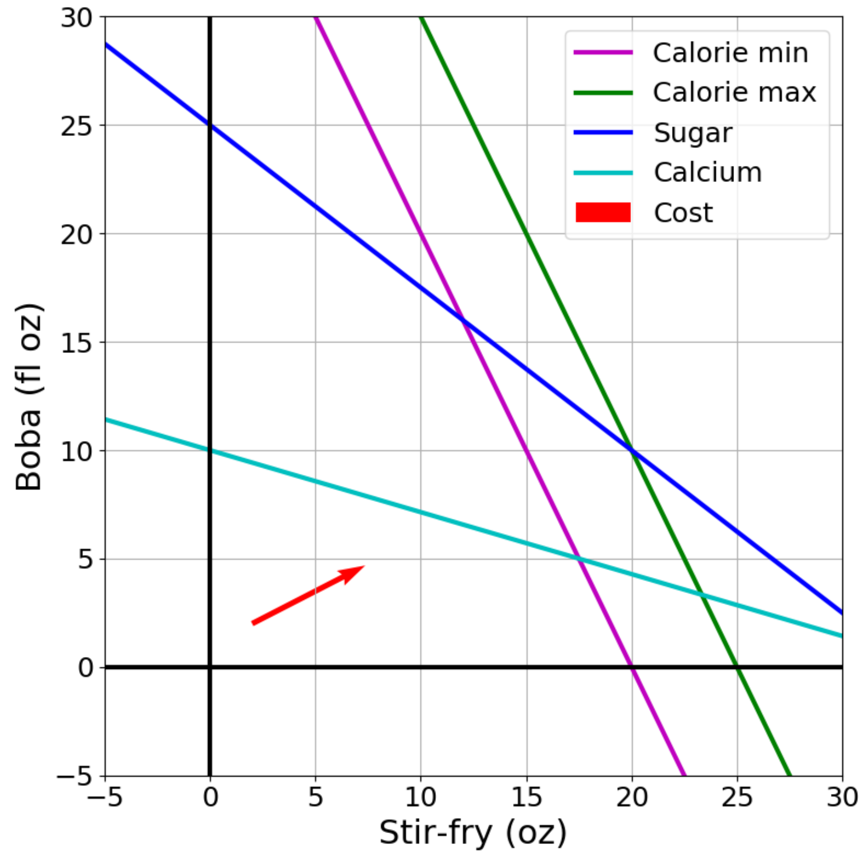
Optimization

Problem Description

Graphical Representation



Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

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Constraint Satisfaction Problems

Map coloring

$$Any \quad \boldsymbol{x}$$

$$\text{s.t.} \quad \boldsymbol{x} \ satisfies \ constraints$$

Constraint Satisfaction Problems

Map coloring

$$Any \quad \boldsymbol{x}$$

$$\text{s.t.} \quad \boldsymbol{x} \;\; satisfies \;\; constraints$$

such that

$\boldsymbol{x}$ $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$

Notation Alert!

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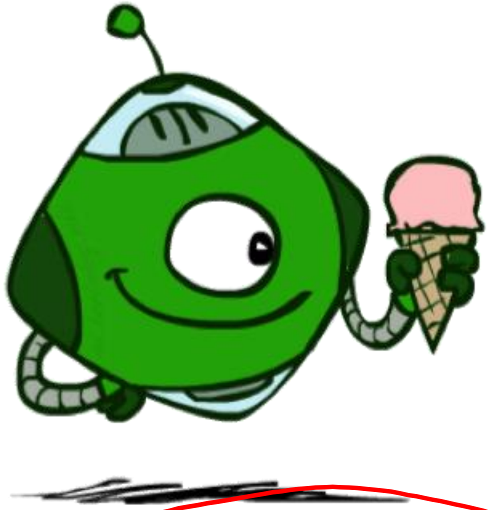
Optimization Formulation

Diet Problem

$$\begin{aligned} &Any \quad \boldsymbol{x} \\ &\text{s.t.} \quad \boldsymbol{x}\ satisfies\ constraints \end{aligned}$$



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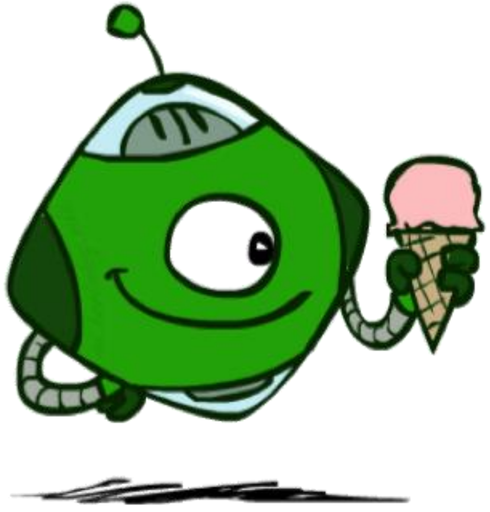
Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad cost(\boldsymbol{x}) \qquad \text{Objective}$$

$$\text{s.t.} \quad \boldsymbol{x} \ satisfies \ constraints$$



Healthy Squad Goals

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Notation Alert!

Optimization Formulation

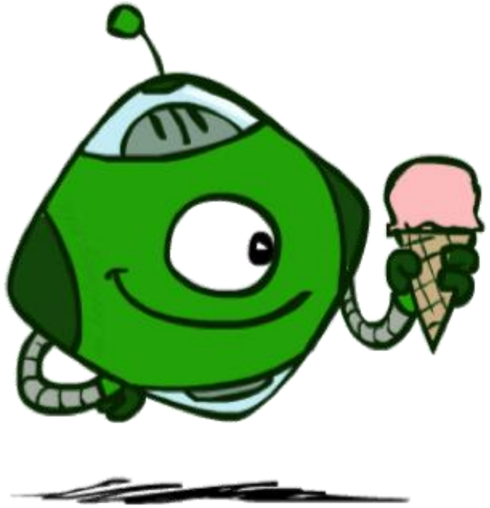
Diet Problem

$$\min_{\boldsymbol{x}} \quad cost(\boldsymbol{x})$$

$$\text{s.t.} \quad calories(\boldsymbol{x})\ contained$$

$$sugar(\boldsymbol{x}) \leq limit$$

$$\text{calcium}(\boldsymbol{x}) \geq limit$$



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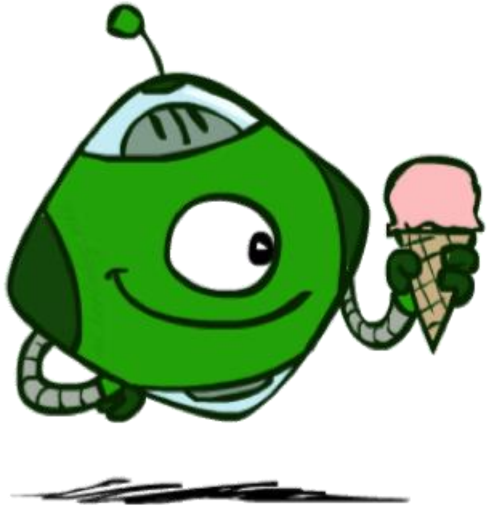
Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Optimization Formulation

Diet Problem

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$$\text{s.t.} \quad \begin{aligned} 100\, x_1 + 50\, x_2 &\geq 2000 \\ 100\, x_1 + 50\, x_2 &\leq 2500 \\ 3\, x_1 + 4\, x_2 &\leq 100 \\ 20\, x_1 + 70\, x_2 &\geq 700 \end{aligned}$$



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Notation Alert!

Optimization Formulation

Diet Problem

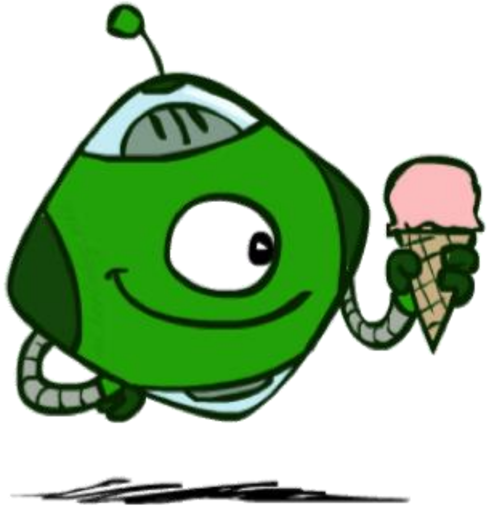
$$\min_{x_1, x_2} \quad c_1 x_1 + c_2 x_2 \rightarrow c^T x$$

$$\text{s.t.} \quad a_{1,1} x_1 + a_{1,2} x_2 \geq b_1$$

$$a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$$

$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$

$$a_{4,1} x_1 + a_{4,2} x_2 \geq b_4$$



Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry, Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

$$\boldsymbol{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

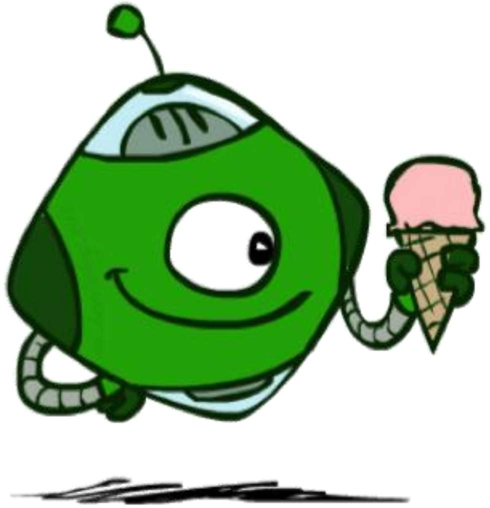
Calorie min
Calorie max
Sugar
Calcium

Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad a_{1,1}\, x_1 + a_{1,2}\, x_2 \geq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \geq b_4$$



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Calorie min
Calorie max
Sugar
Calcium

Notation Alert!

Optimization Formulation

Diet Problem

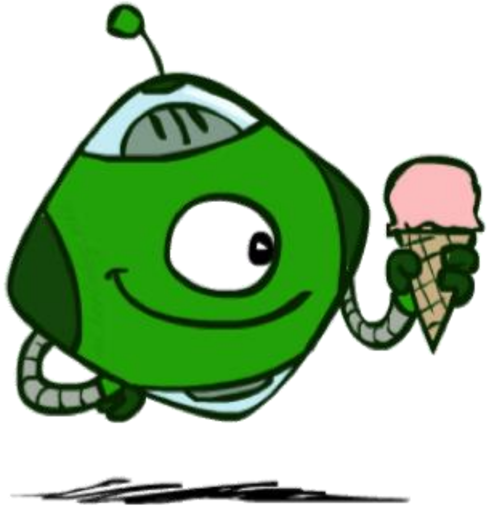
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$

$$\text{s.t.} \quad -a_{1,1}\,x_1 - a_{1,2}\,x_2 \leq -b_1$$

$$a_{2,1}\,x_1 + a_{2,2}\,x_2 \leq b_2$$

$$a_{3,1}\,x_1 + a_{3,2}\,x_2 \leq b_3$$

$$-a_{4,1}\,x_1 - a_{4,2}\,x_2 \leq -b_4$$



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$$c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

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Limit

$$b = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

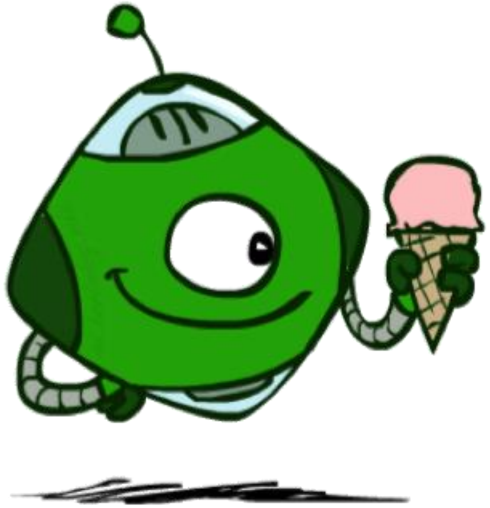
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry Boba

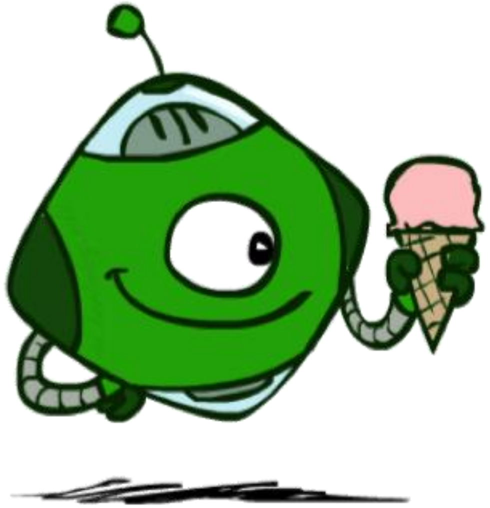
Limit

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry, Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

Limit

$$\boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

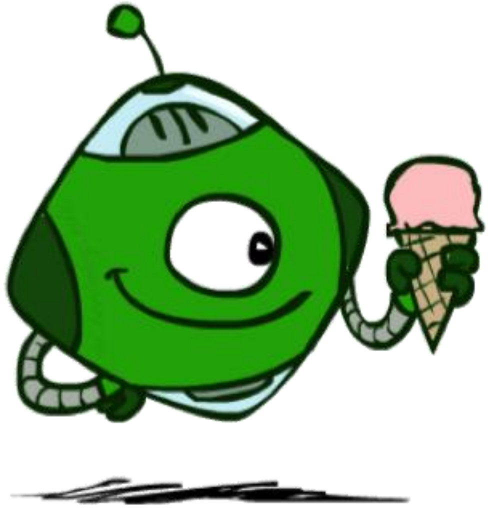
Calorie min
Calorie max
Sugar
Calcium

Notation Alert!

Piazza Poll 1

What has to increase to add more nutrition constraints?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

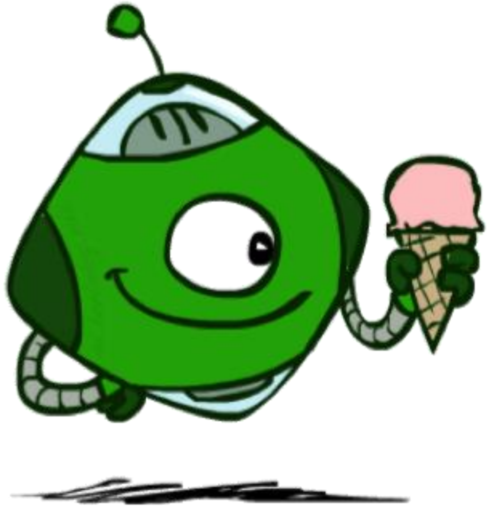
C) height A

D) width A

E) length $\boldsymbol{b}$

Piazza Poll 1

What has to increase to add more nutrition constraints?



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

M = 6
N = 2

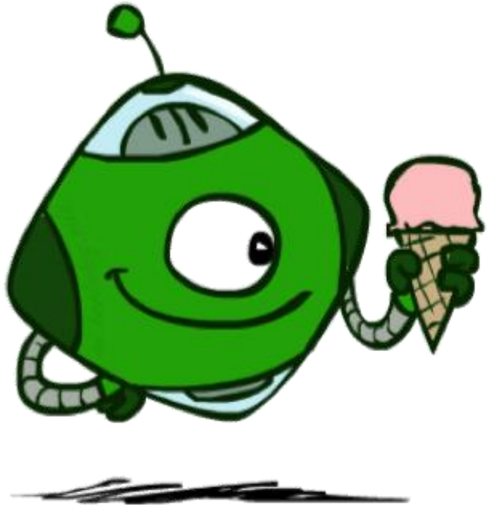
6×2 2×1 → 6×1

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Piazza Poll 2

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

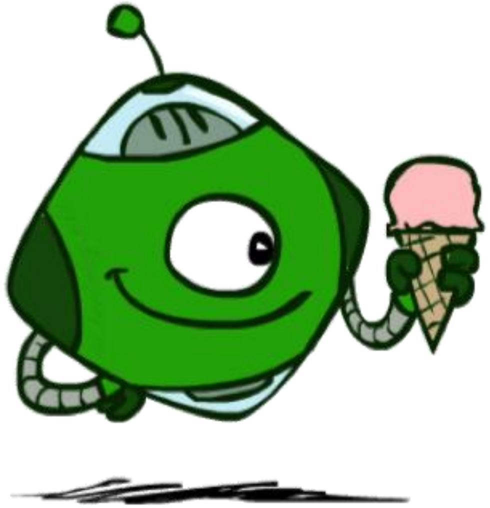
C) height A

D) width A

E) length $\boldsymbol{b}$

Piazza Poll 2

What has to increase to add more nutrition constraints?



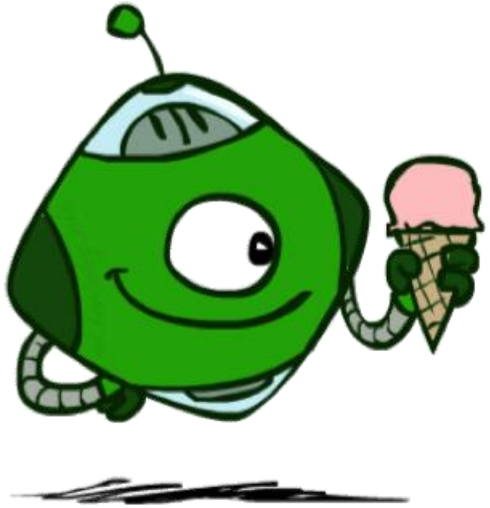
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Question

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

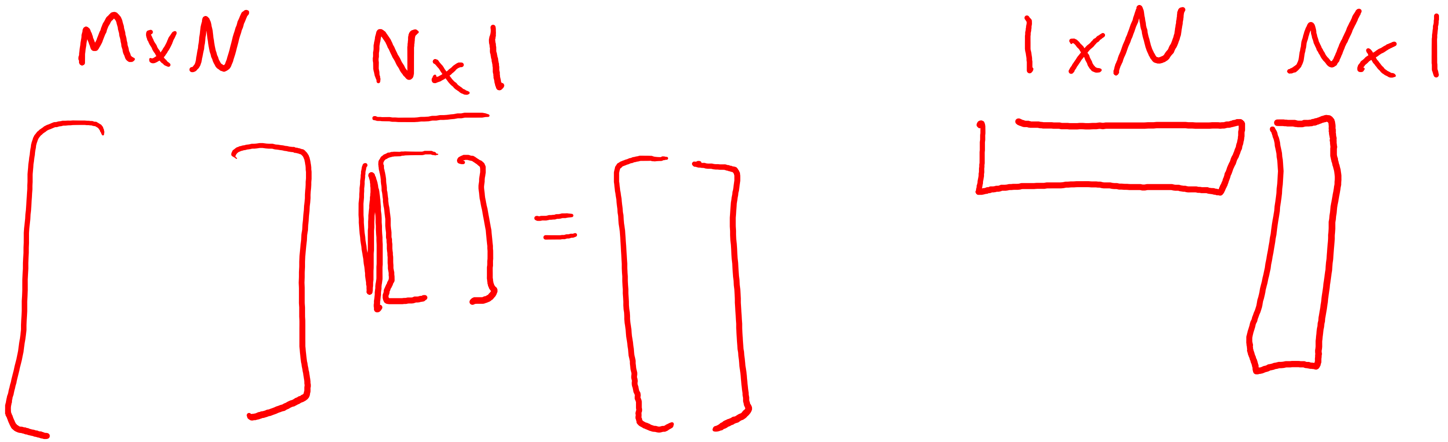


Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$

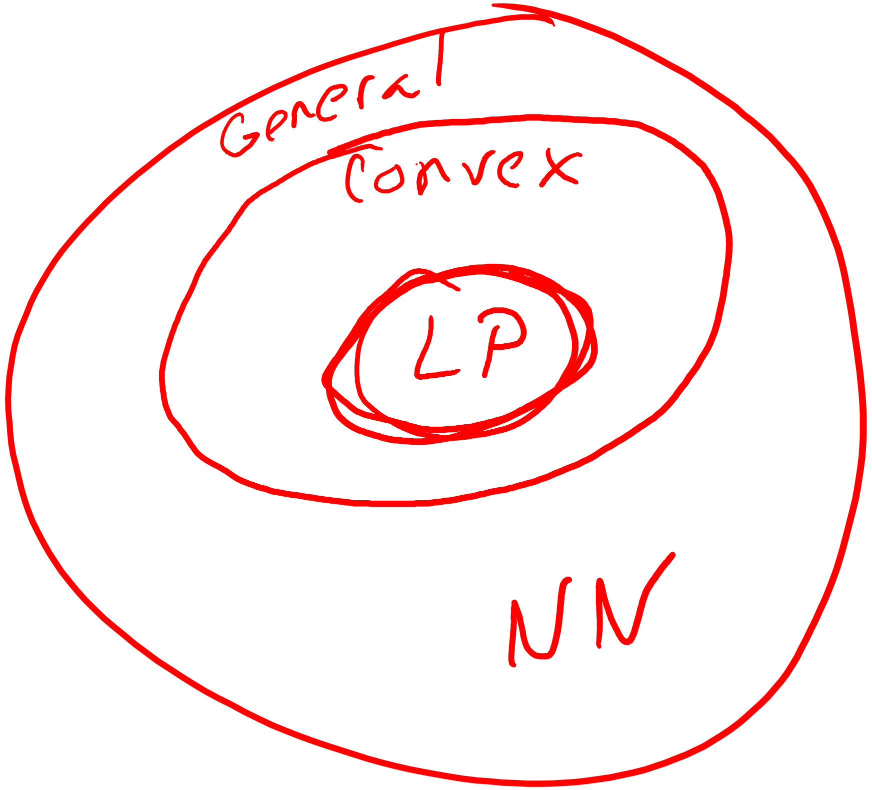


Notation Alert!

Linear Programming

Linear objective with linear constraints

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



As opposed to general optimization

$$\min_{\boldsymbol{x}}. \quad f_0(\boldsymbol{x})$$
$$\text{s.t.} \quad f_i(\boldsymbol{x}) \leq 0, \quad i = 1 \dots M$$
$$\boldsymbol{a}_i^T \boldsymbol{x} = \boldsymbol{b}_i, \quad i = 1 \dots P$$

$x_1^2 + x_2^2$

Linear Programming

Different formulations

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Important to pay attention to form!

Linear Programming

Different formulations

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Can switch between formulations!

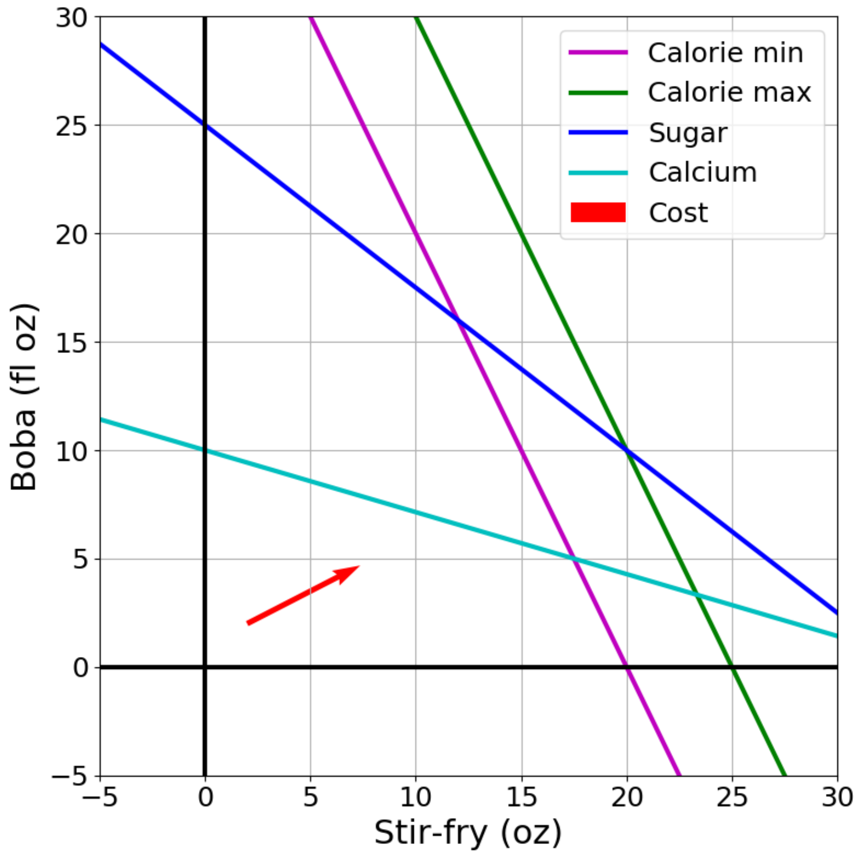
Optimization

Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



Graphics Representation

Geometry / Algebra I Quiz

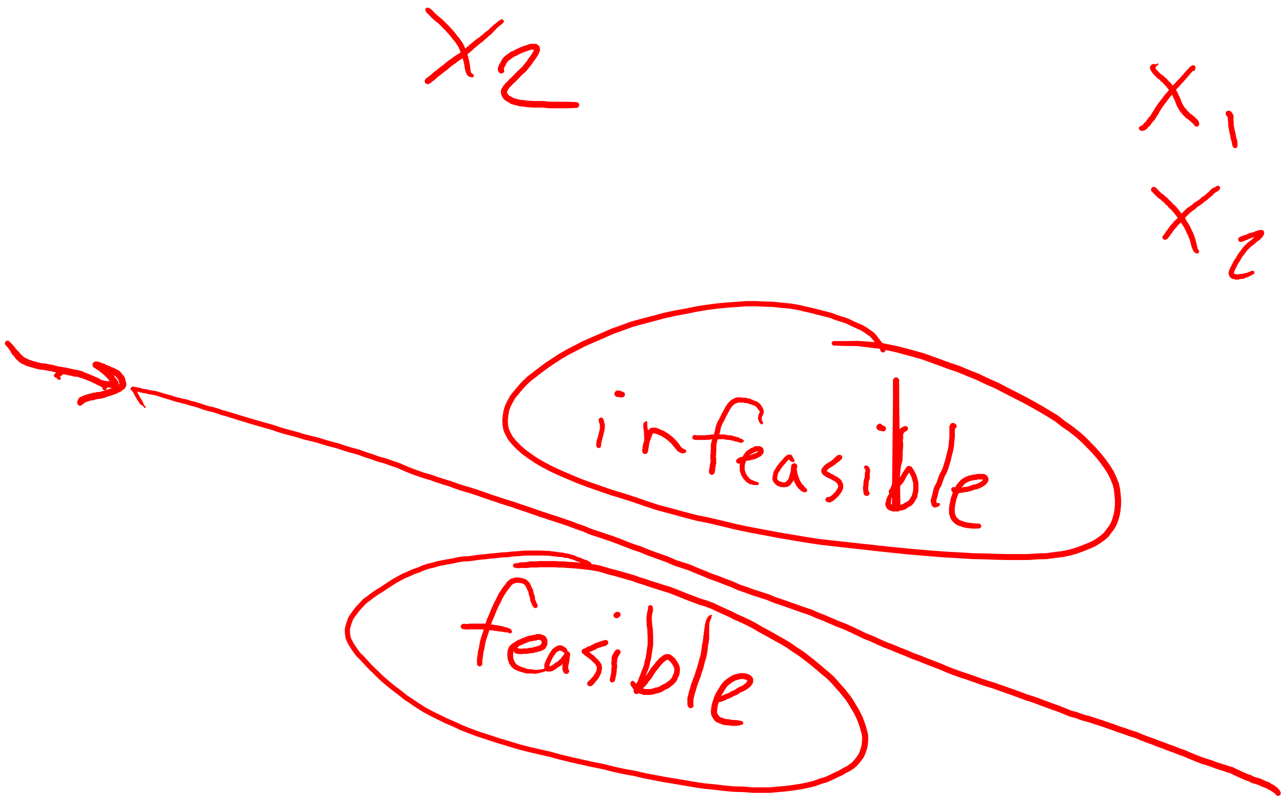
What shape does this inequality represent?

$$a_1 \, x_1 + a_2 \, x_2 \leq b_1$$

$=$

$x_1 \geq 0$

$x_2 \geq 0$



Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

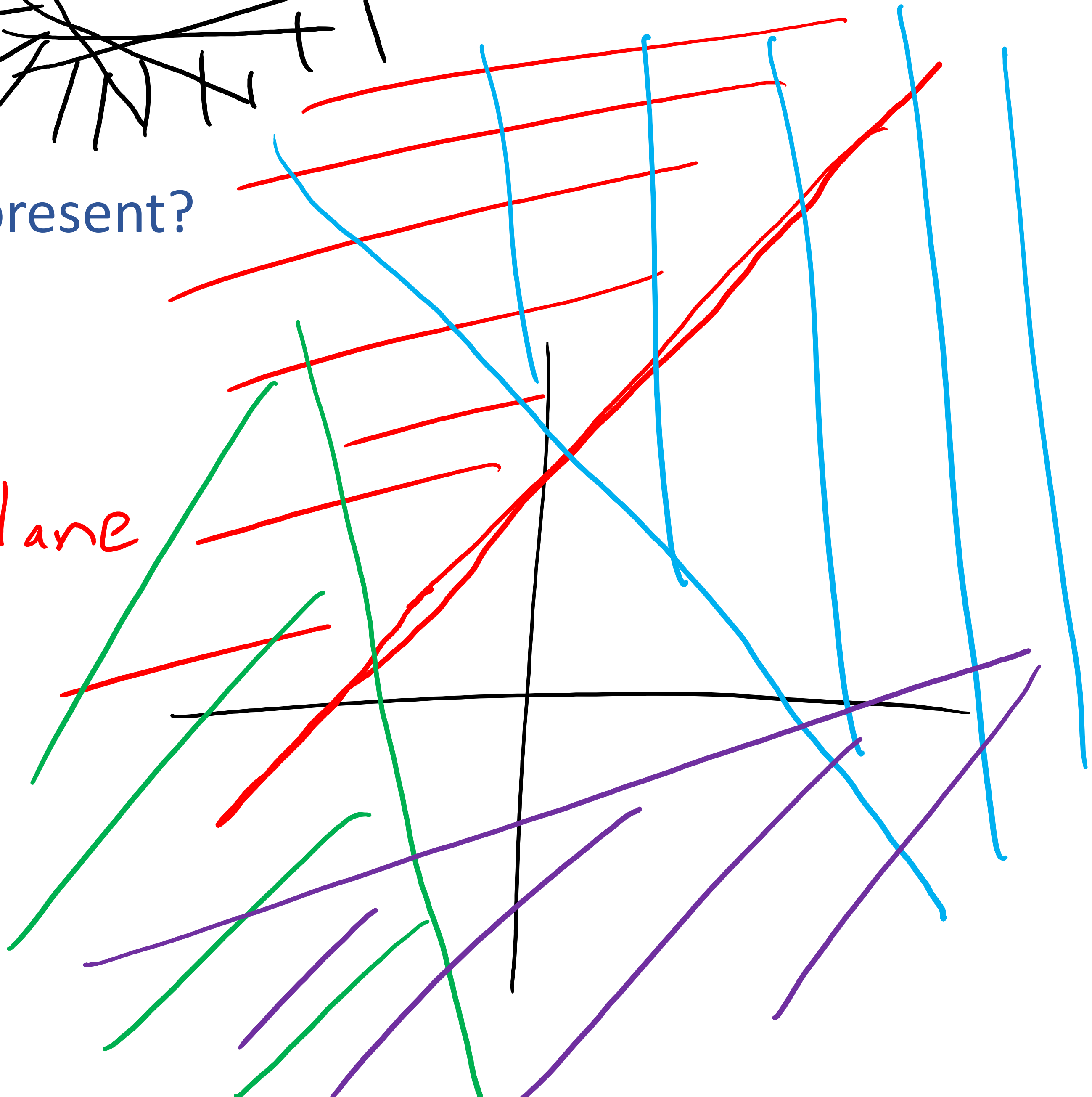
$$a_1\, x_1 + a_2\, x_2 = b_1$$

line

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

half plane

$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



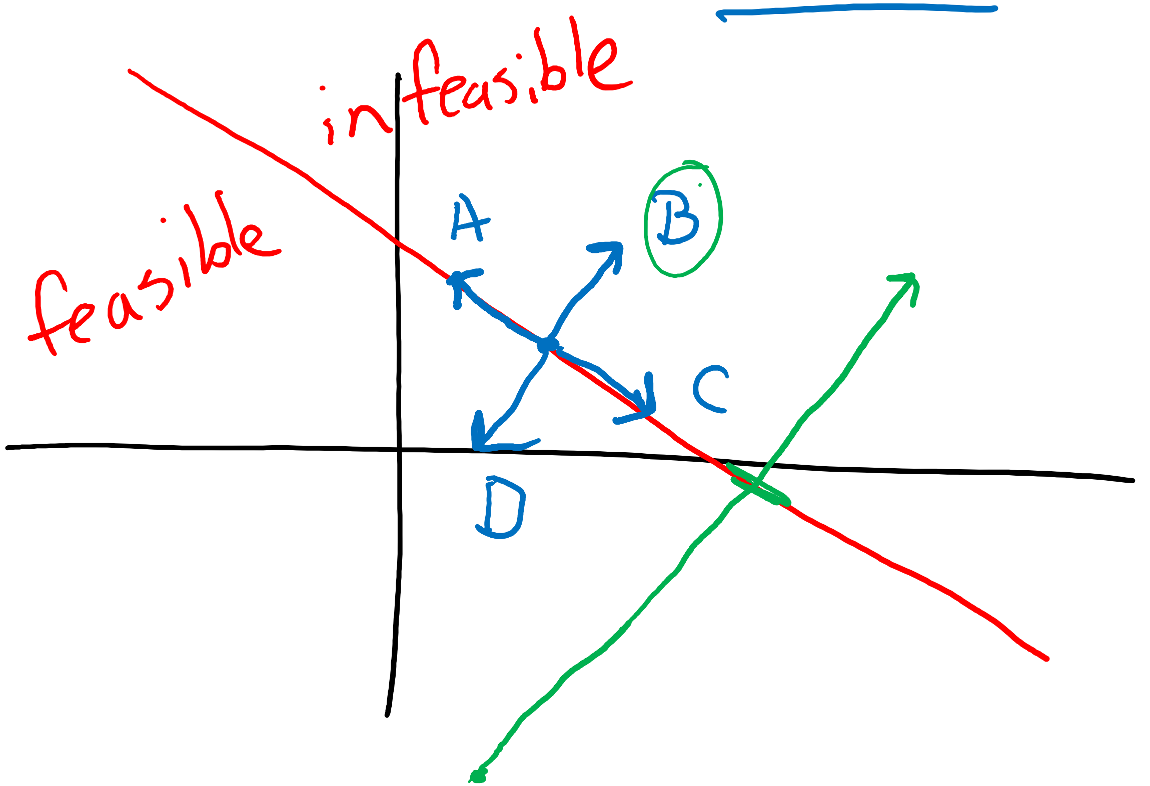
Piazza Poll 3

What is the relationship between the half plane:

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

and the vector:

$$[a_1, a_2]^T$$



A)
B)
C)
D)

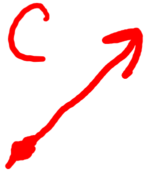
Piazza Poll 4

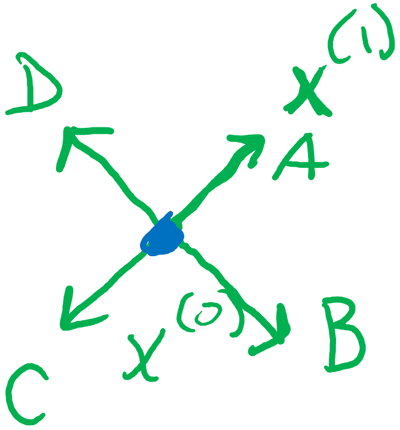
Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle\, \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle\, \boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?

$\min \; c^T x$





A)

B)

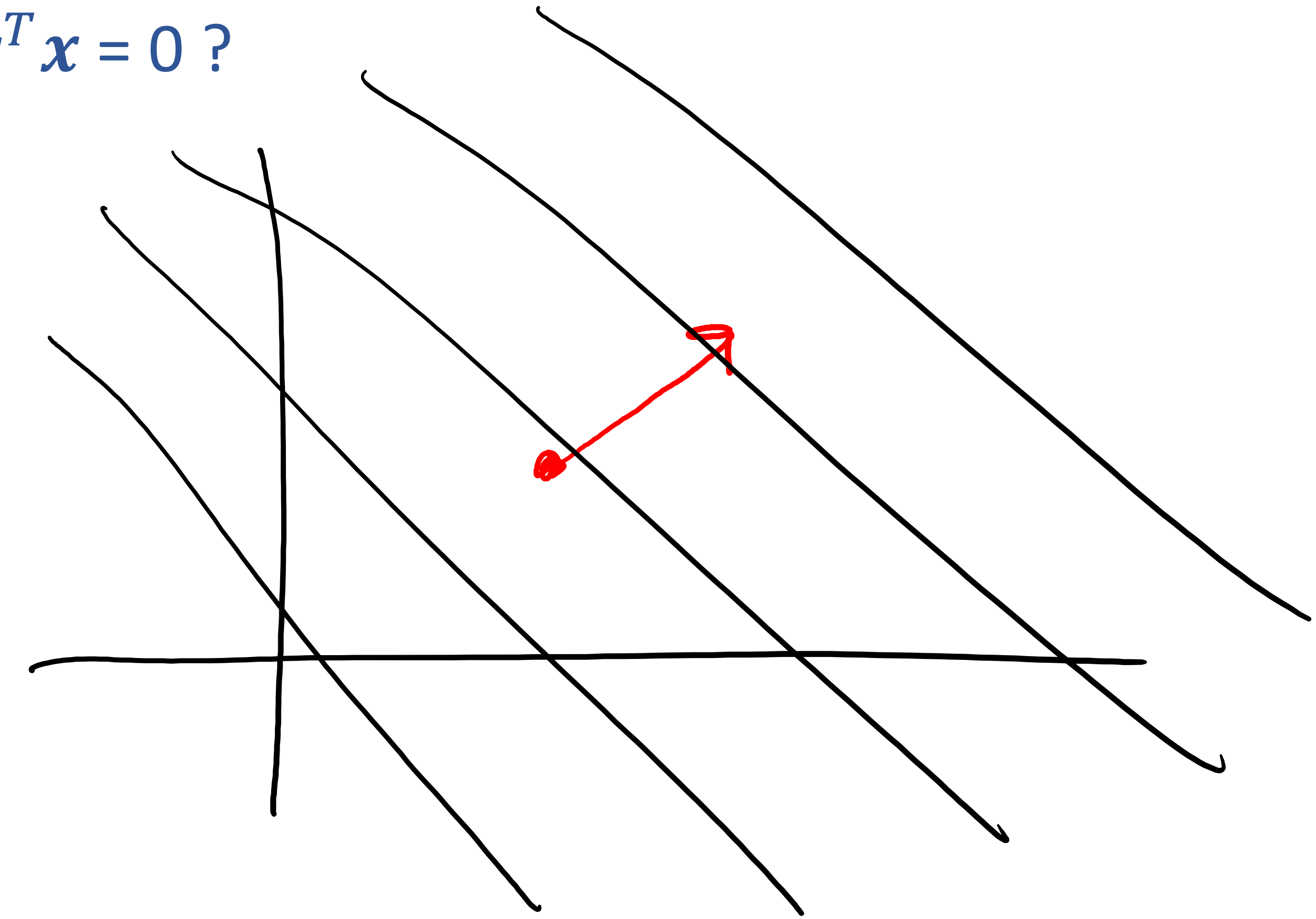
C)

D)

Notation Alert!

Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will $\boldsymbol{c}^T\boldsymbol{x} = 0$?



Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -2 ?

LP Graphical Representation

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

LP Graphical Representation

Inequality form, with no constraints

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$

LP Graphical Representation

Inequality form, with no constraints

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad a_1x_1 + a_2x_2 \leq b$$

Piazza Poll 5

True or False: An minimizing LP with exactly on constraint, will always have a minimum objective at $-\infty$.

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

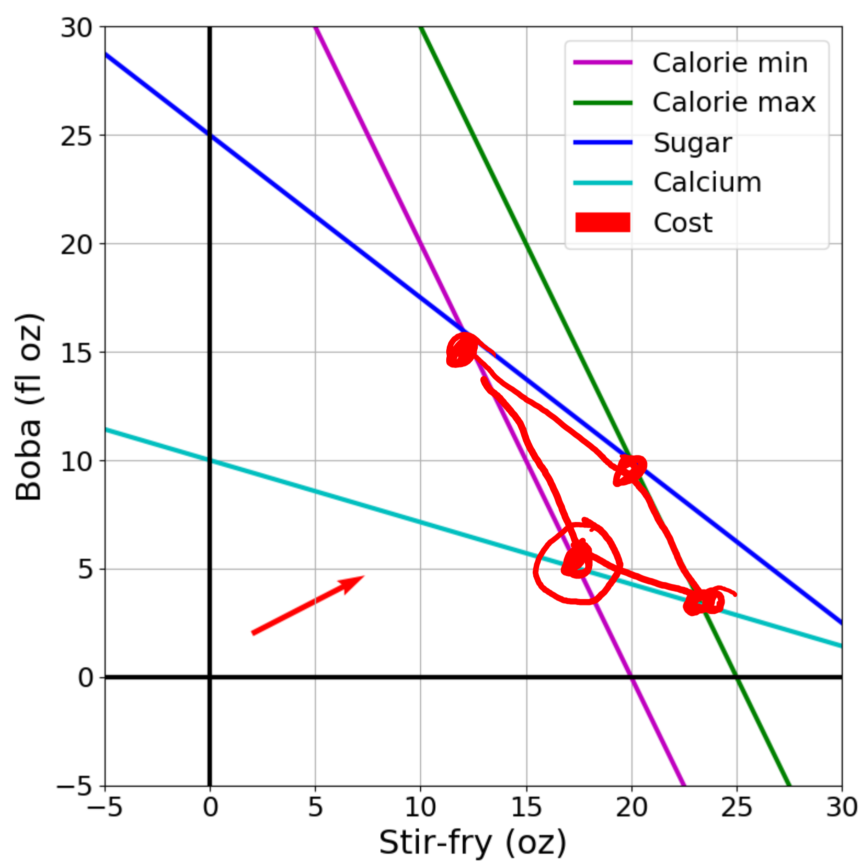
Optimization

Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

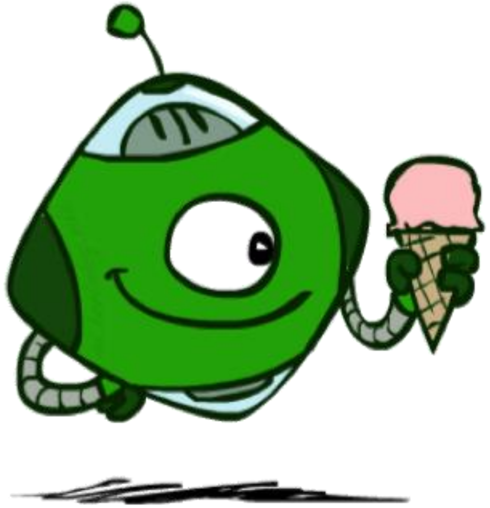
How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

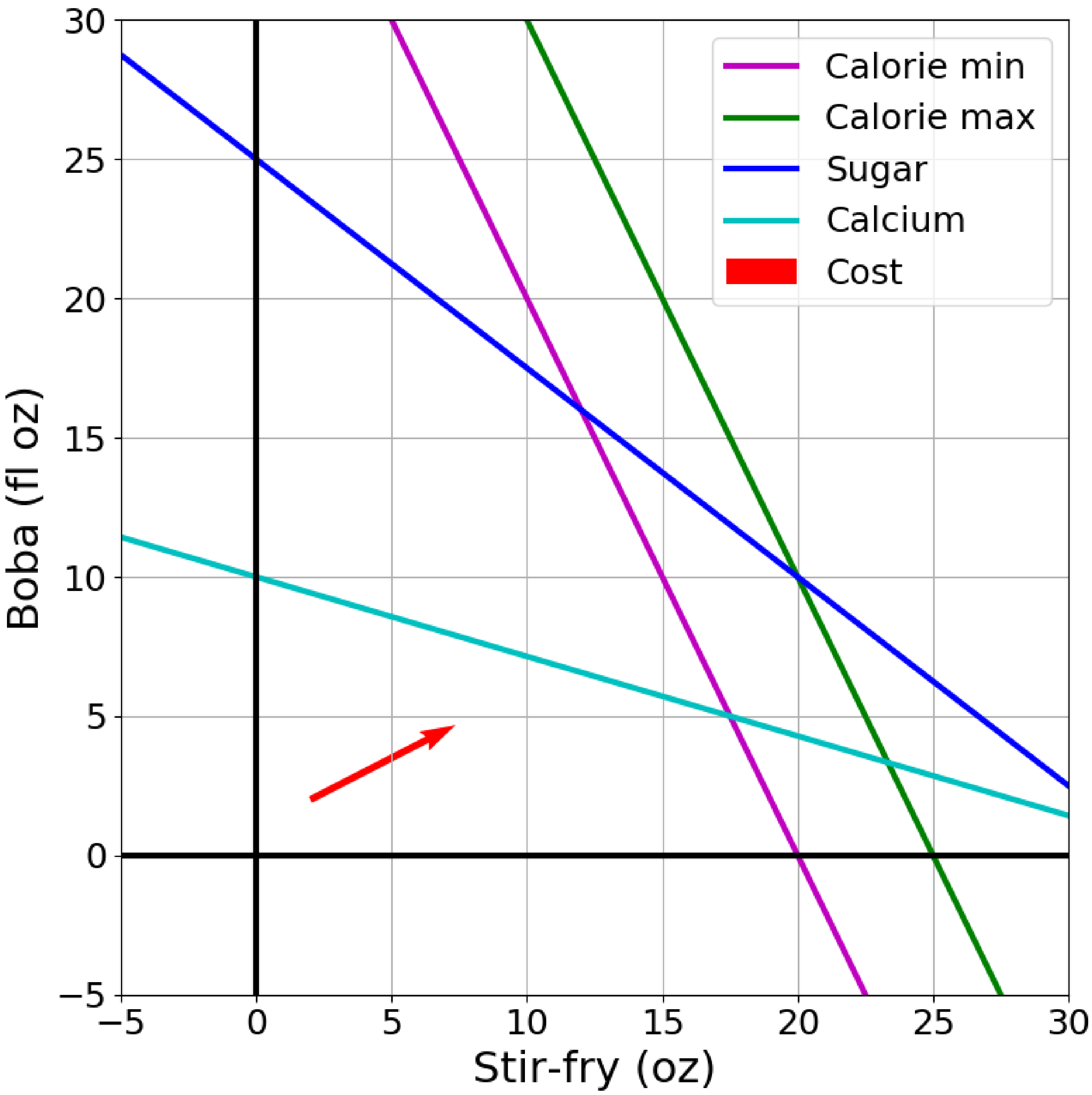
Stir-fry Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

Limit

$$\boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

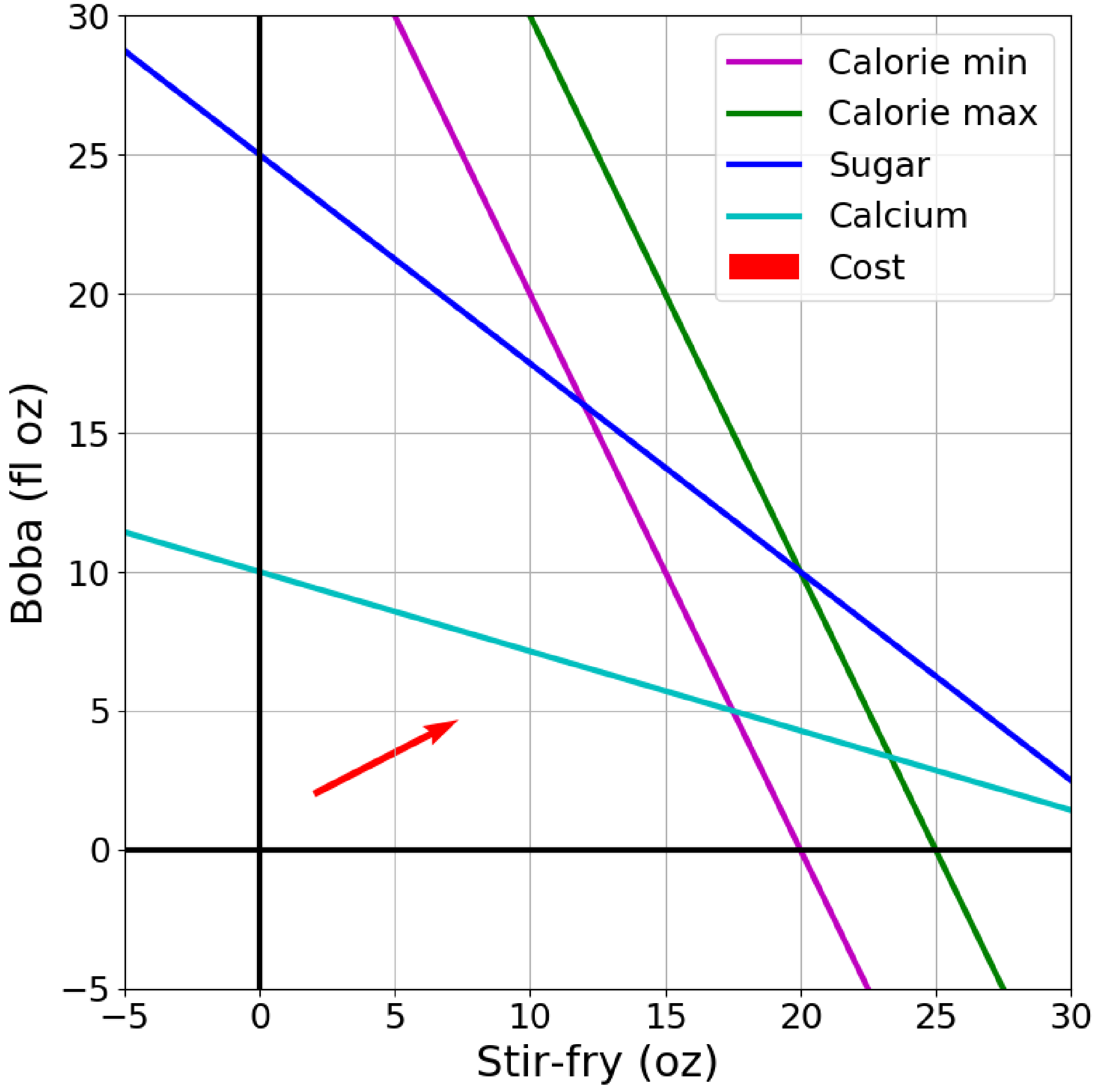
Calorie min
Calorie max
Sugar
Calcium



Calorie min
Calorie max
Sugar
Calcium
Cost
Boba (fl oz)
Stir-fry (oz)
30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30

Solving an LP

Solutions are at feasible intersections of constraint boundaries!!



Solving an LP

Solutions are at feasible intersections of constraint boundaries!!