Warm Up

How would you search for moves in Tic Tac Toe?

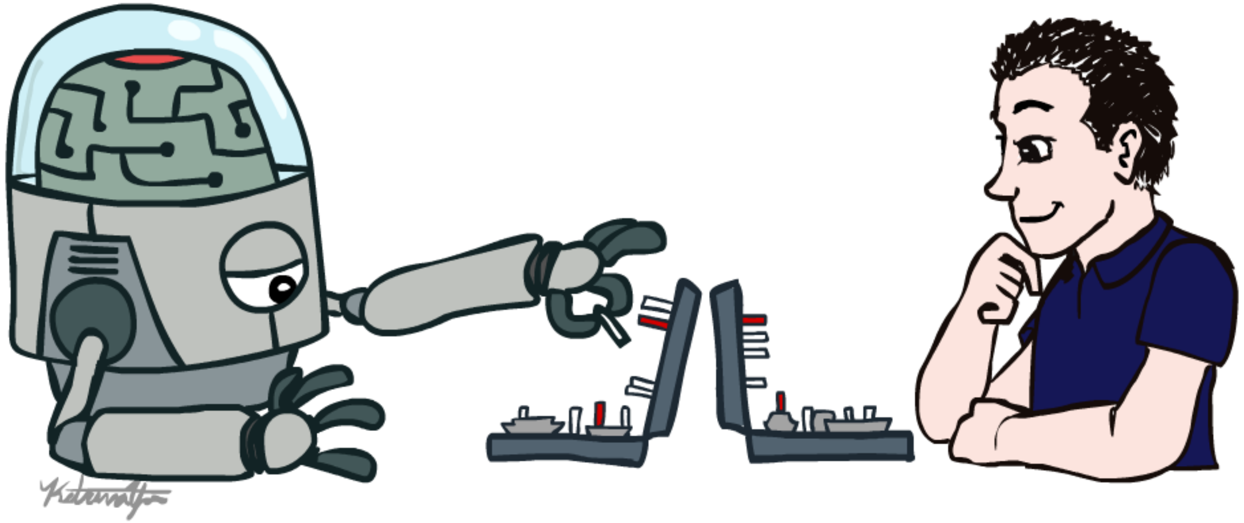
X	O	X
	O	X
	O	

X	O	X
O	O	X
X	X	O

X	O	X
	X	
X	O	O

AI: Representation and Problem Solving

Adversarial Search



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: Pat Virtue, http://ai.berkeley.edu

Announcements

- Homework 2 due tonight!
- Homework 3 out this evening!
- P1 due 2/7, work in pairs!

Warm Up

How would you search for moves in Tic Tac Toe?

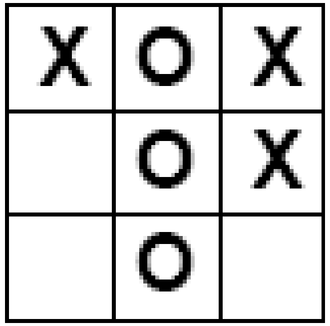
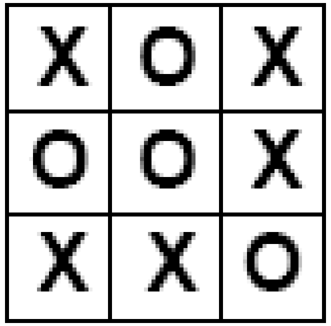
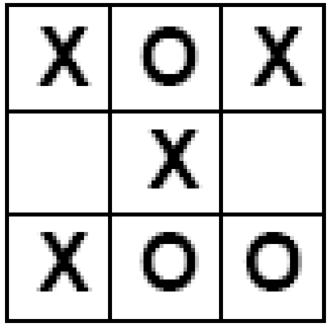
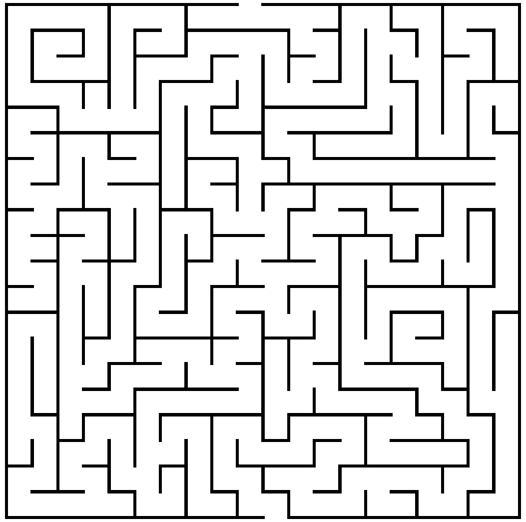
X	O	X
	O	X
	O	

X	O	X
O	O	X
X	X	O

X	O	X
	X	
X	O	O

Warm Up

How is Tic Tac Toe different from maze search?

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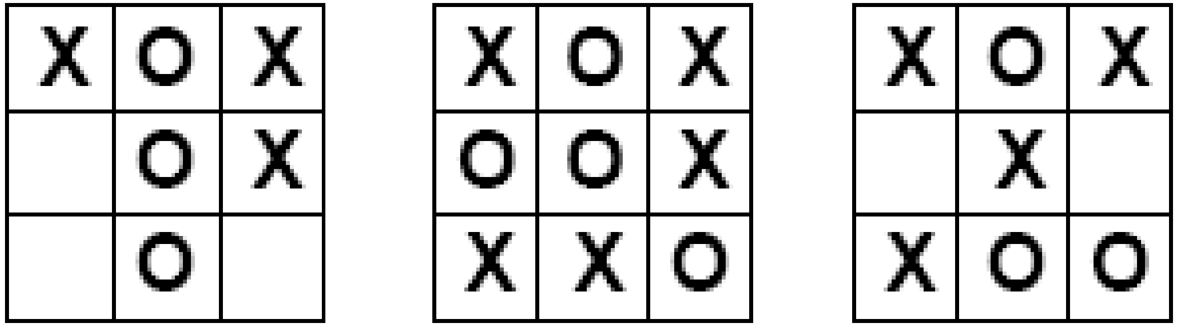
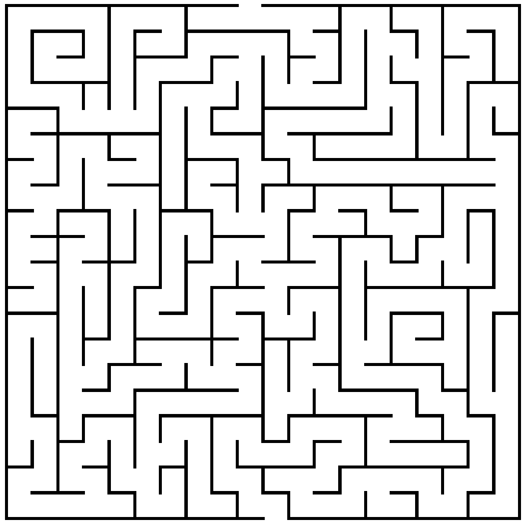
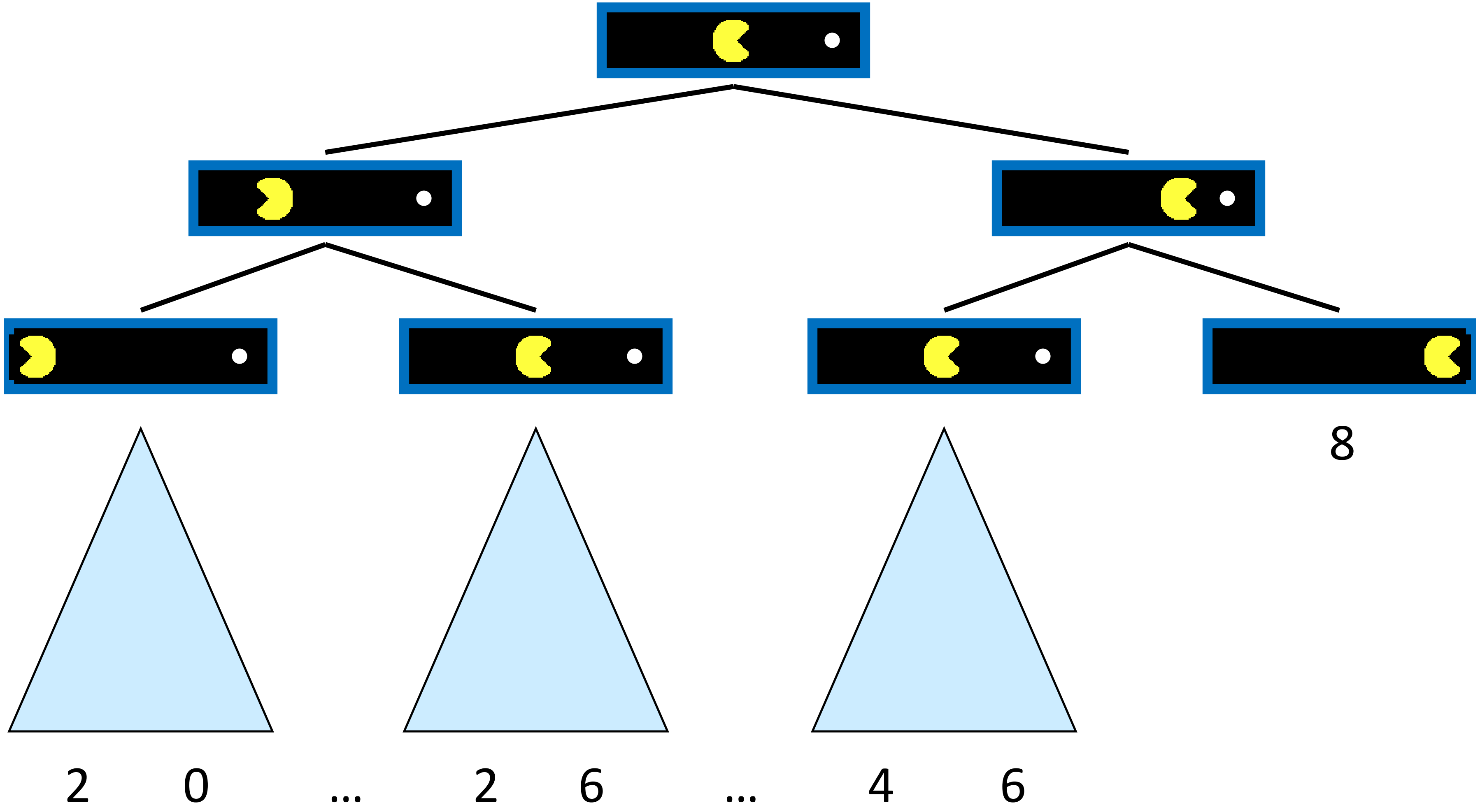


Multi-Agent, Adversarial, Zero Sum



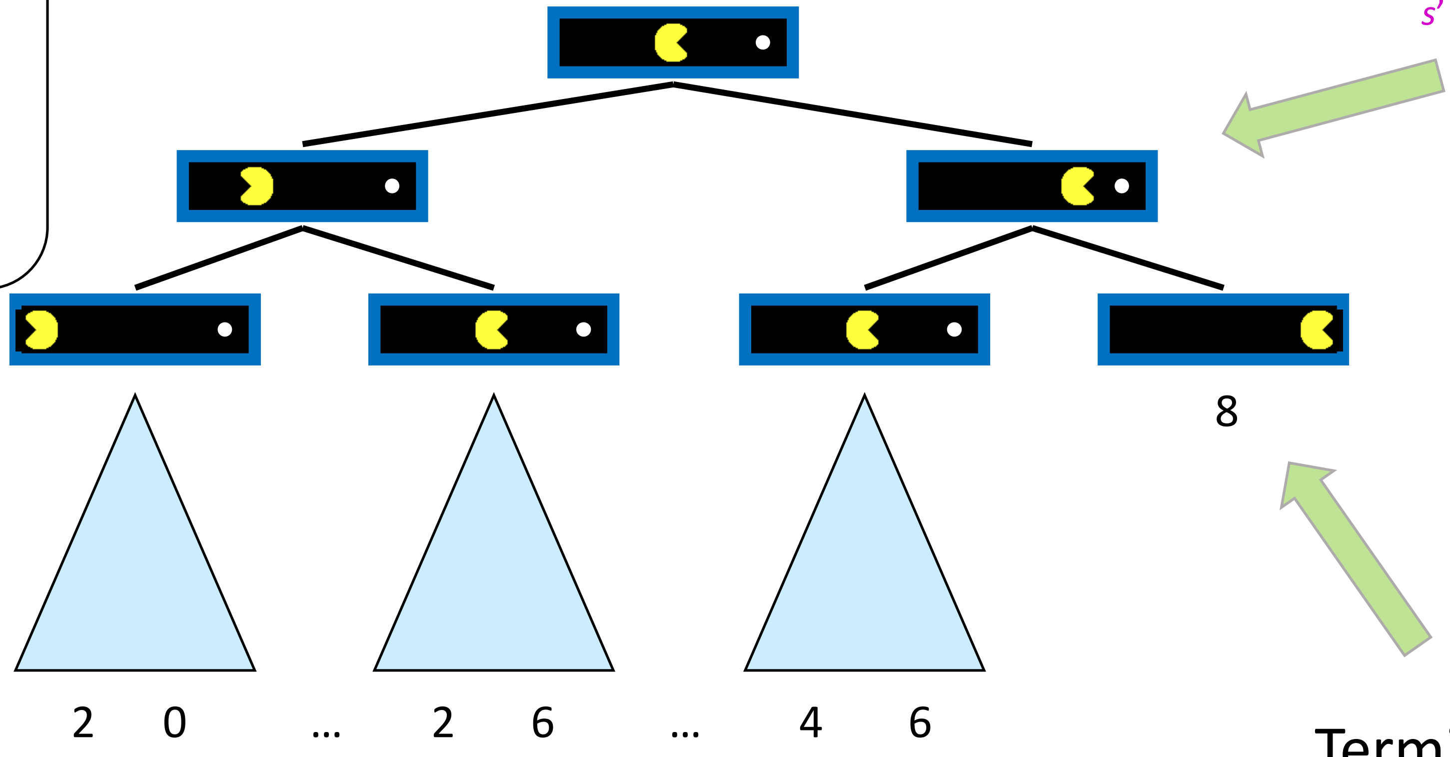
Single Agent

Single-Agent Trees



Value of a State

Value of a state:
The best achievable outcome (utility) from that state



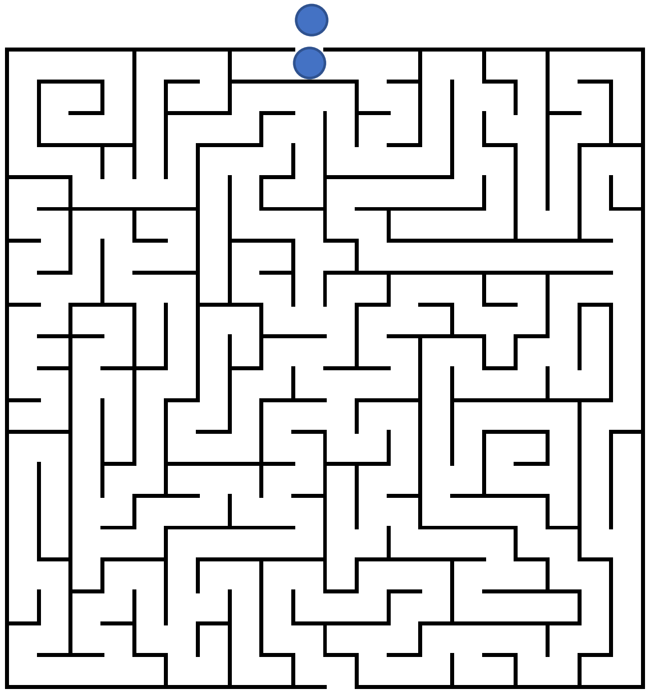
Non-Terminal States:

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

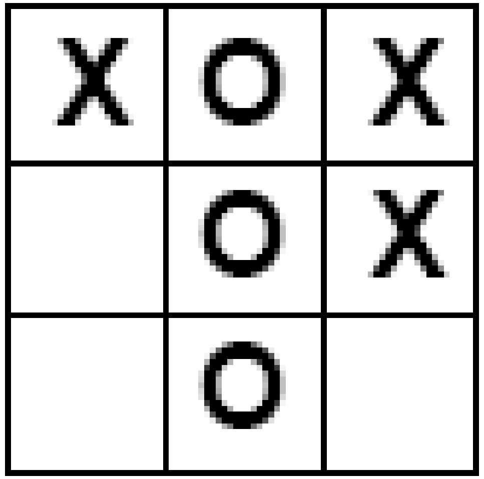
Terminal States:

$V(s)$ = known

Multi-Agent Applications



Collaborative Maze Solving

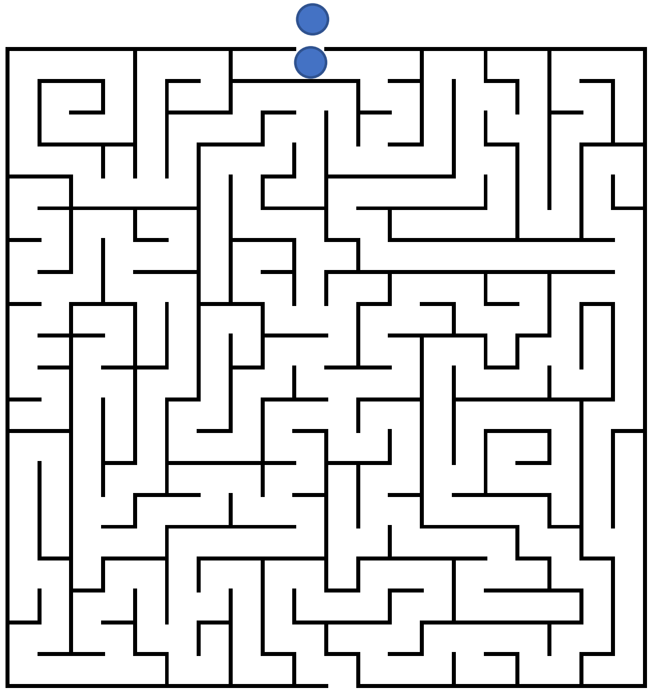


Adversarial

(Football)

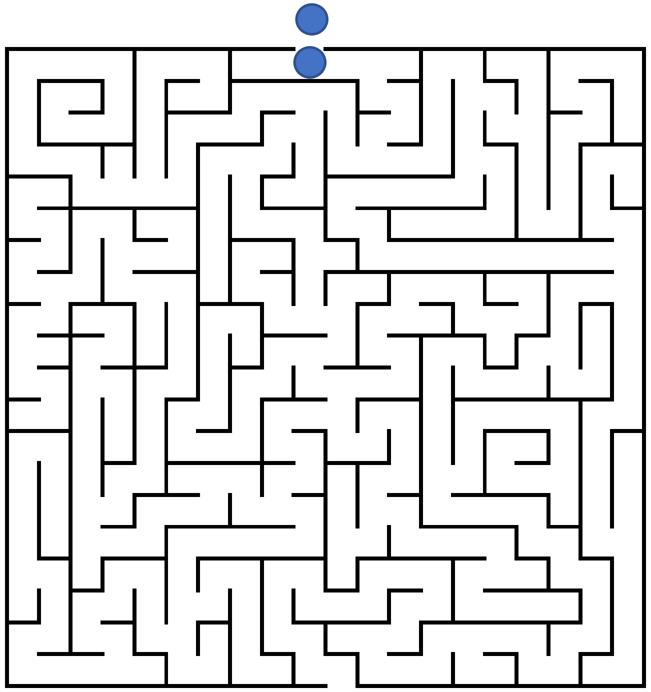
Team: Collaborative
Competition: Adversarial

How could we model multi-agent collaborative problems?



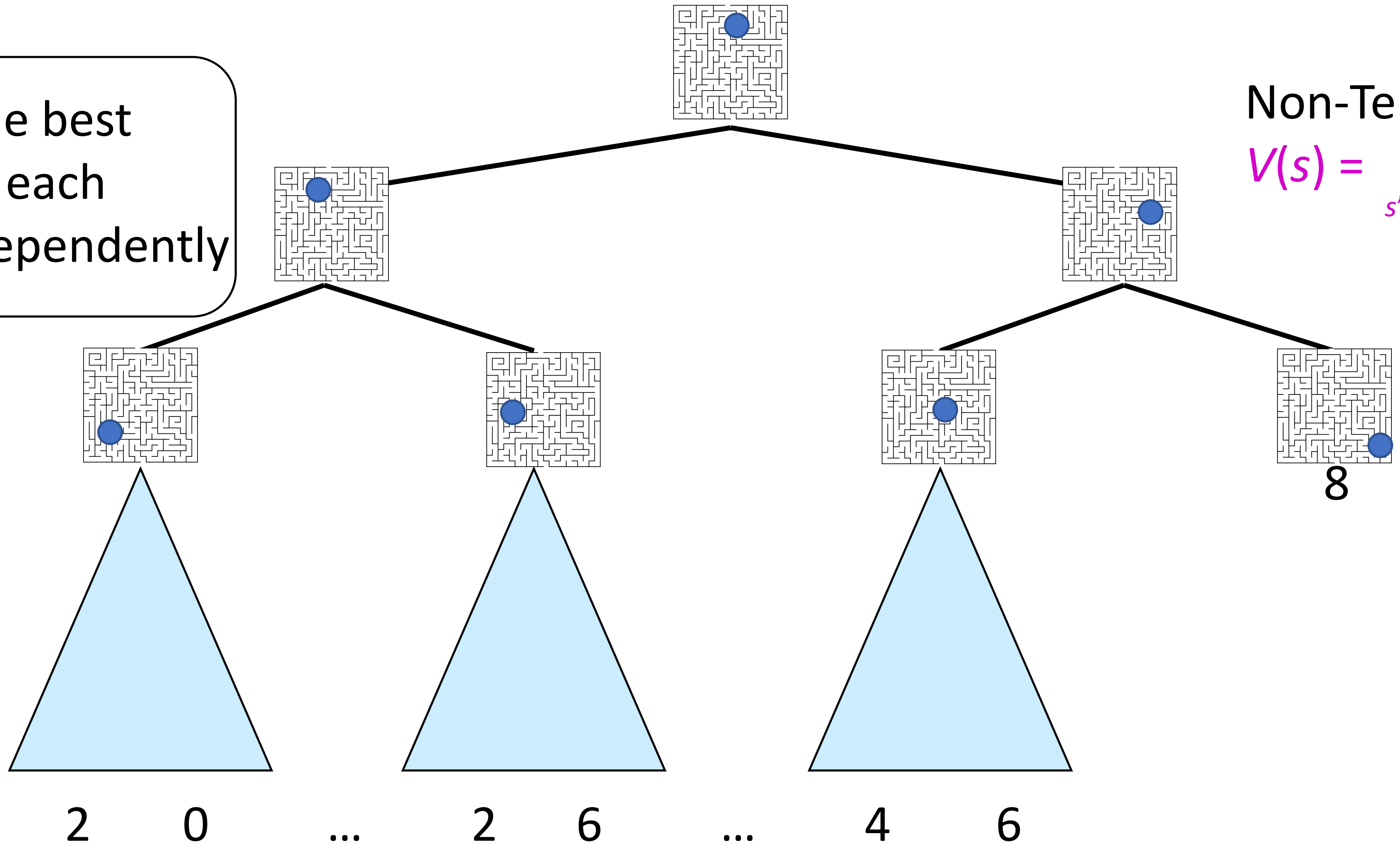
How could we model multi-agent problems?

Simplest idea: each agent plans their own actions separately from others.



Many Single-Agent Trees

Choose the best action for each agent independently



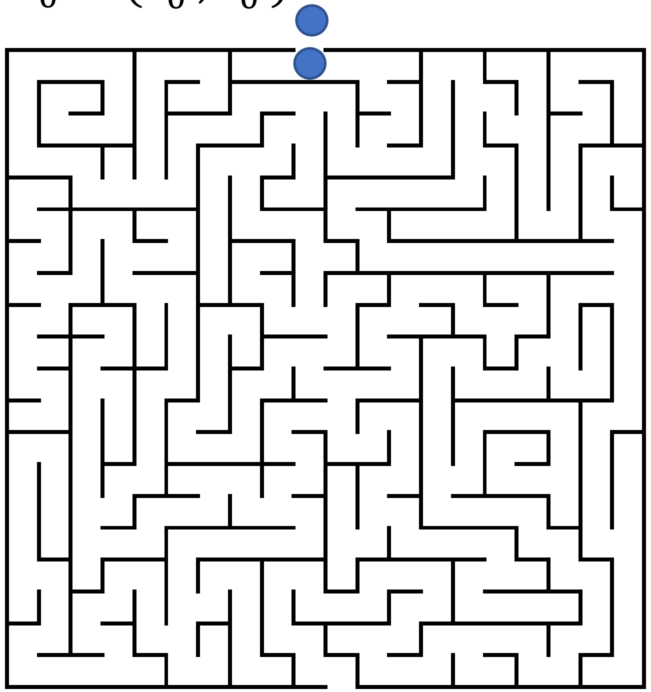
Non-Terminal States:

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

Idea 2: Joint State/Action Spaces

Combine the states and actions of the N agents

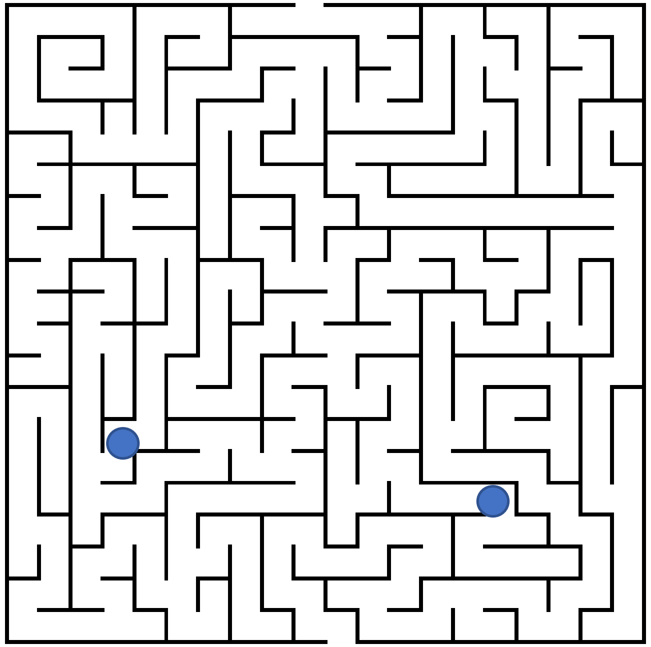
$S_0 = (S_0^A, S_0^B)$



Idea 2: Joint State/Action Spaces

Combine the states and actions of the N agents

$$S_K = (S_K^A, S_K^B)$$

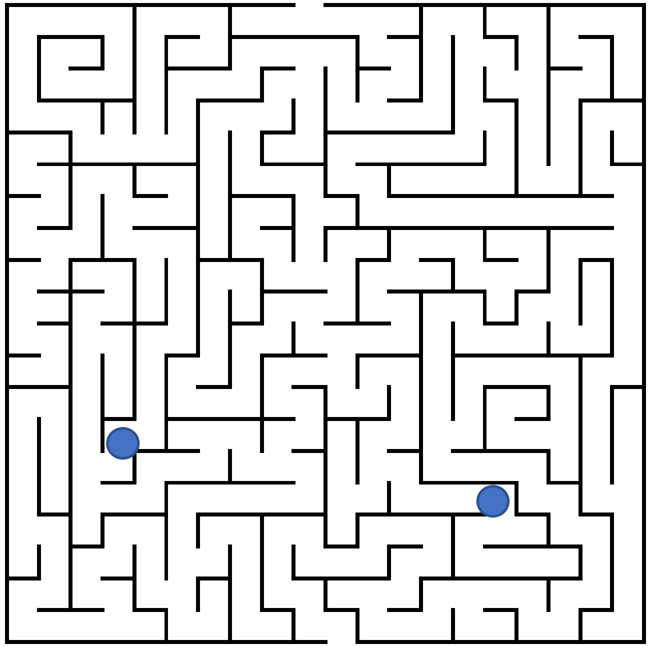


Idea 2: Joint State/Action Spaces

Search looks through all combinations of all agents' states and actions

Think of one brain controlling many agents

$$S_K = (S_K^A, S_K^B)$$



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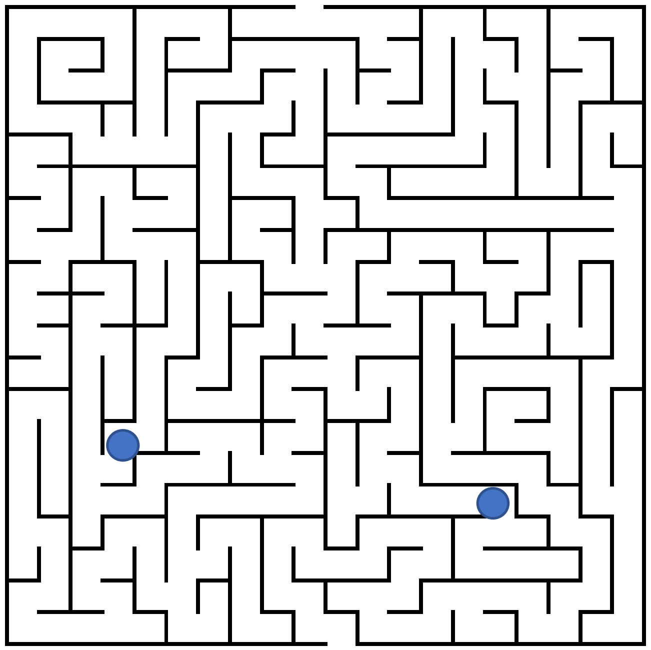
Think of one brain controlling many agents

What is the size of the state space?



What is the size of the action space?

What is the size of the search tree?

Idea 3: Centralized Decision Making

Each agent proposes their actions and computer confirms the joint plan

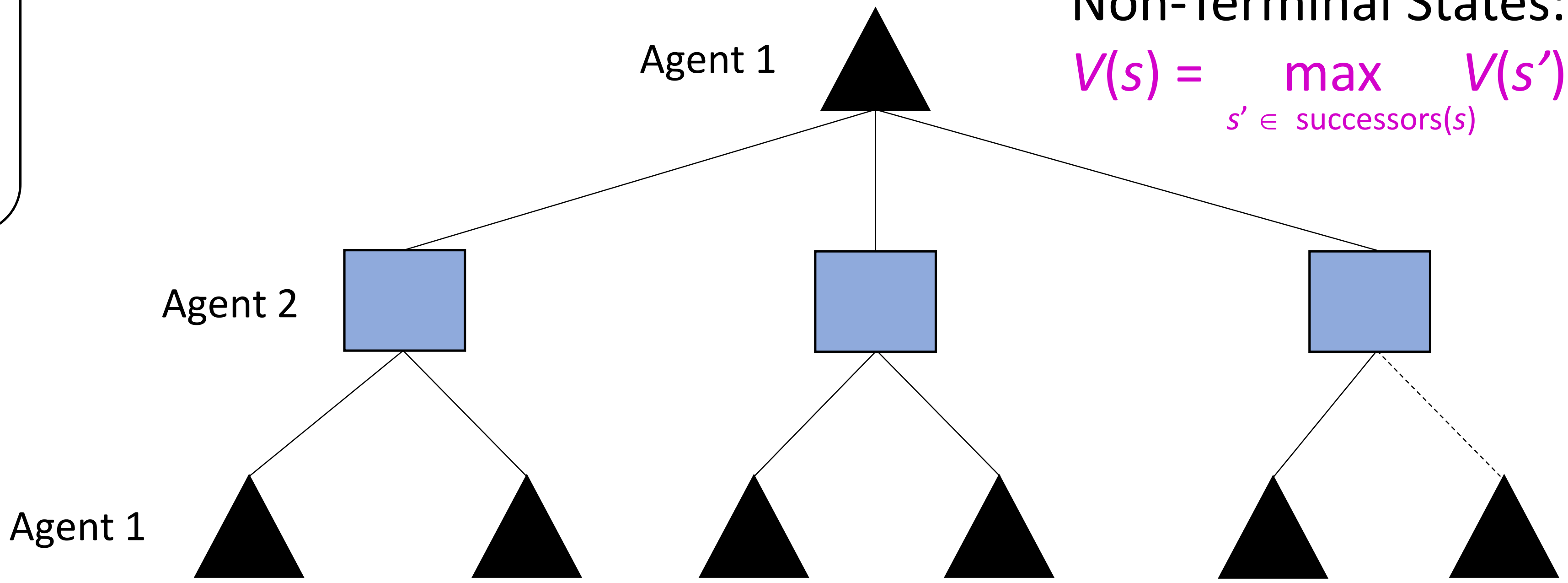
Example: Autonomous driving through intersections

https://www.youtube.com/watch?v=4pbAI40dK0A

Idea 4: Alternate Searching One Agent at a Time

Search one agent's actions from a state, search the next agent's actions from those resulting states , etc…

Choose the best cascading combination of actions



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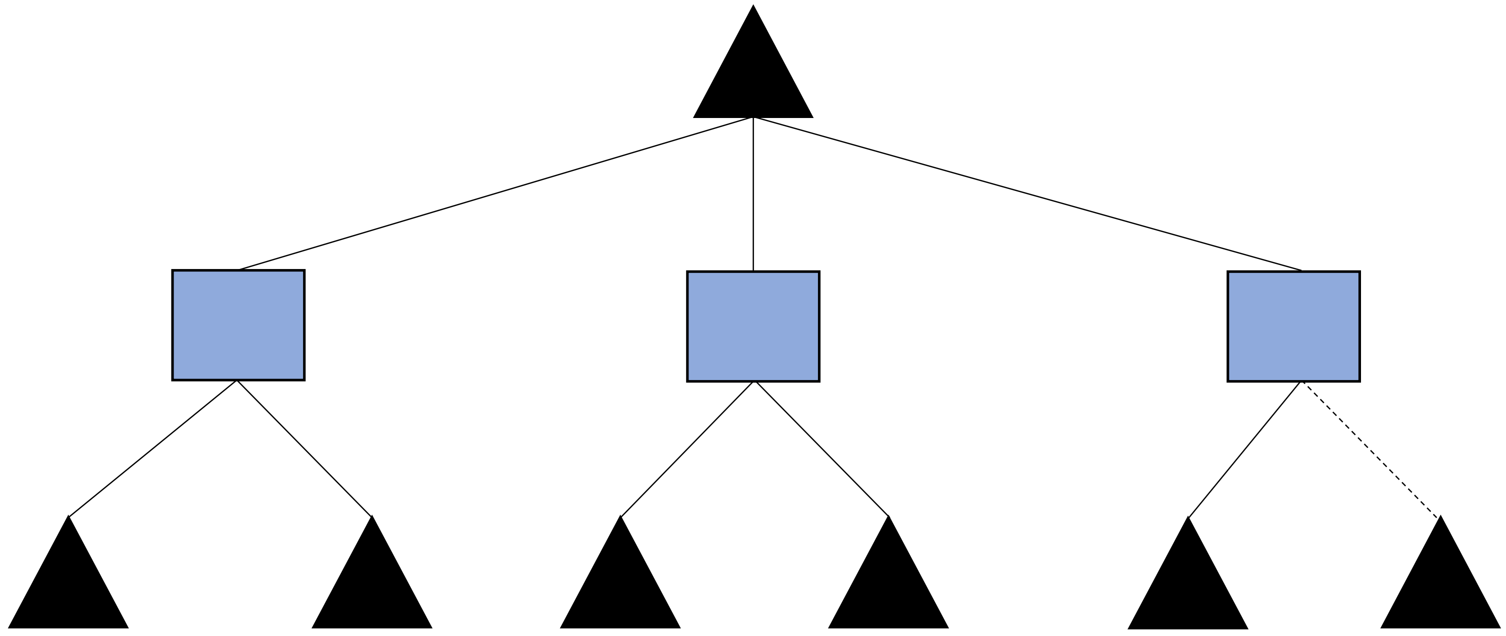
Idea 4: Alternate Searching One Agent at a Time

Search one agent’s actions from a state, search the next agent’s actions from those resulting states , etc…

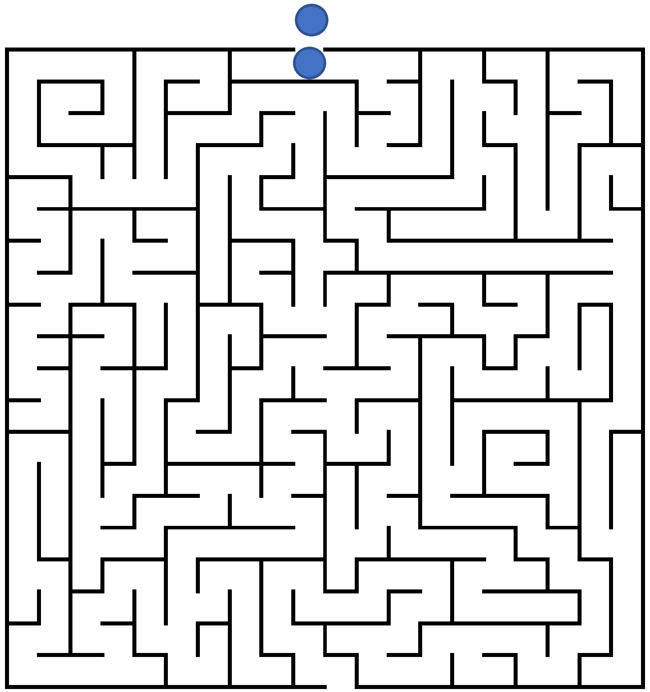
What is the size of the state space?

What is the size of the action space?

What is the size of the search tree?



Multi-Agent Applications



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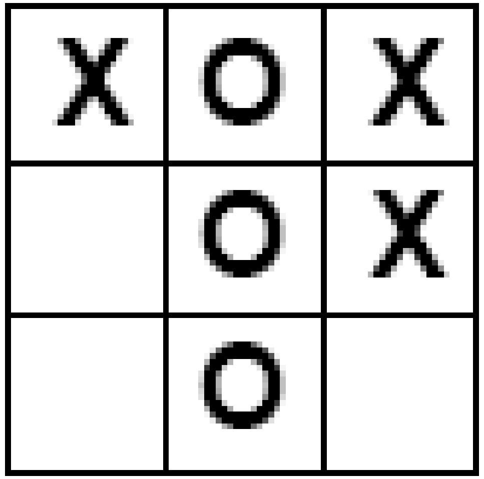


Adversarial

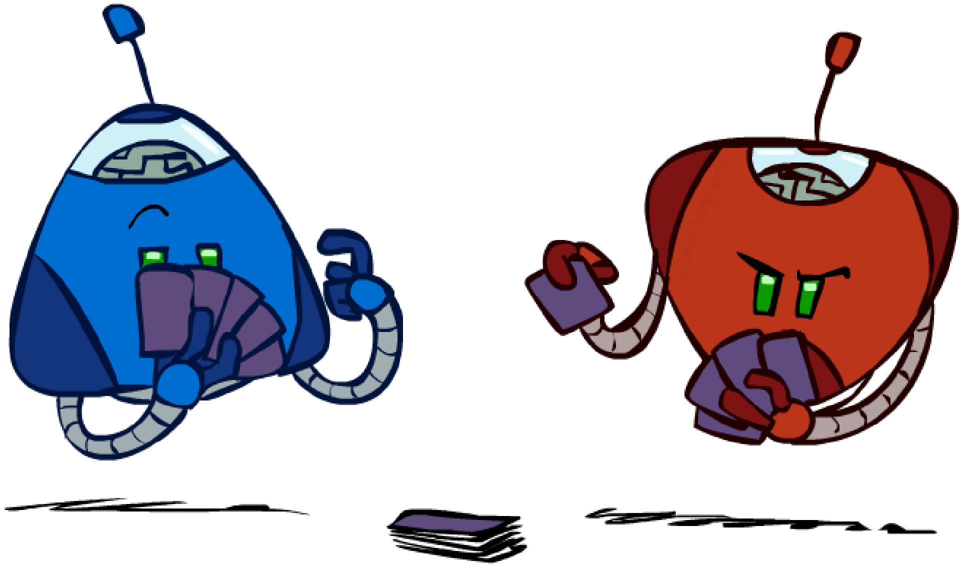
(Football)

Team: Collaborative
Competition: Adversarial

Games

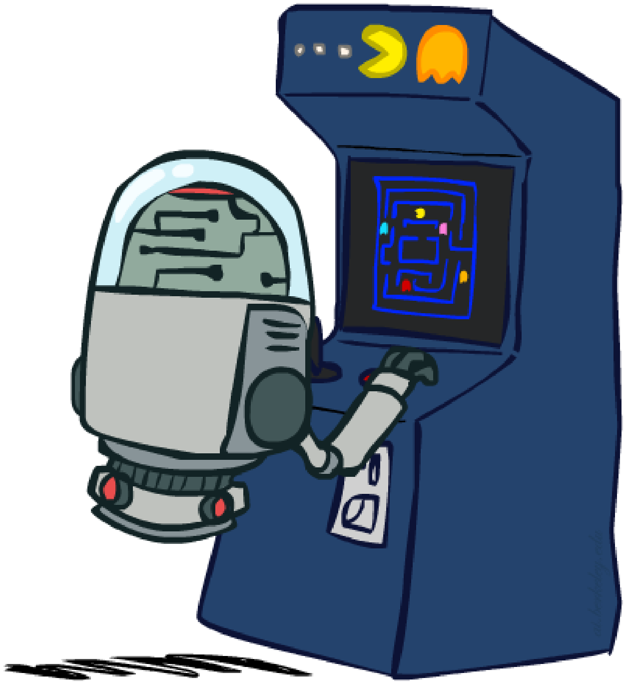
Types of Games

- Deterministic or stochastic?
- Perfect information (fully observable)?
- One, two, or more players?
- Turn-taking or simultaneous?
- Zero sum?

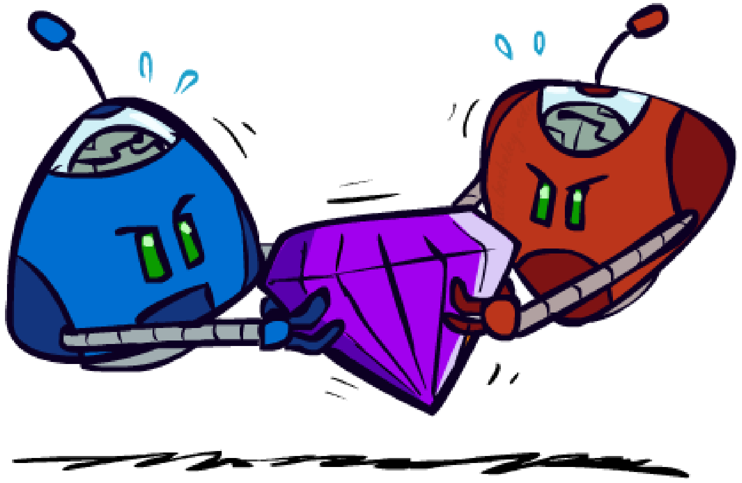


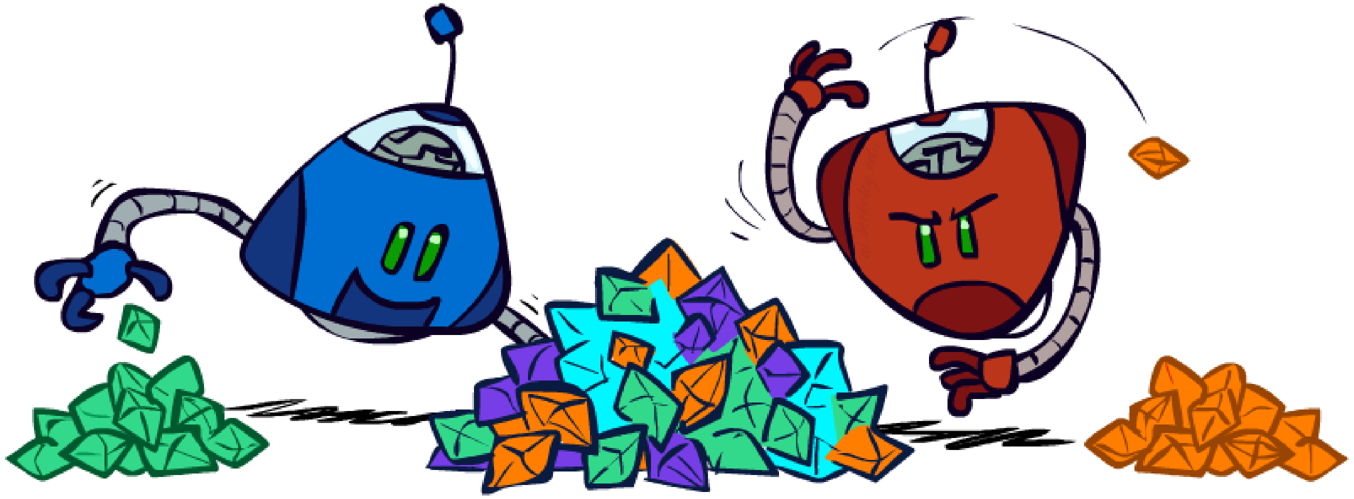
Standard Games

- Standard games are deterministic, observable, two-player, turn-taking, zero-sum
- Game formulation:
  - Initial state: s_0
  - Players: Player(s) indicates whose move it is
  - Actions: Actions(s) for player on move
  - Transition model: Result(s,a)
  - Terminal test: Terminal-Test(s)
  - Terminal values: Utility(s,p) for player p
    - Or just Utility(s) for player making the decision at root



Zero-Sum Games





Zero-Sum Games

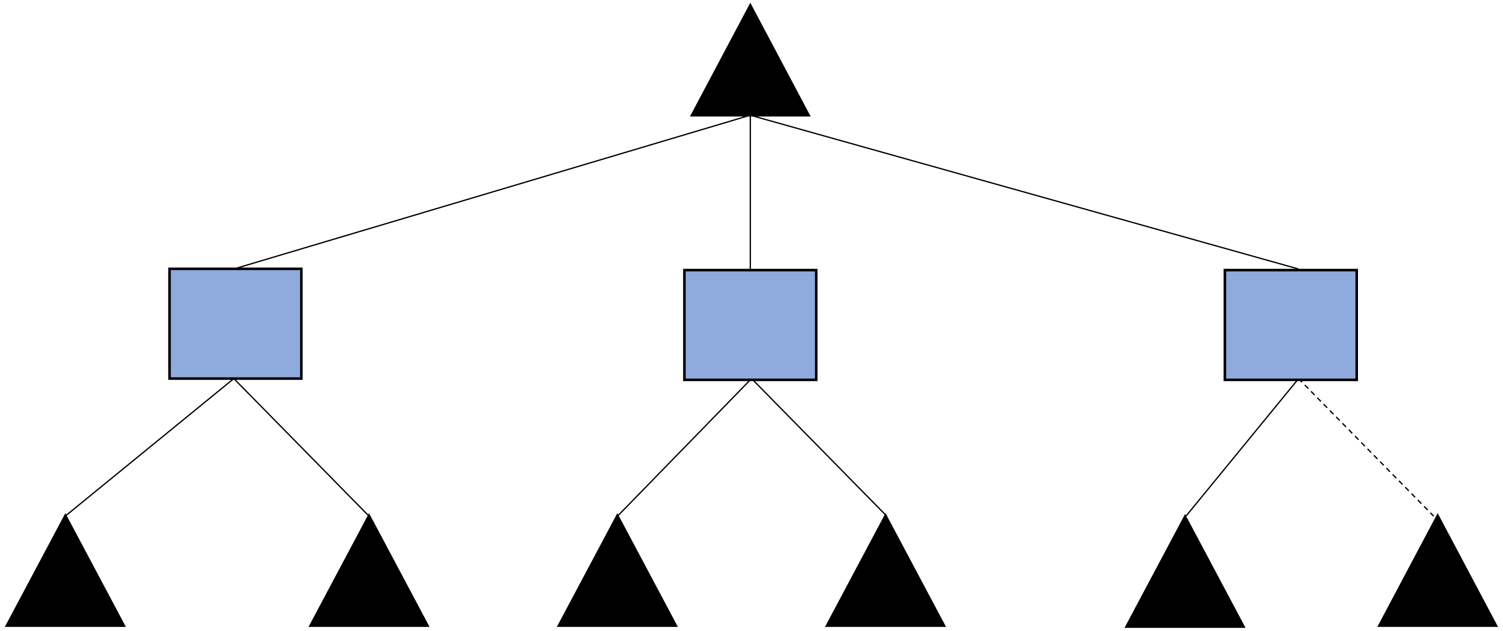
- Agents have ***opposite*** utilities
- Pure competition:
  - One ***maximizes***, the other ***minimizes***

General Games

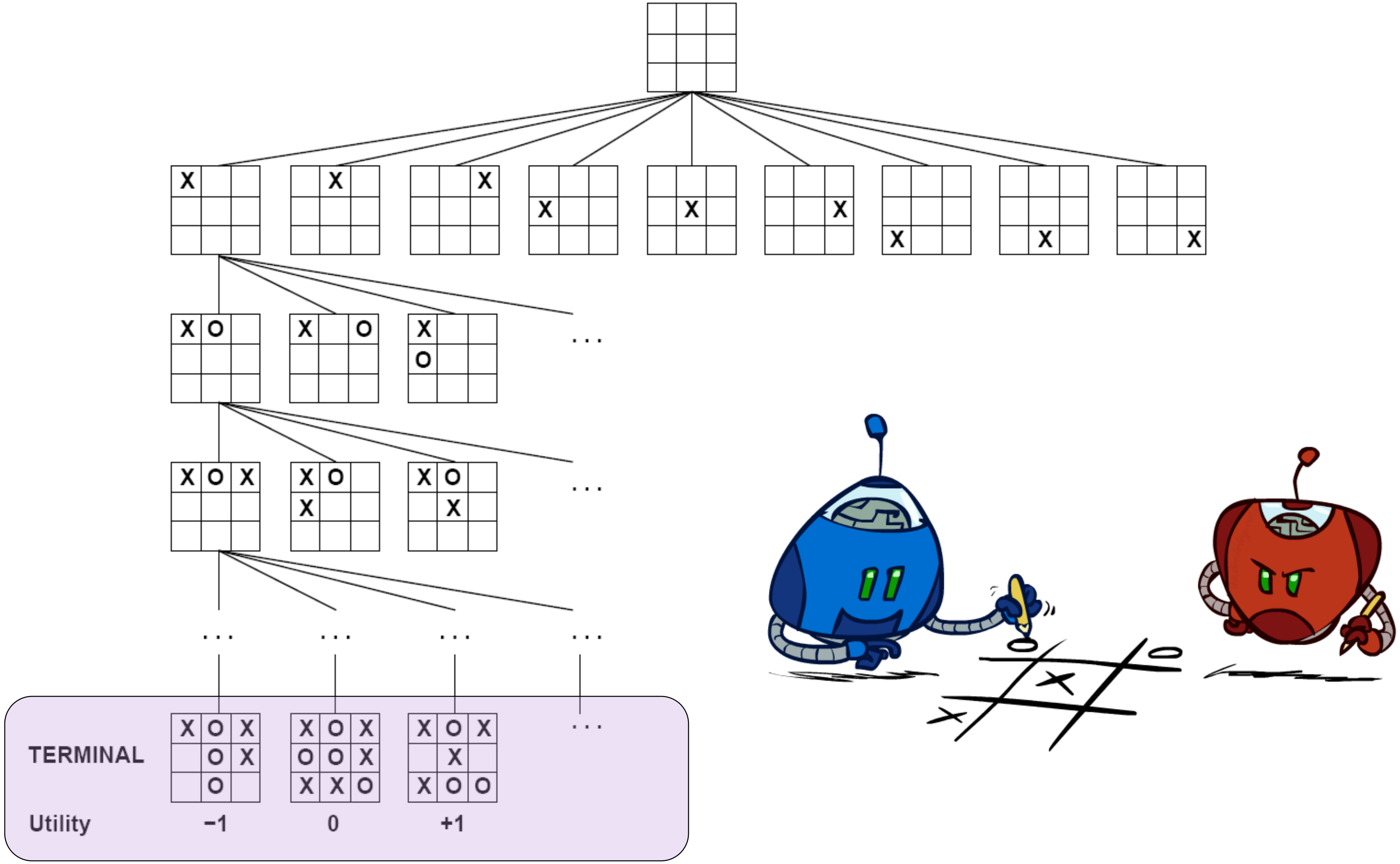
- Agents have ***independent*** utilities
- Cooperation, indifference, competition, shifting alliances, and more are all possible

Game Trees

Search one agent’s actions from a state, search the competitor’s actions from those resulting states , etc…



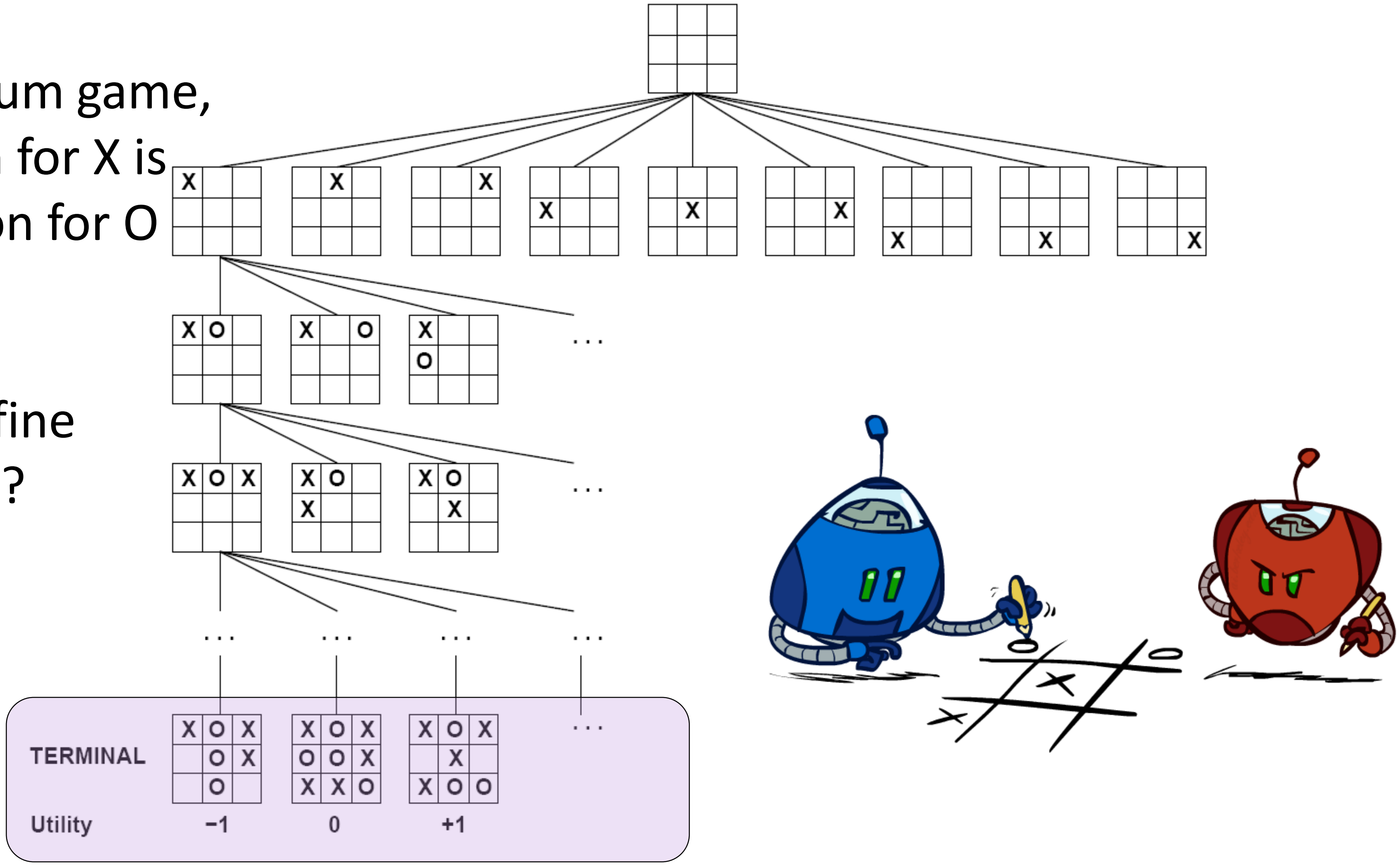
Tic-Tac-Toe Game Tree



Tic-Tac-Toe Game Tree

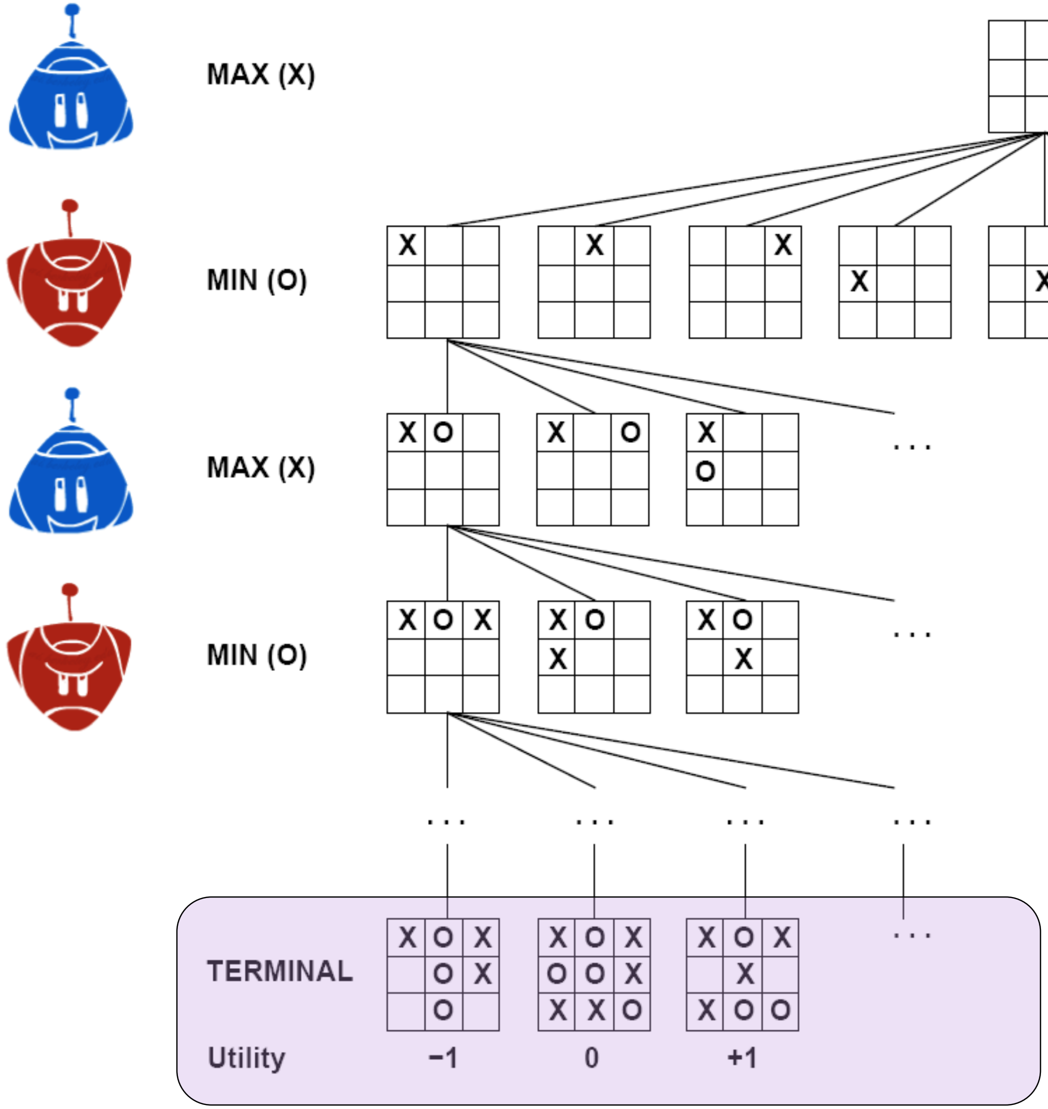
This is a zero-sum game, the best action for X is the worst action for O and vice versa

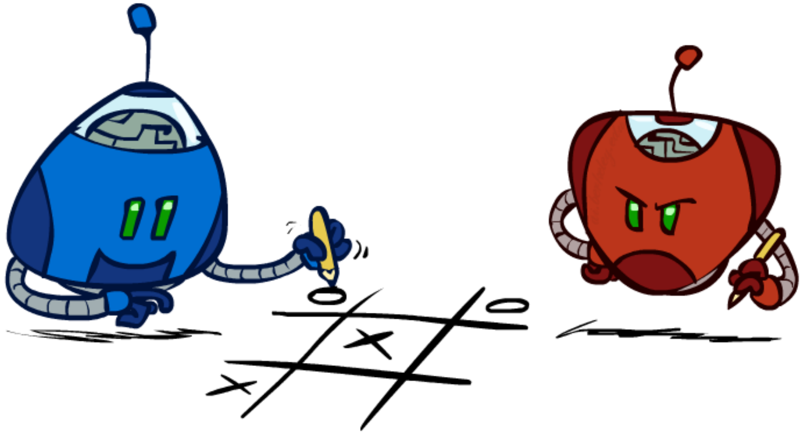
How do we define best and worst?



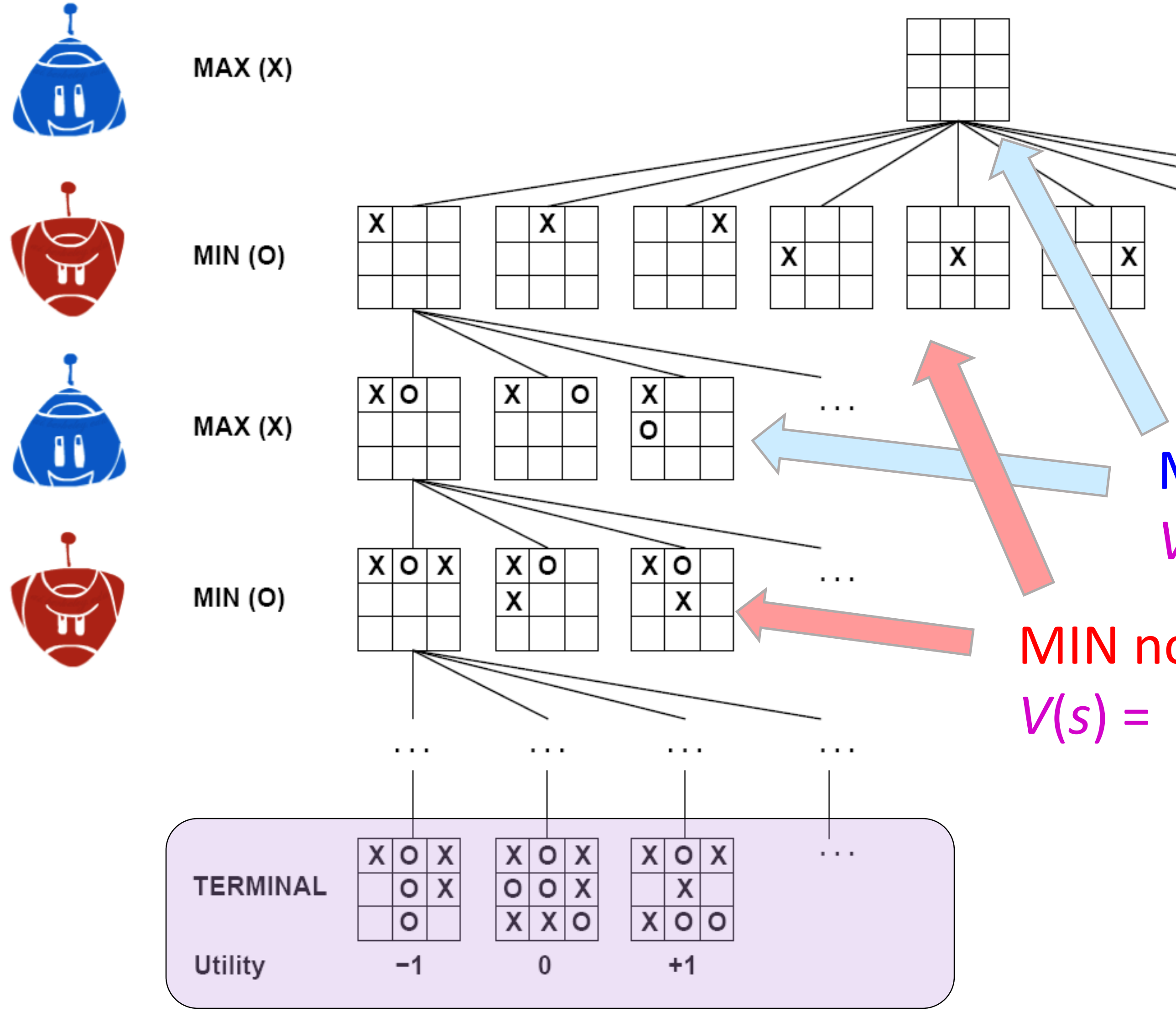
Tic-Tac-Toe Game Tree

Instead of taking the max utility at every level, alternate max and min





Tic-Tac-Toe Minimax



MAX nodes: under Agent's control

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

MIN nodes: under Opponent's control

$$V(s) = \min_{s' \in \text{successors}(s)} V(s')$$

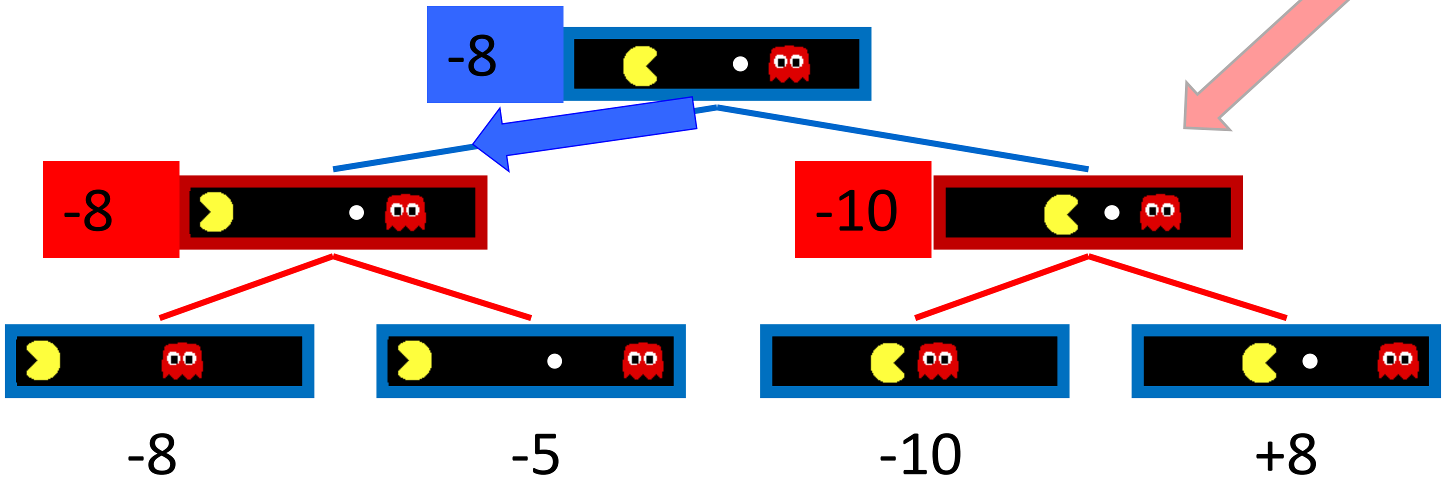
Small Pacman Example

MAX nodes: under Agent's control

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

MIN nodes: under Opponent's control

$$V(s) = \min_{s' \in \text{successors}(s)} V(s')$$



Terminal States:

$V(s)$ = known

Minimax Implementation

function minimax-decision(s) returns action

return the action a in Actions(s) with the highest min-value(Result(s,a))

Result(s,a)→s'

```
function max-value(s) returns value
    if Terminal-Test(s) then return Utility(s)
    initialize v = -∞
    for each a in Actions(s):
        v = max(v, min-value(Result(s,a)))
    return v
```

```
function min-value(s) returns value
    if Terminal-Test(s) then return Utility(s)
    initialize v = +∞
    for each a in Actions(state):
        v = min(v, max-value(Result(s,a)))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s) = \min_{s' \in \text{successors}(s)} V(s')$$

Alternative Implementation

function minimax-decision(s) returns an action

return the action a in Actions(s) with the highest value(Result(s,a))

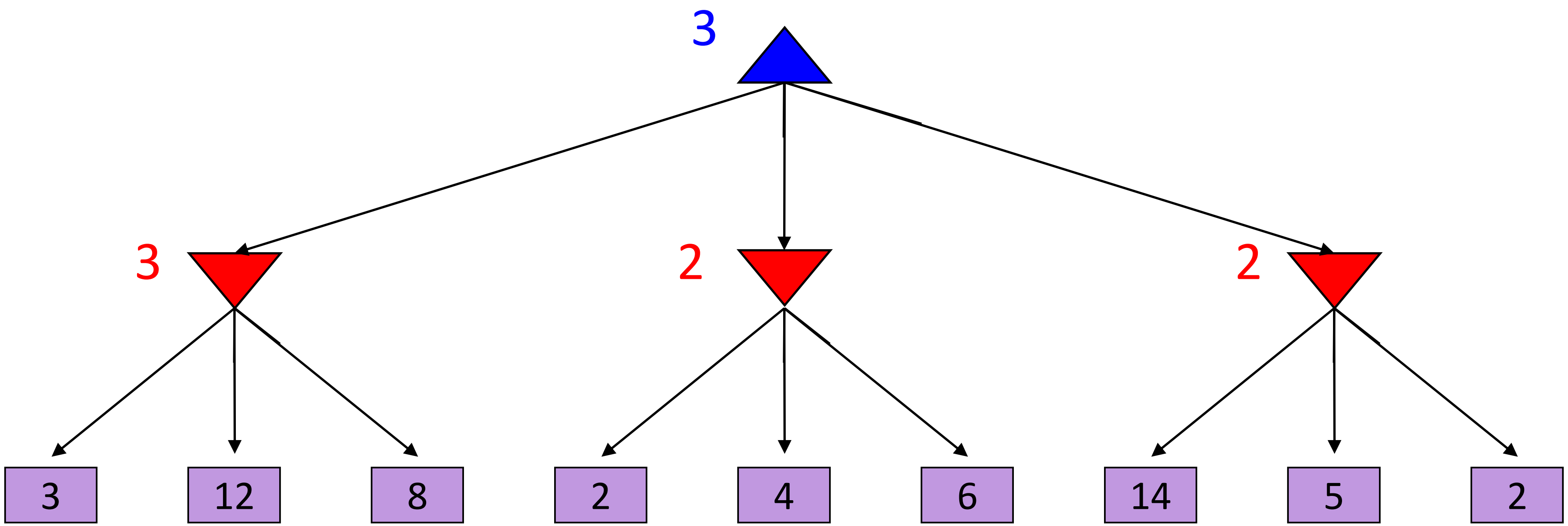
function value(s) returns a value

if Terminal-Test(s) then return Utility(s)

if Player(s) = MAX then return $\max_{a \text{ in Actions}(s)}$ value(Result(s,a))

if Player(s) = MIN then return $\min_{a \text{ in Actions}(s)}$ value(Result(s,a))

Minimax Example



Poll

What kind of search is Minimax Search?

A) BFS
B) DFS
C) UCS
D) A*

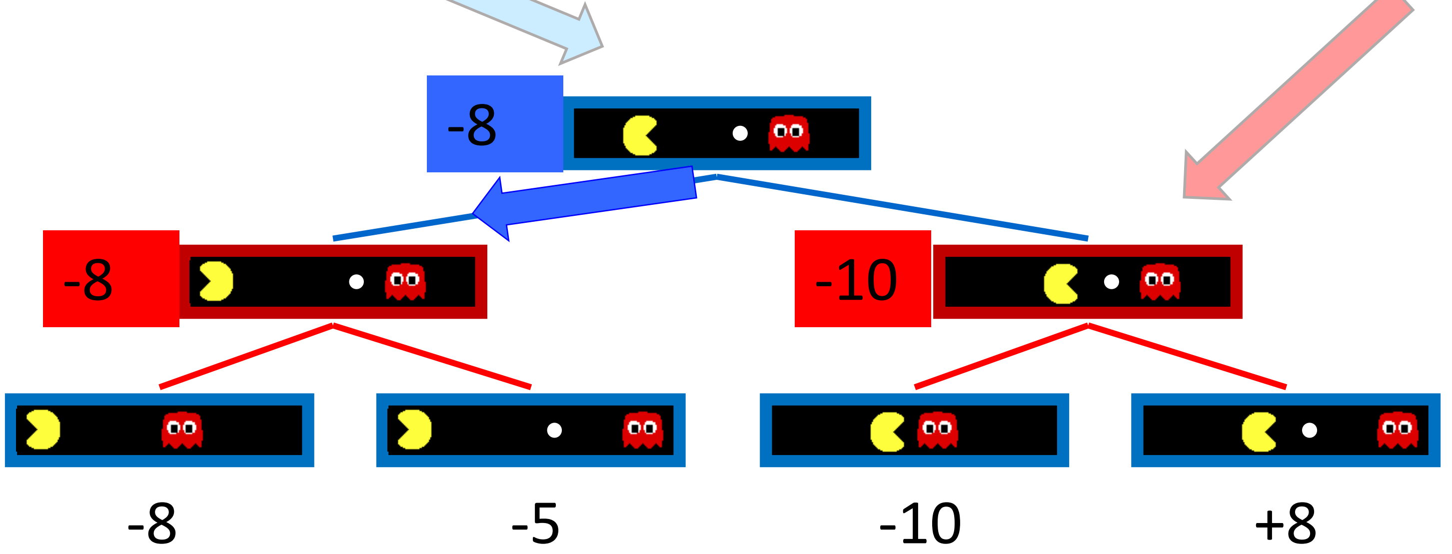
Minimax is Depth-First Search

MAX nodes: under Agent's control

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

MIN nodes: under Opponent's control

$$V(s) = \min_{s' \in \text{successors}(s)} V(s')$$



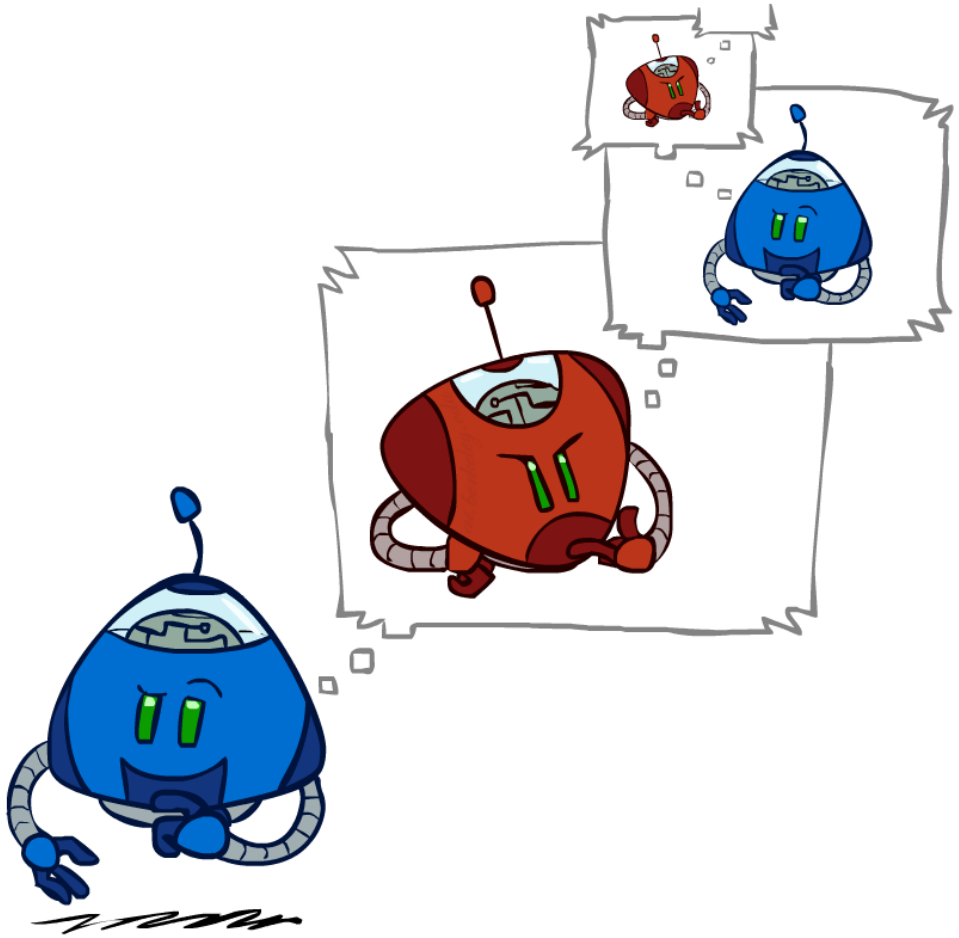
Terminal States:

$V(s)$ = known

Minimax Efficiency

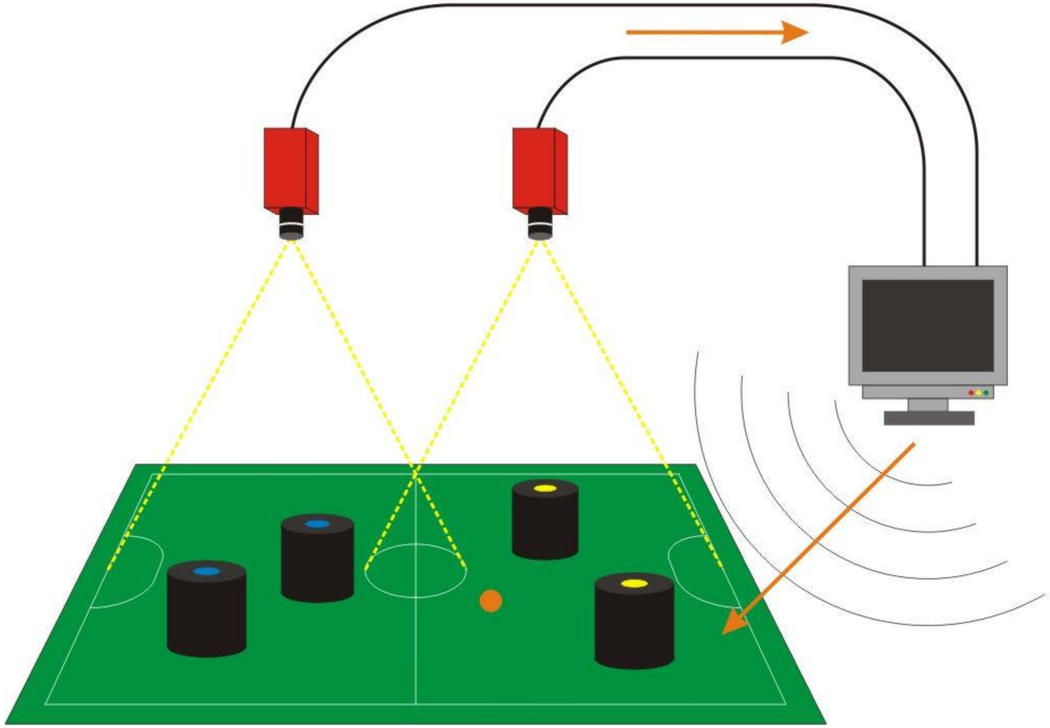
- How efficient is minimax?
  - Just like (exhaustive) DFS
  - Time: $O(b^m)$
  - Space: $O(bm)$

- Example: For chess, $b \approx 35$, $m \approx 100$
  - Exact solution is completely infeasible
  - Humans can't do this either, so how do we play chess?



Small Size Robot Soccer

- Joint State/Action space and search for our team
- Adversarial search to predict the opponent team

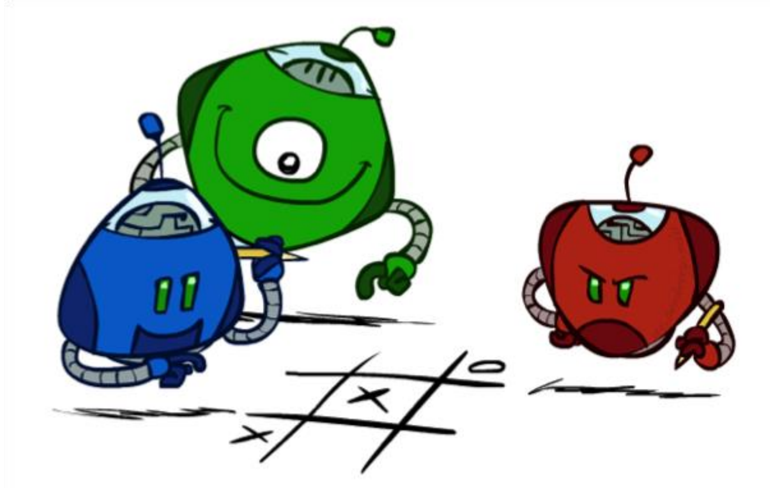


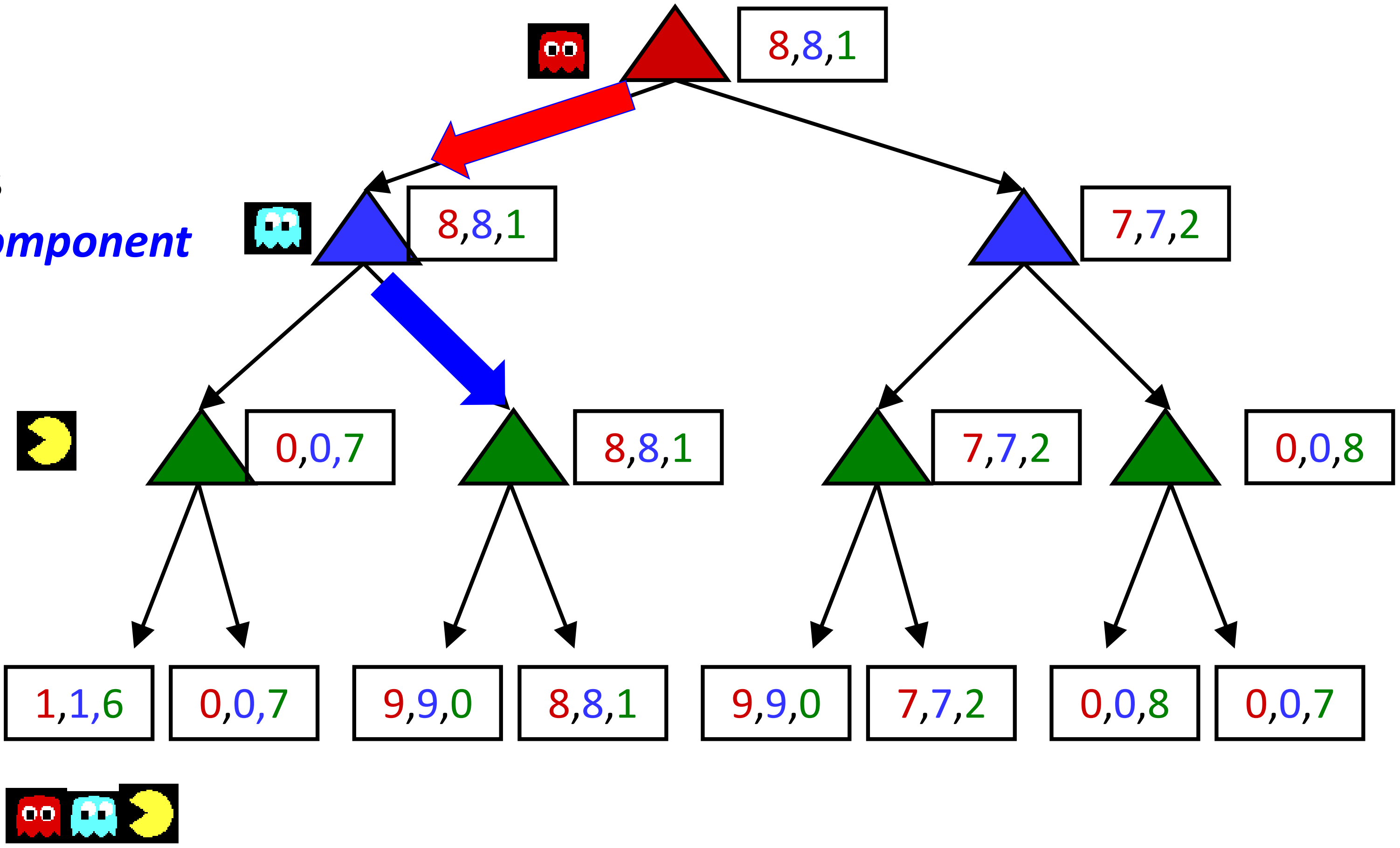
https://www.youtube.com/watch?v=YihJguq26ek

Generalized minimax

- What if the game is not zero-sum, or has multiple players?

- Generalization of minimax:
  - Terminals have ***utility tuples***
  - Node values are also utility tuples
  - ***Each player maximizes its own component***
  - Can give rise to cooperation and competition dynamically…





Three Person Chess

Resource Limits

Resource Limits

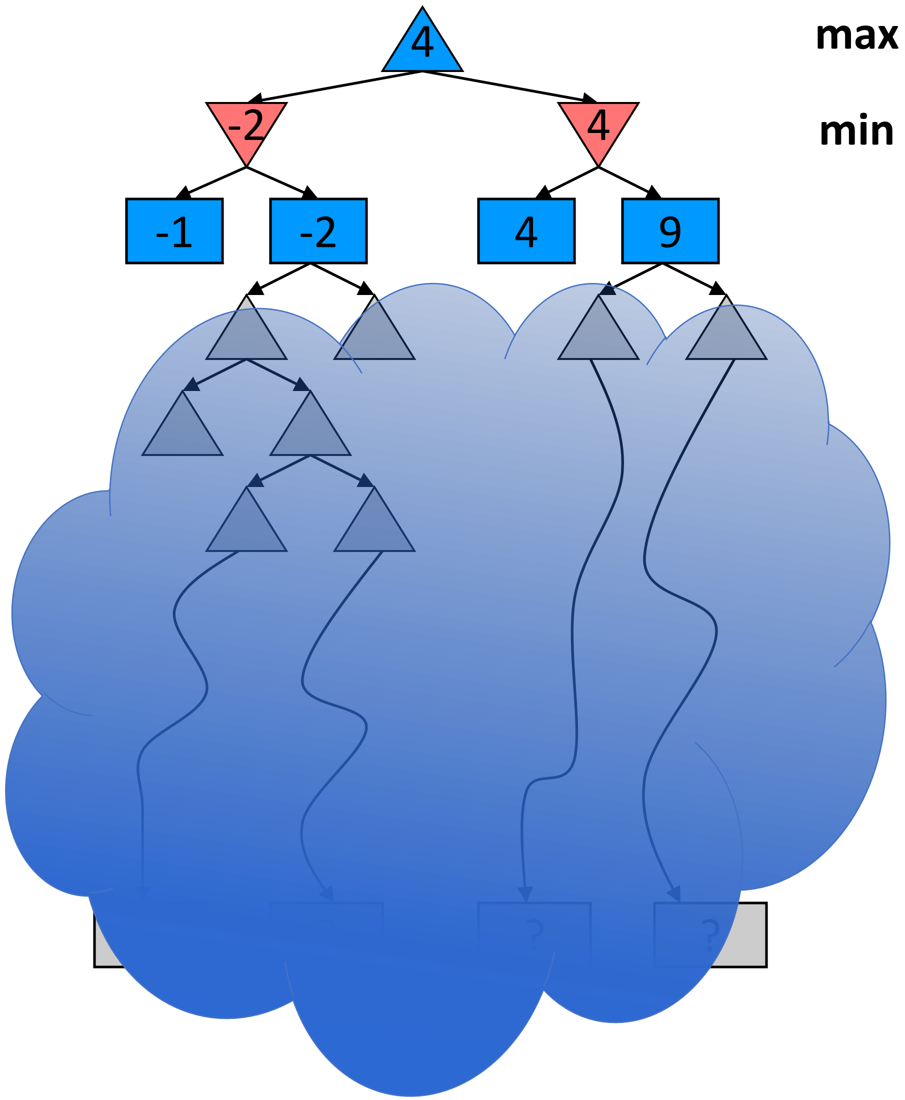
- Problem: In realistic games, cannot search to leaves!

- Solution 1: Bounded lookahead
  - Search only to a preset ***depth limit*** or ***horizon***
  - Use an ***evaluation function*** for non-terminal positions

- Guarantee of optimal play is gone

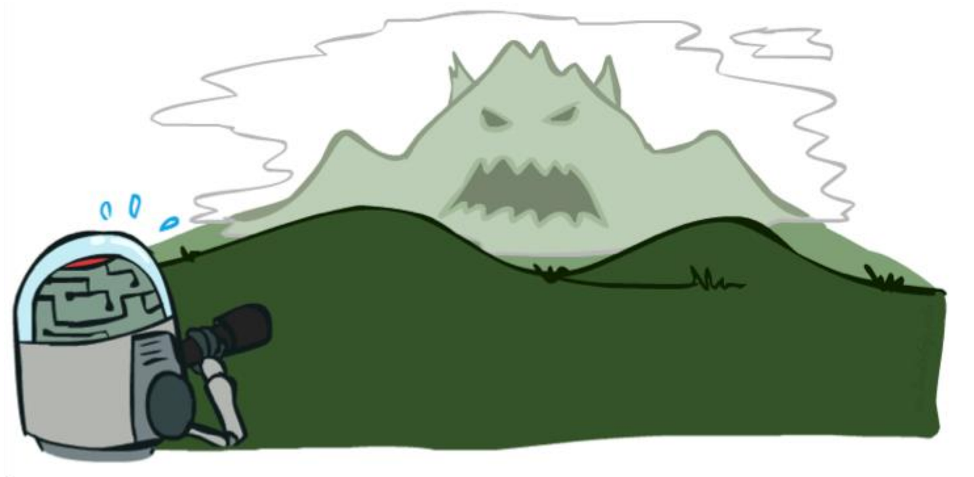
- More plies make a BIG difference

- Example:
  - Suppose we have 100 seconds, can explore 10K nodes / sec
  - So can check 1M nodes per move
  - For chess, b=~35 so reaches about depth 4 – not so good



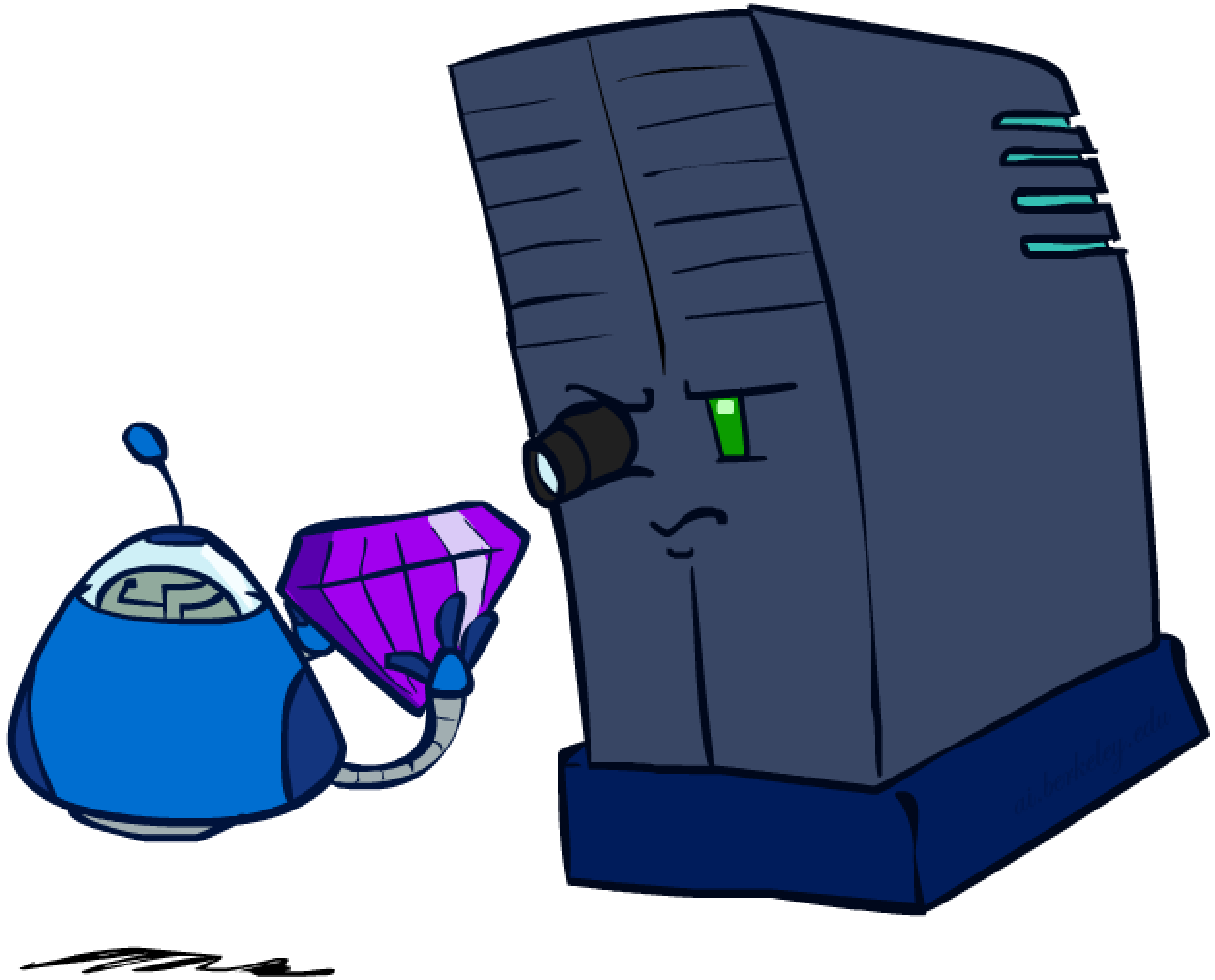
Depth Matters

- Evaluation functions are always imperfect
- Deeper search => better play (usually)
- Or, deeper search gives same quality of play with a less accurate evaluation function
- An important example of the tradeoff between complexity of features and complexity of computation



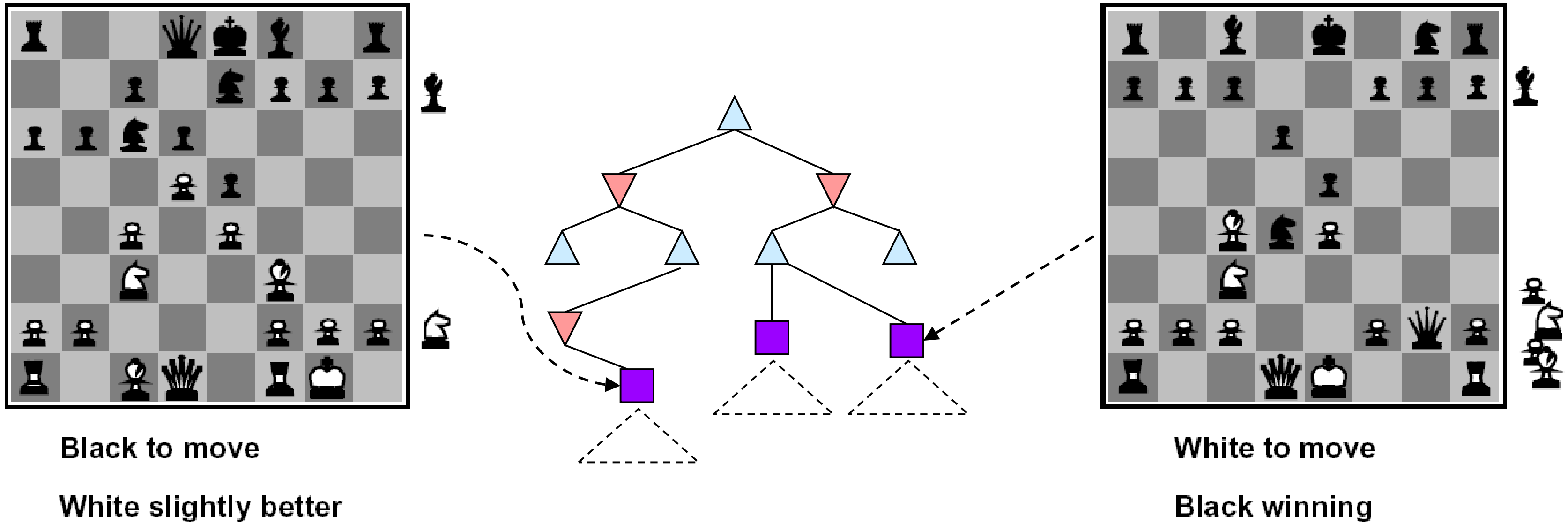
[Demo: depth limited (L6D4, L6D5)]

Evaluation Functions



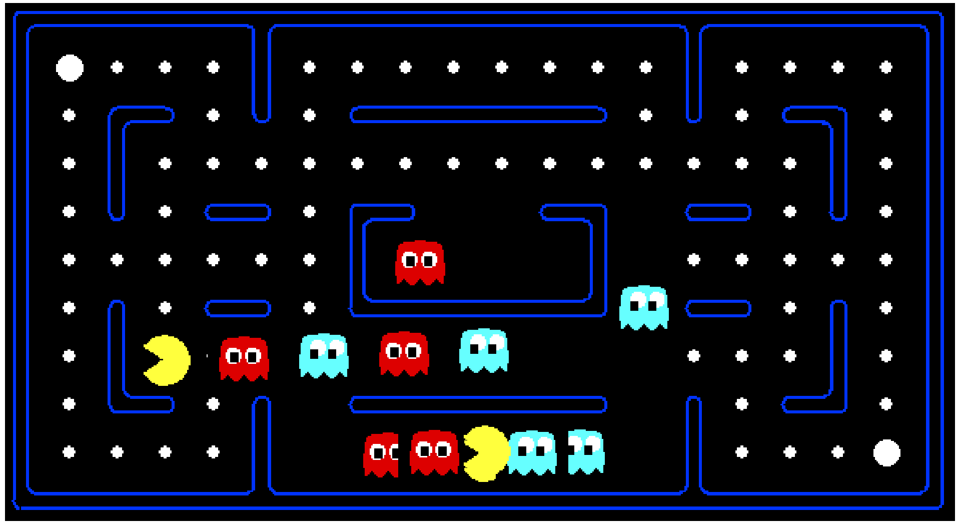
Evaluation Functions

- Evaluation functions score non-terminals in depth-limited search



- Ideal function: returns the actual minimax value of the position
- In practice: typically weighted linear sum of features:
  - EVAL(s) = $w_1 f_1(s) + w_2 f_2(s) + ... + w_n f_n(s)$
  - E.g., $w_1 = 9$, $f_1(s)$ = (num white queens – num black queens), etc.
- Terminate search only in ***quiescent*** positions, i.e., no major changes expected in feature values

Evaluation for Pacman



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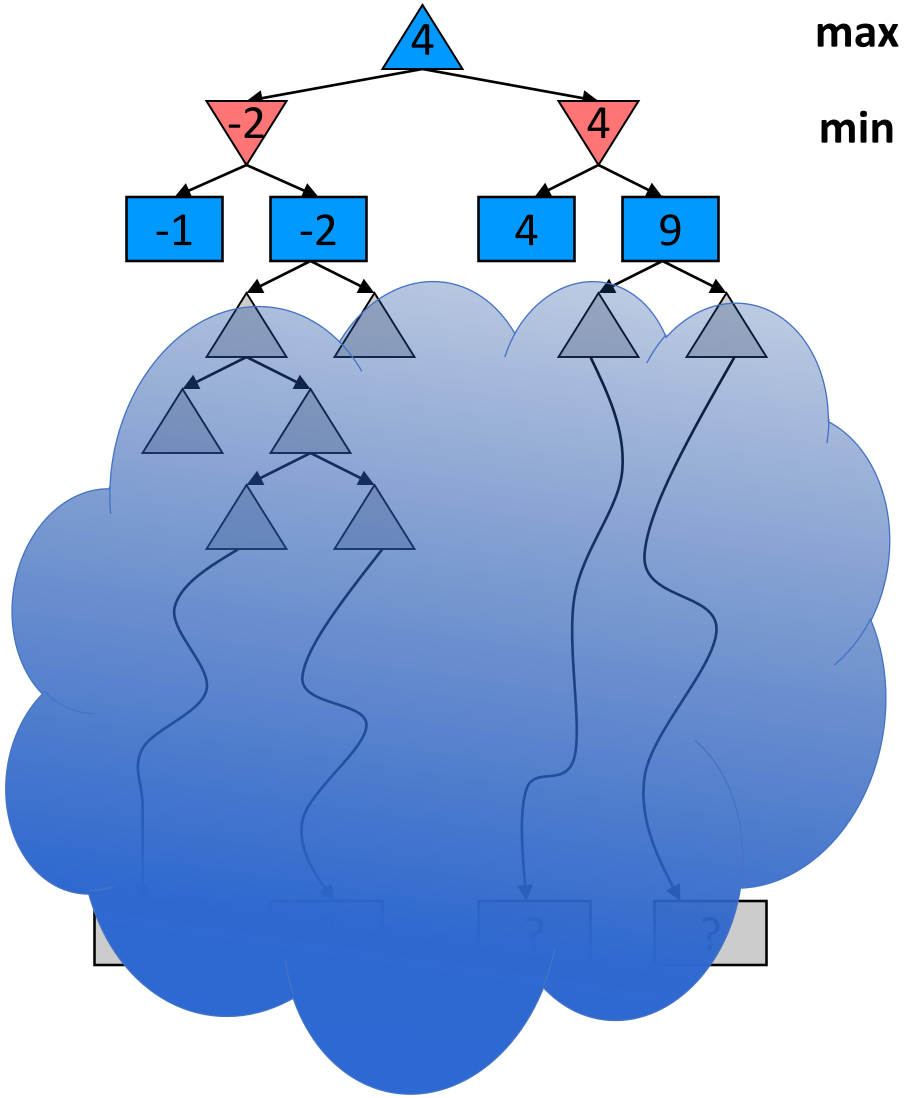
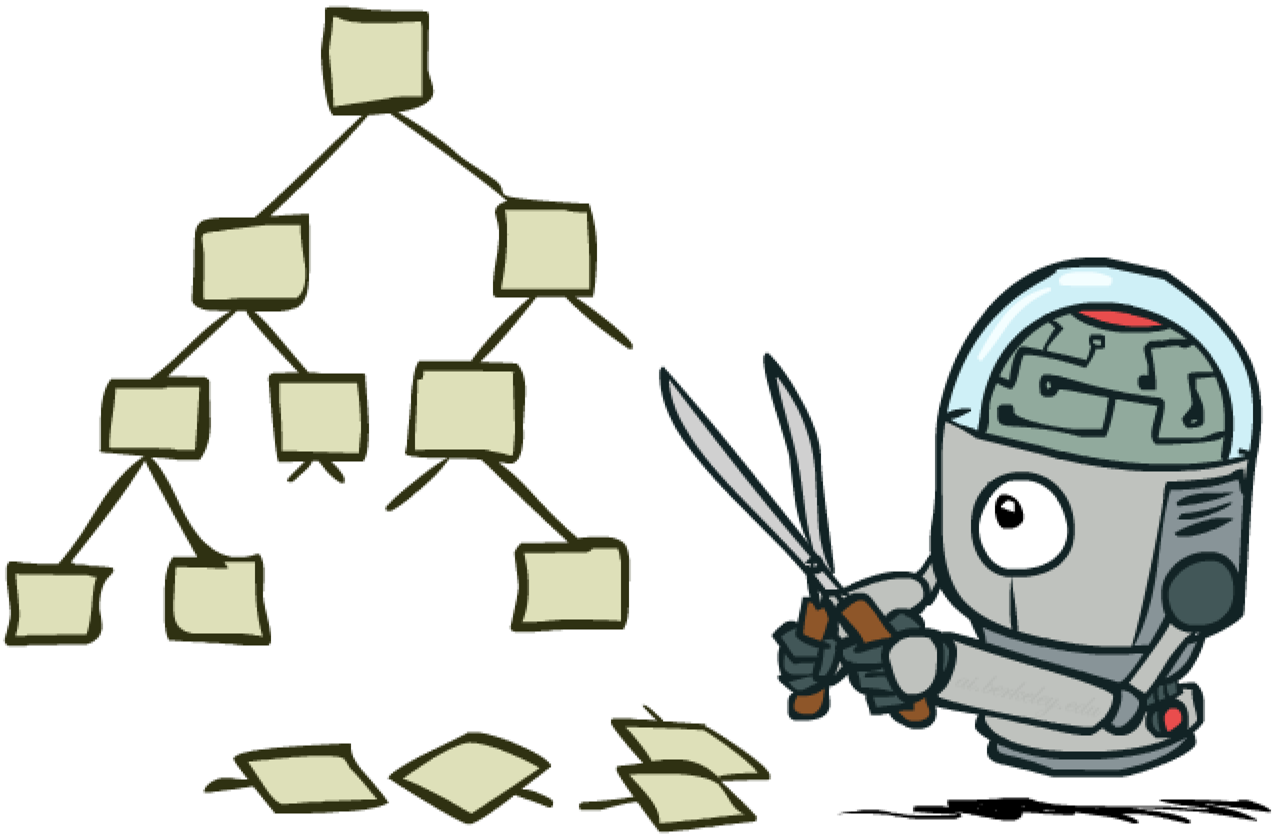
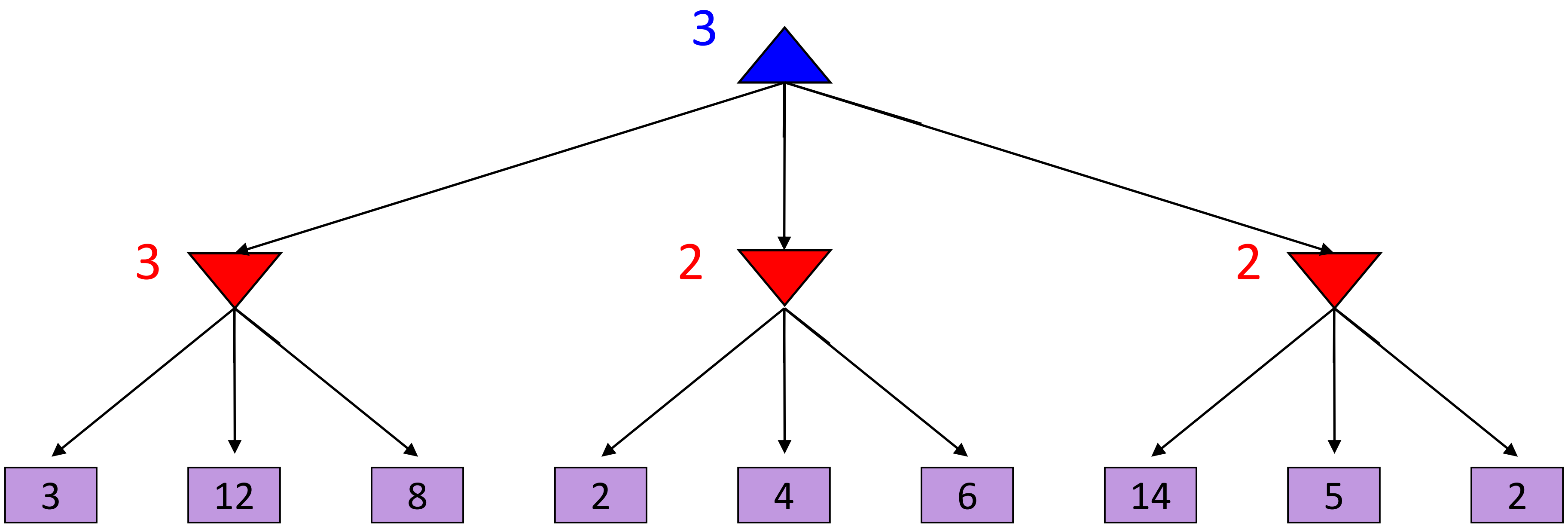


Solution 2: Game Tree Pruning

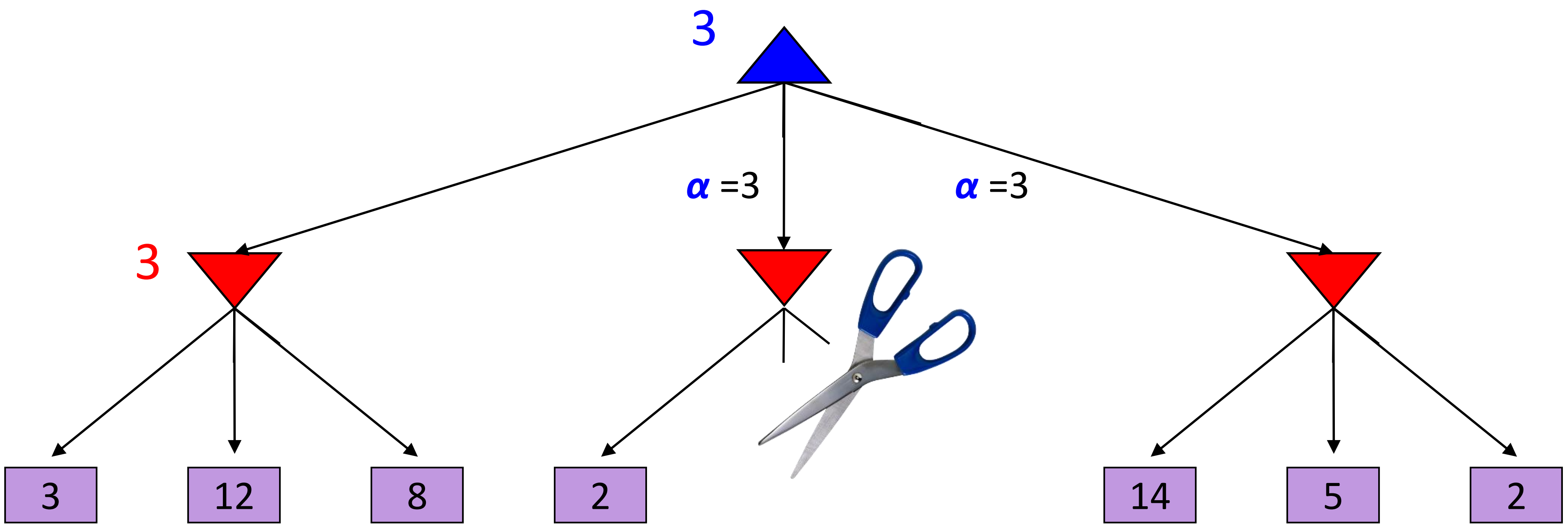


Intuition: prune the branches that can't be chosen



Alpha-Beta Pruning Example

α = best option so far from any MAX node on this path



We can prune when: min node won't be higher than 2, while parent max has seen something larger in another branch

The order of generation matters: more pruning is possible if good moves come first

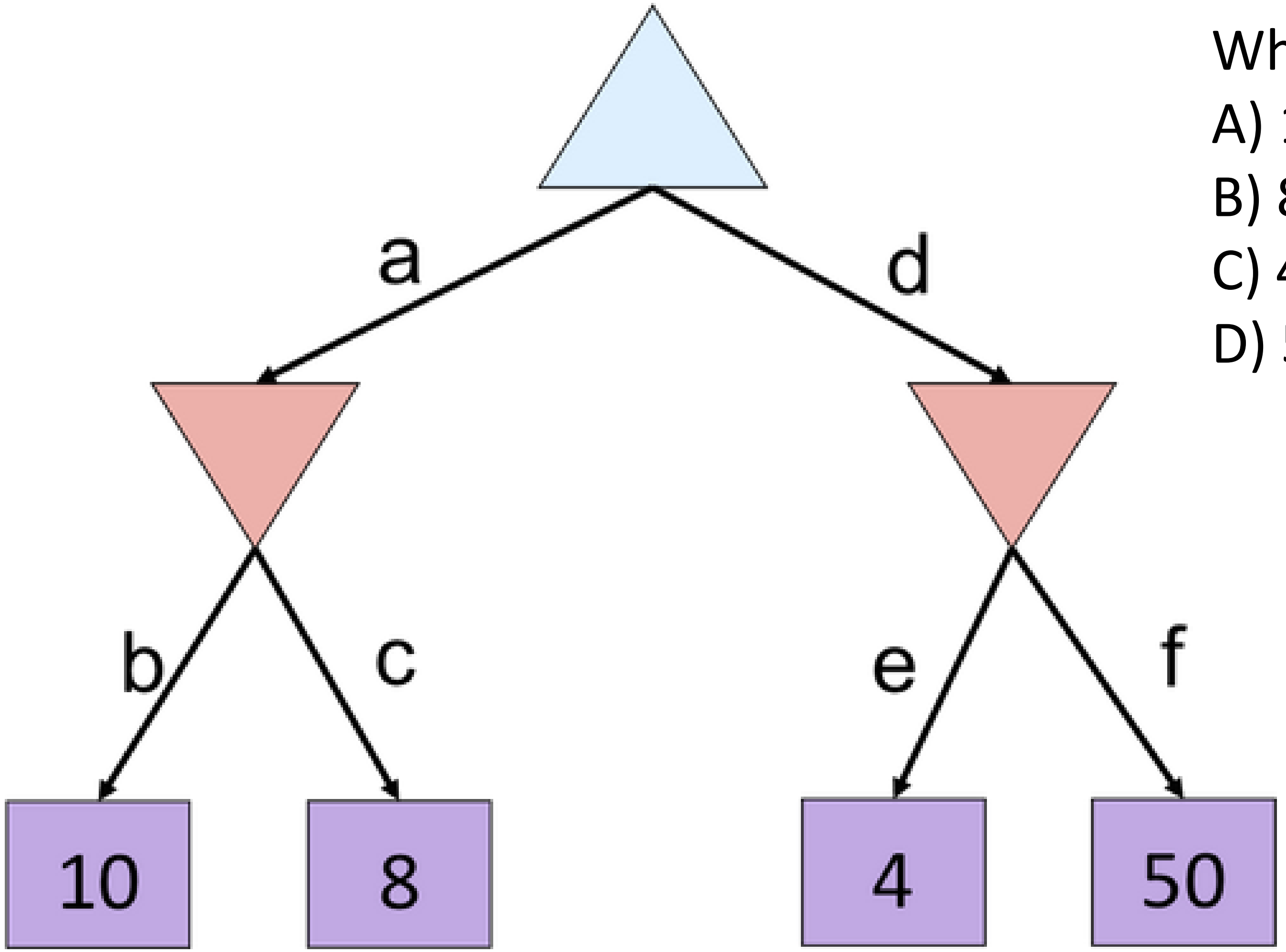
Alpha-Beta Implementation

α: MAX’s best option on path to root
β: MIN’s best option on path to root

```
def max-value(state, α, β):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor, α, β))
        if v ≥ β
            return v
        α = max(α, v)
    return v
```

```
def min-value(state , α, β):
    initialize v = +∞
    for each successor of state:
        v = min(v, value(successor, α, β))
        if v ≤ α
            return v
        β = min(β, v)
    return v
```

Quiz: Minimax Example



What is the value of the blue triangle?

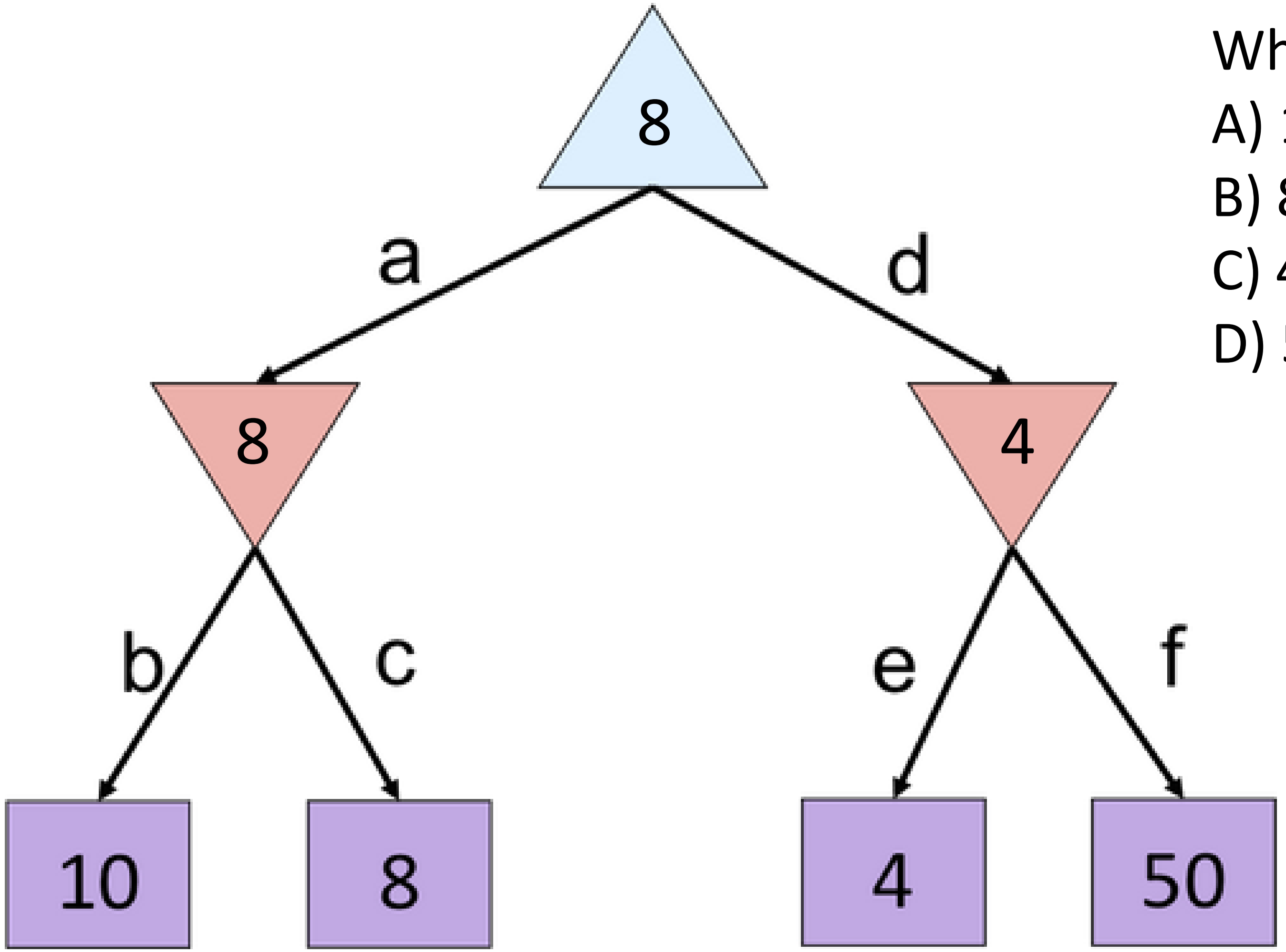
A) 10

B) 8

C) 4

D) 50

Quiz: Minimax Example



What is the value of the blue triangle?

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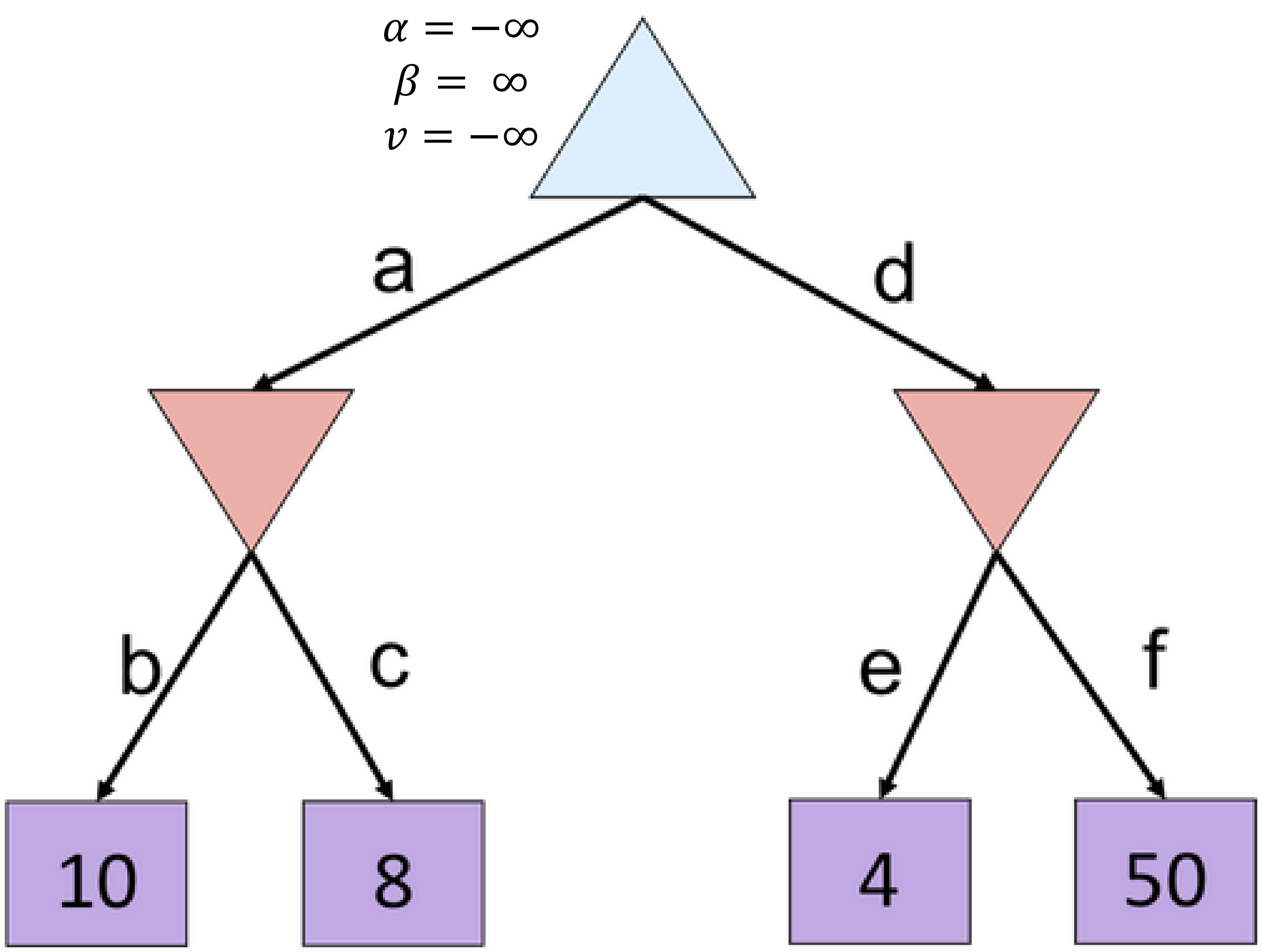
Alpha-Beta Small Example

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        if v ≥ β
            return v
        α = max(α, v)
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def min-value(state , α, β):
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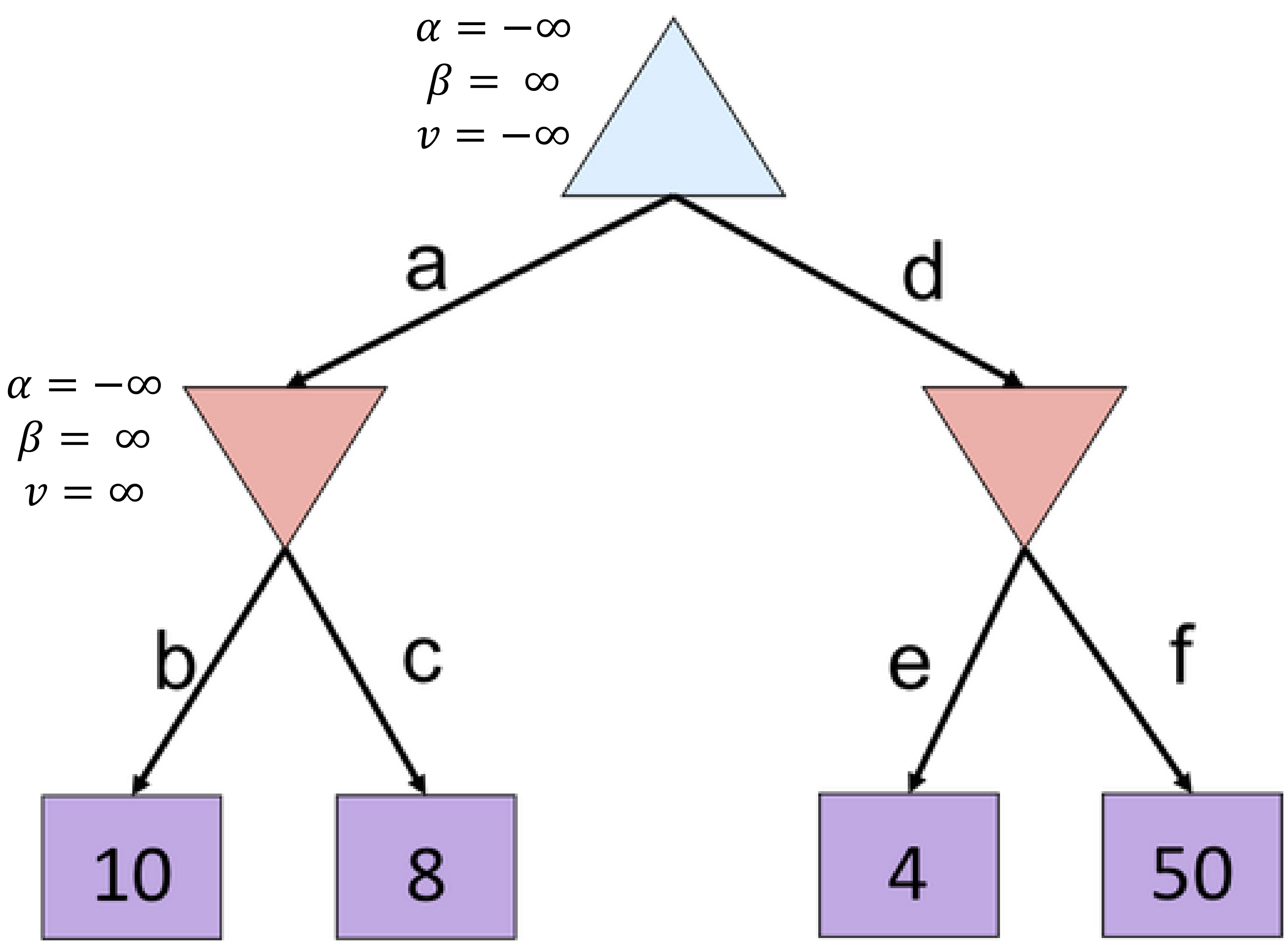
Alpha-Beta Small Example

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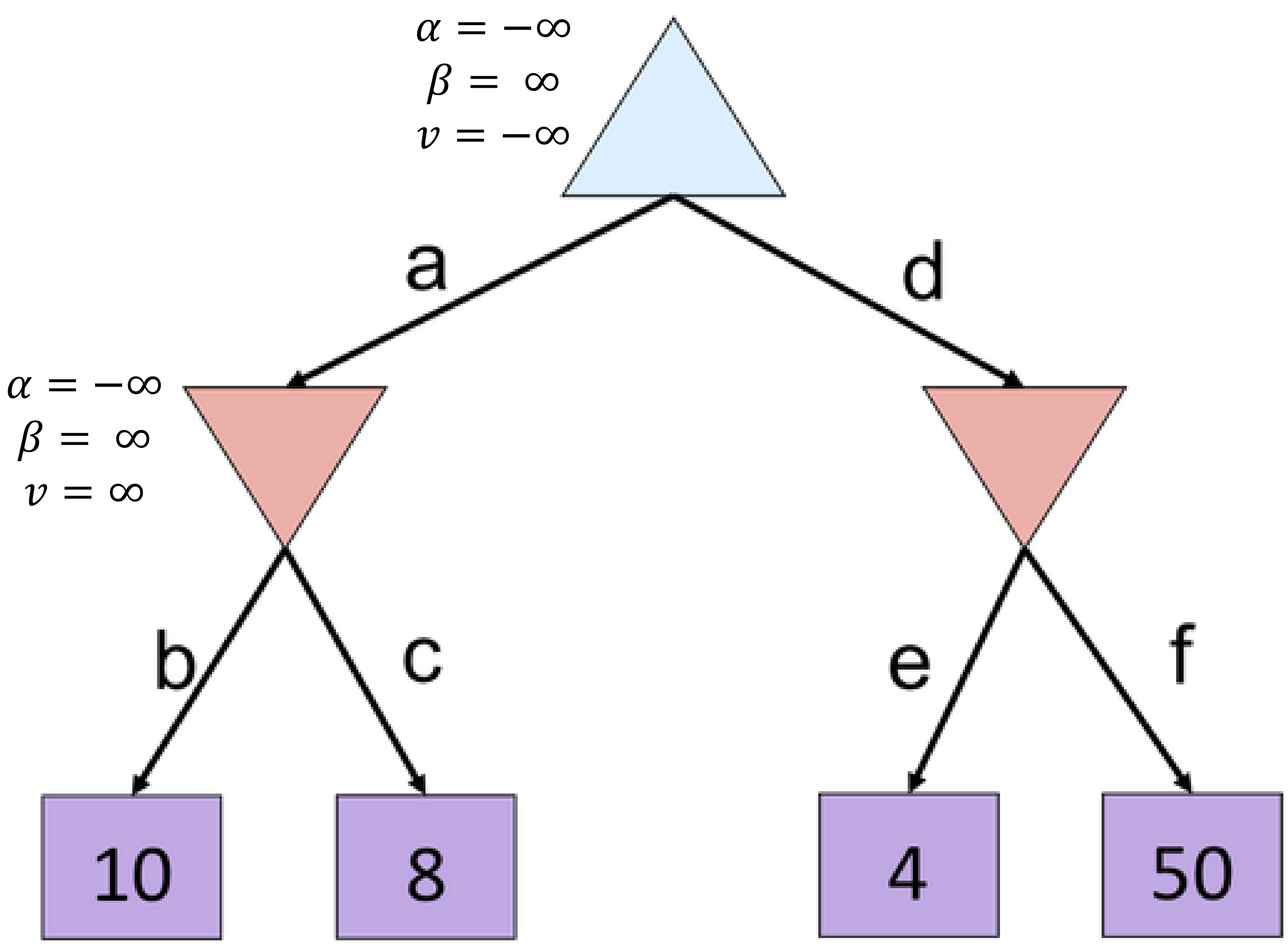


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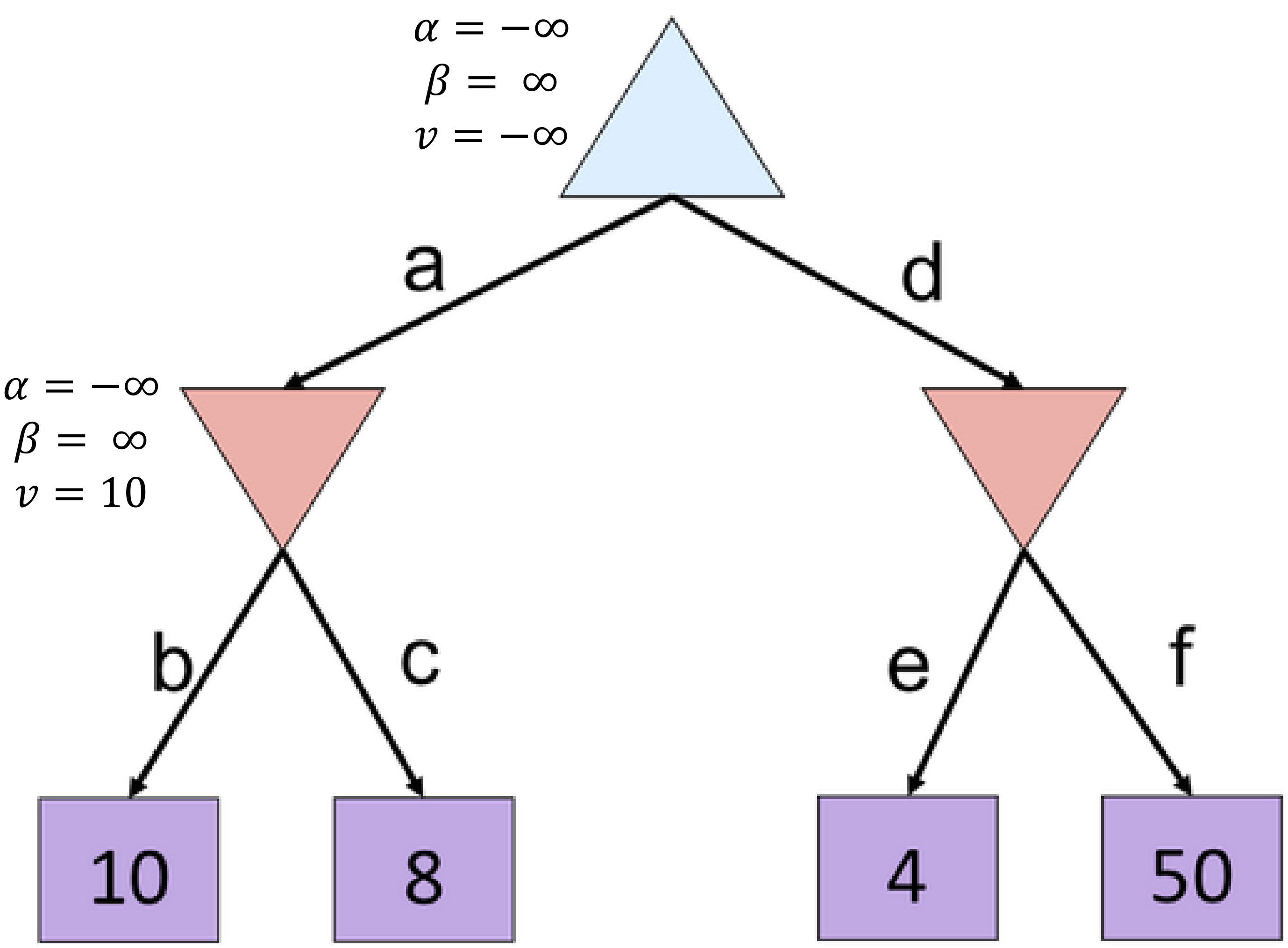


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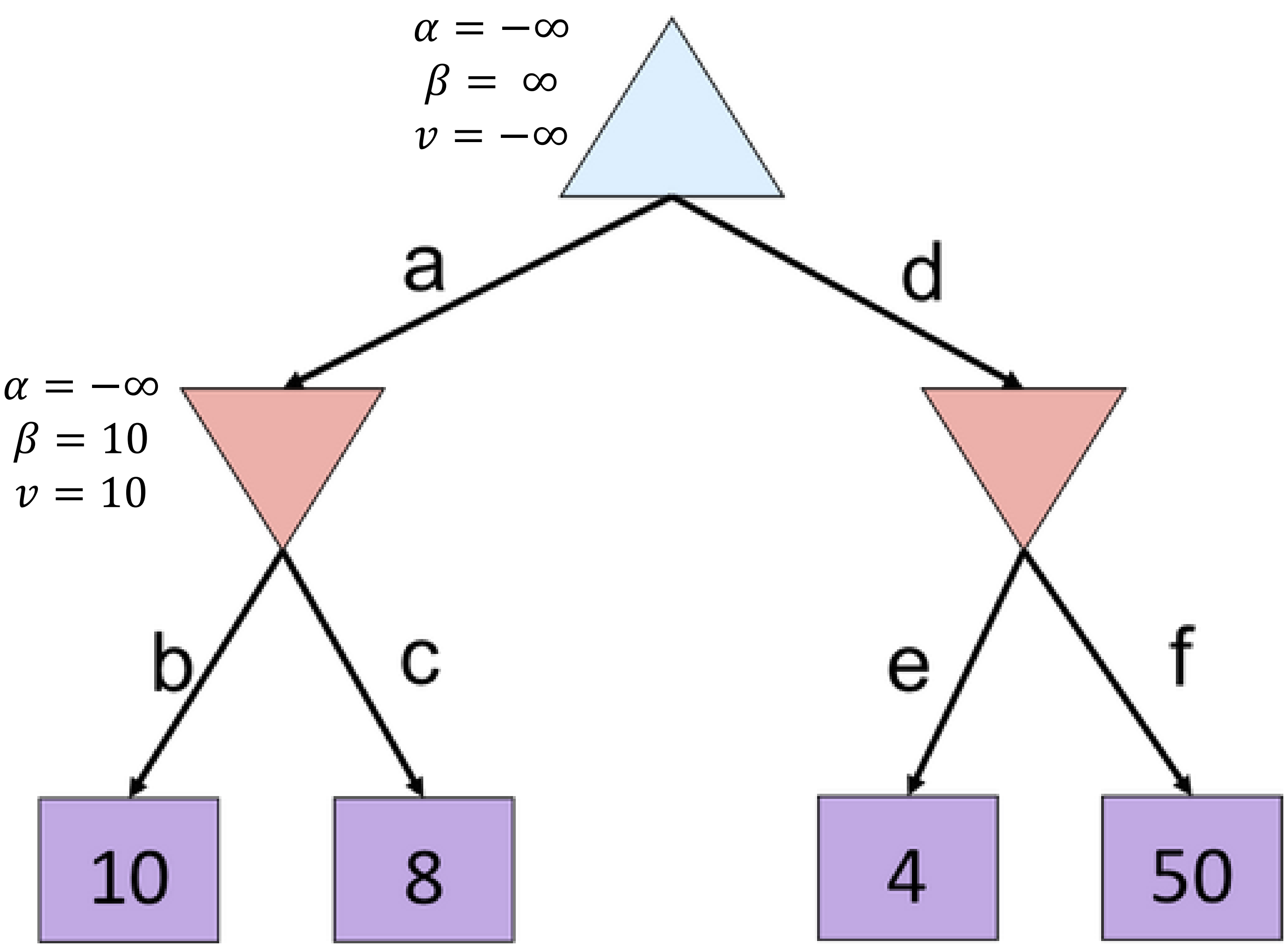


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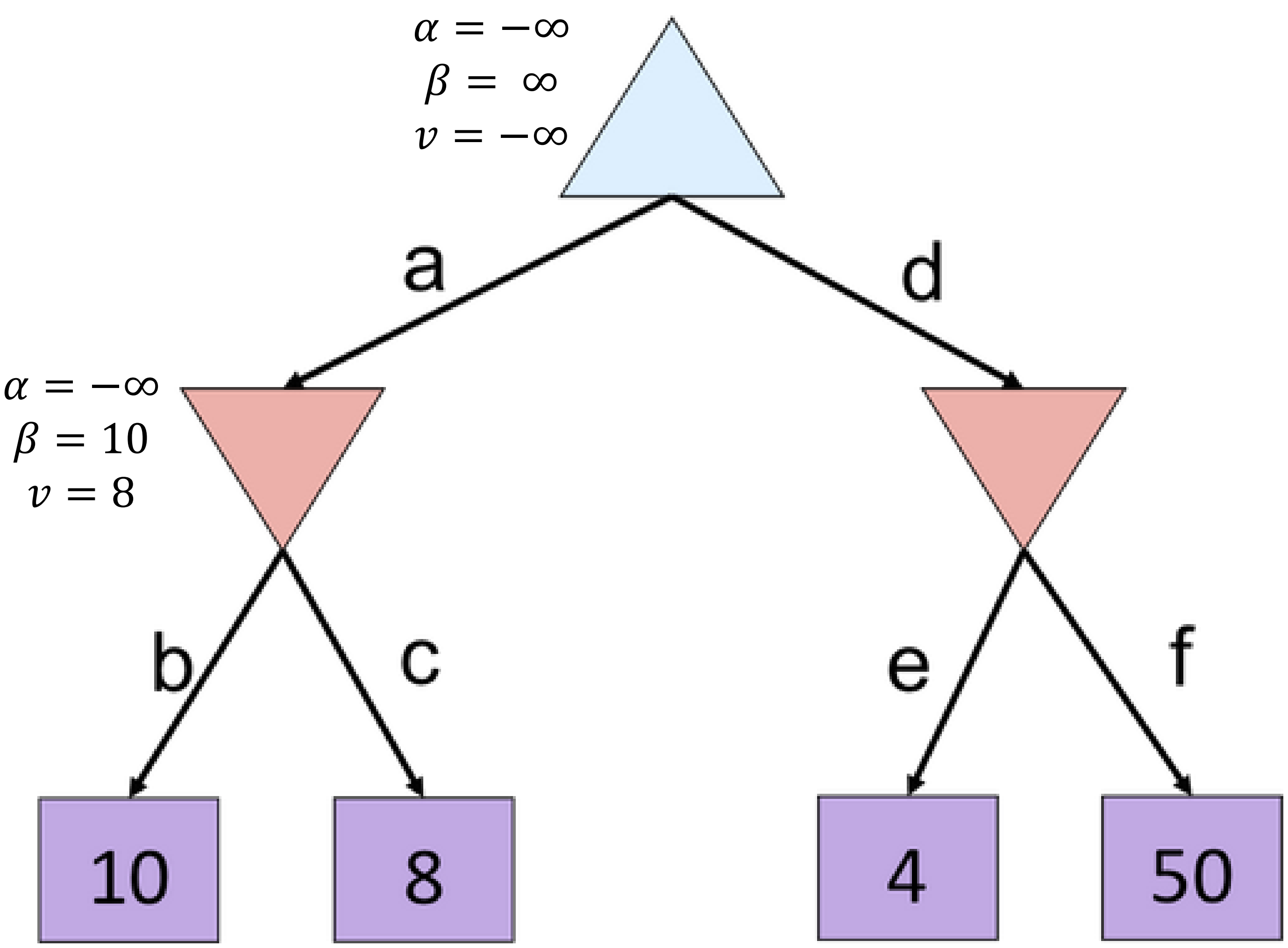


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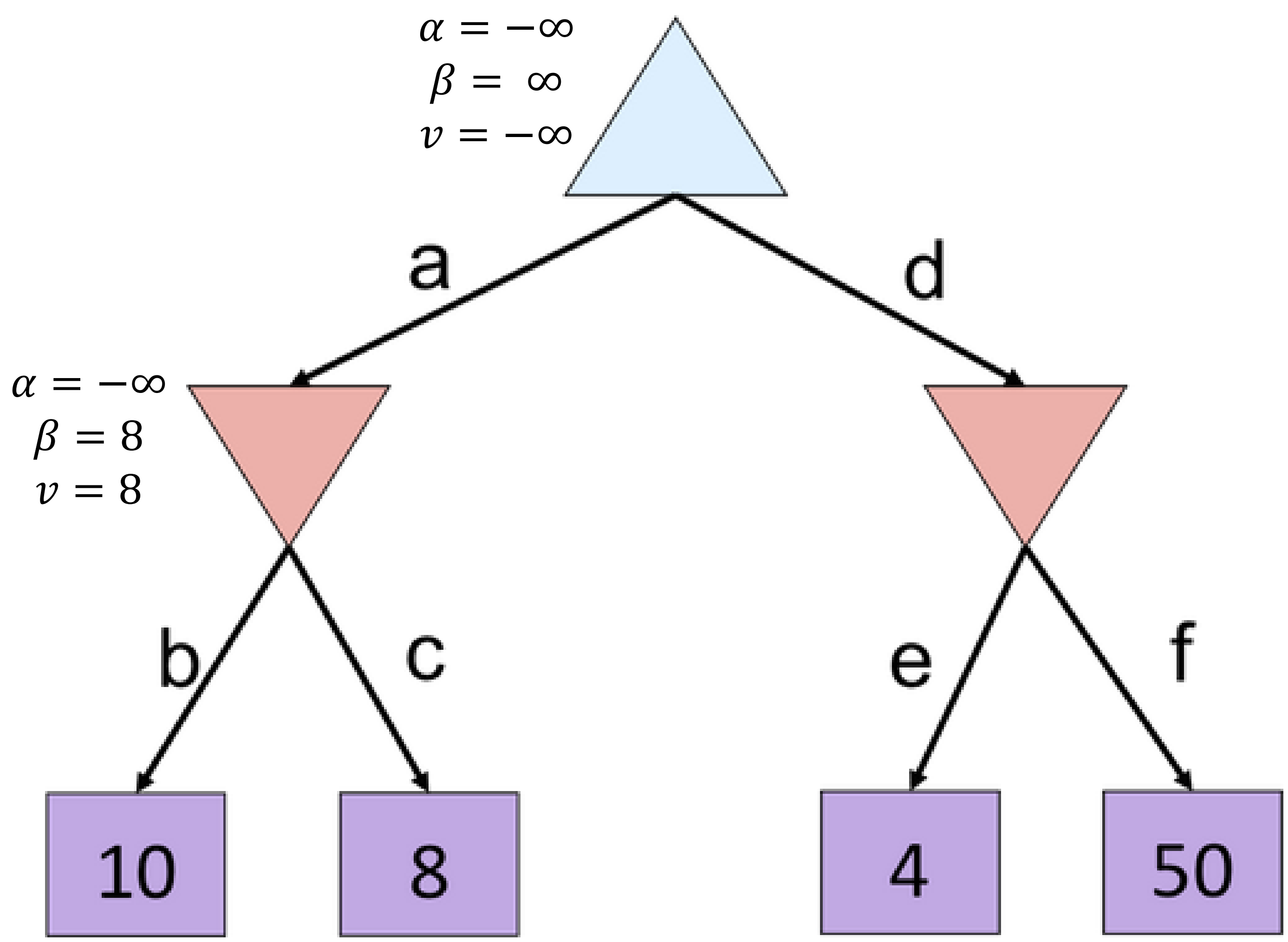


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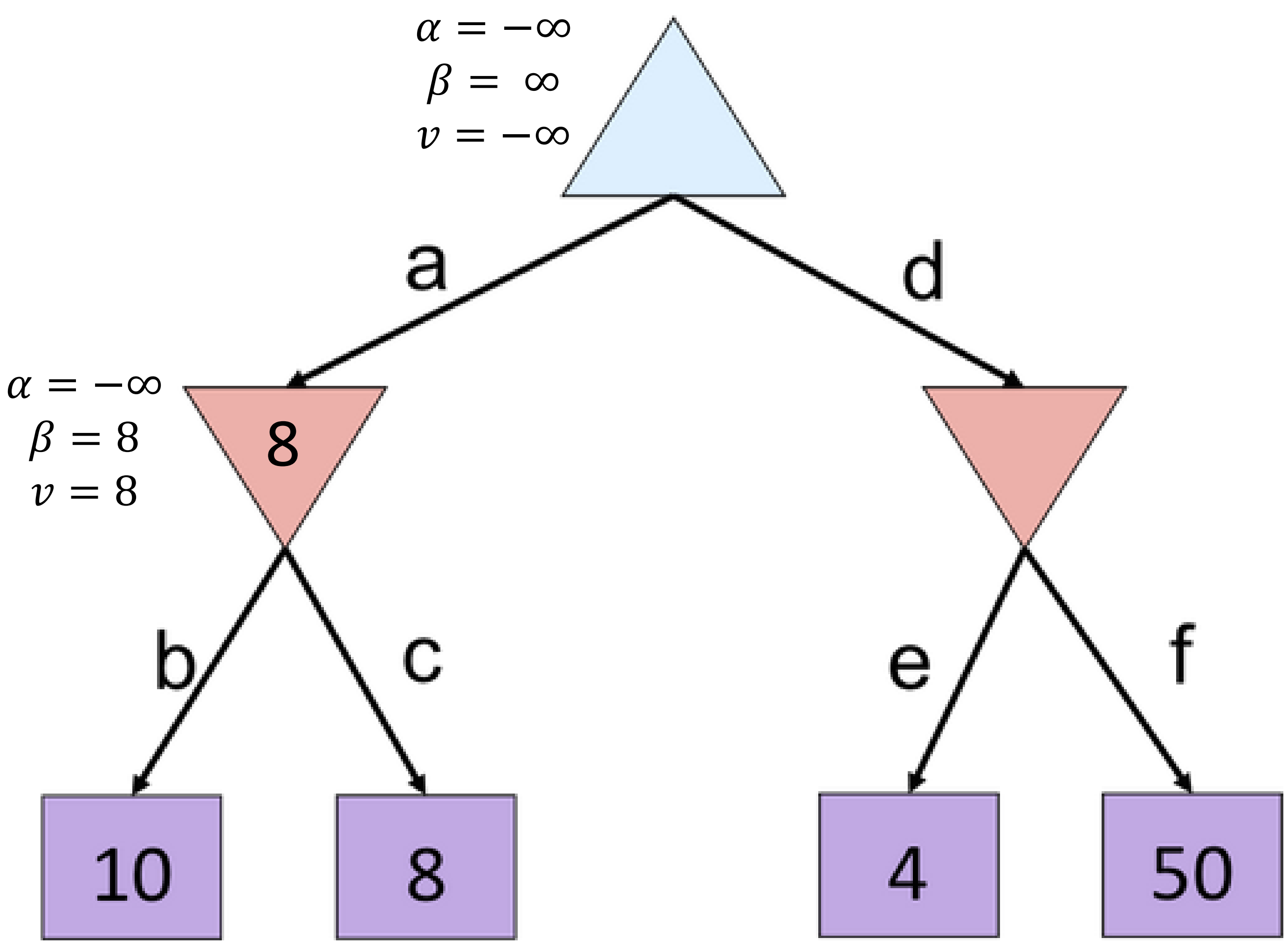

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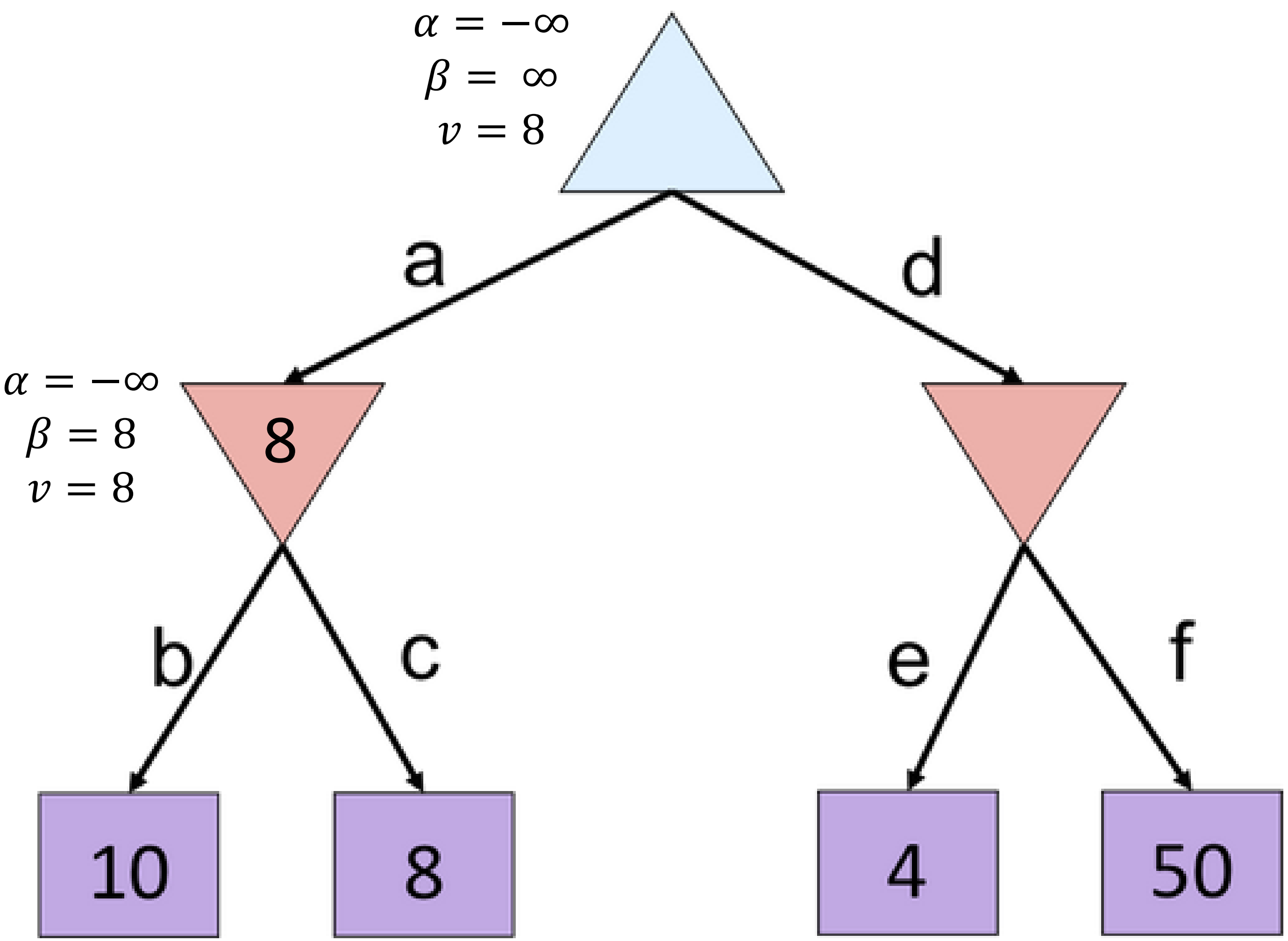


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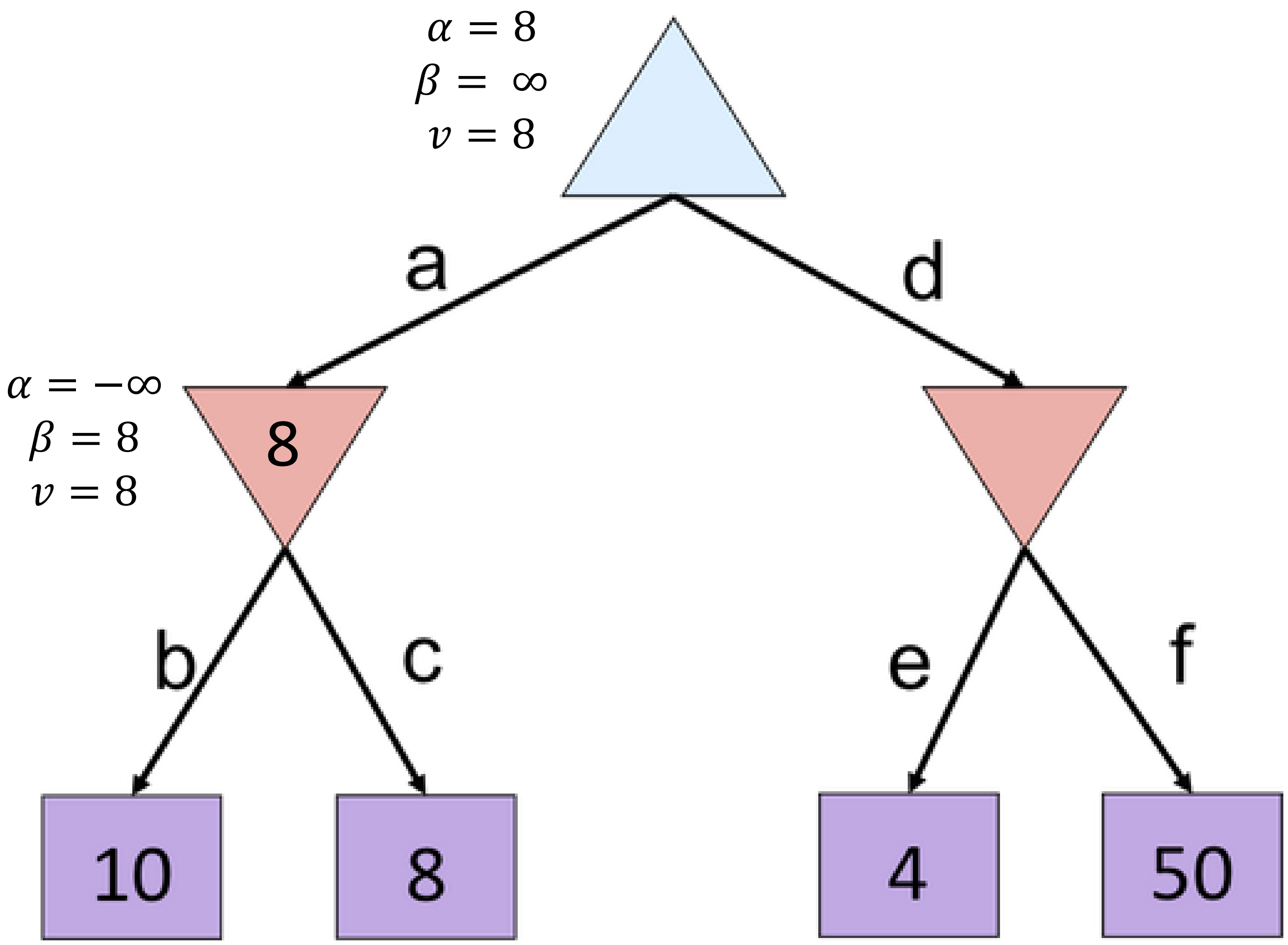


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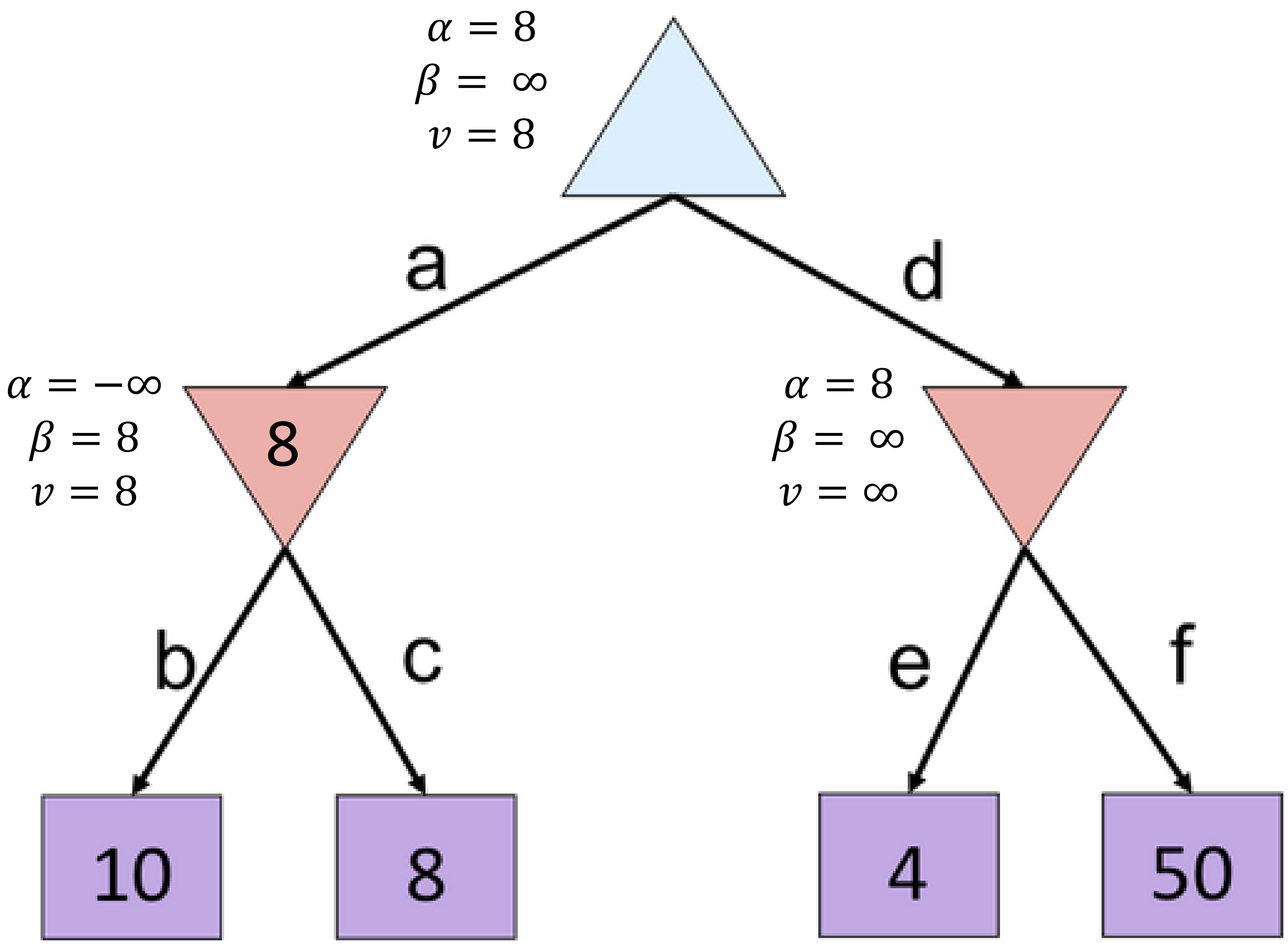


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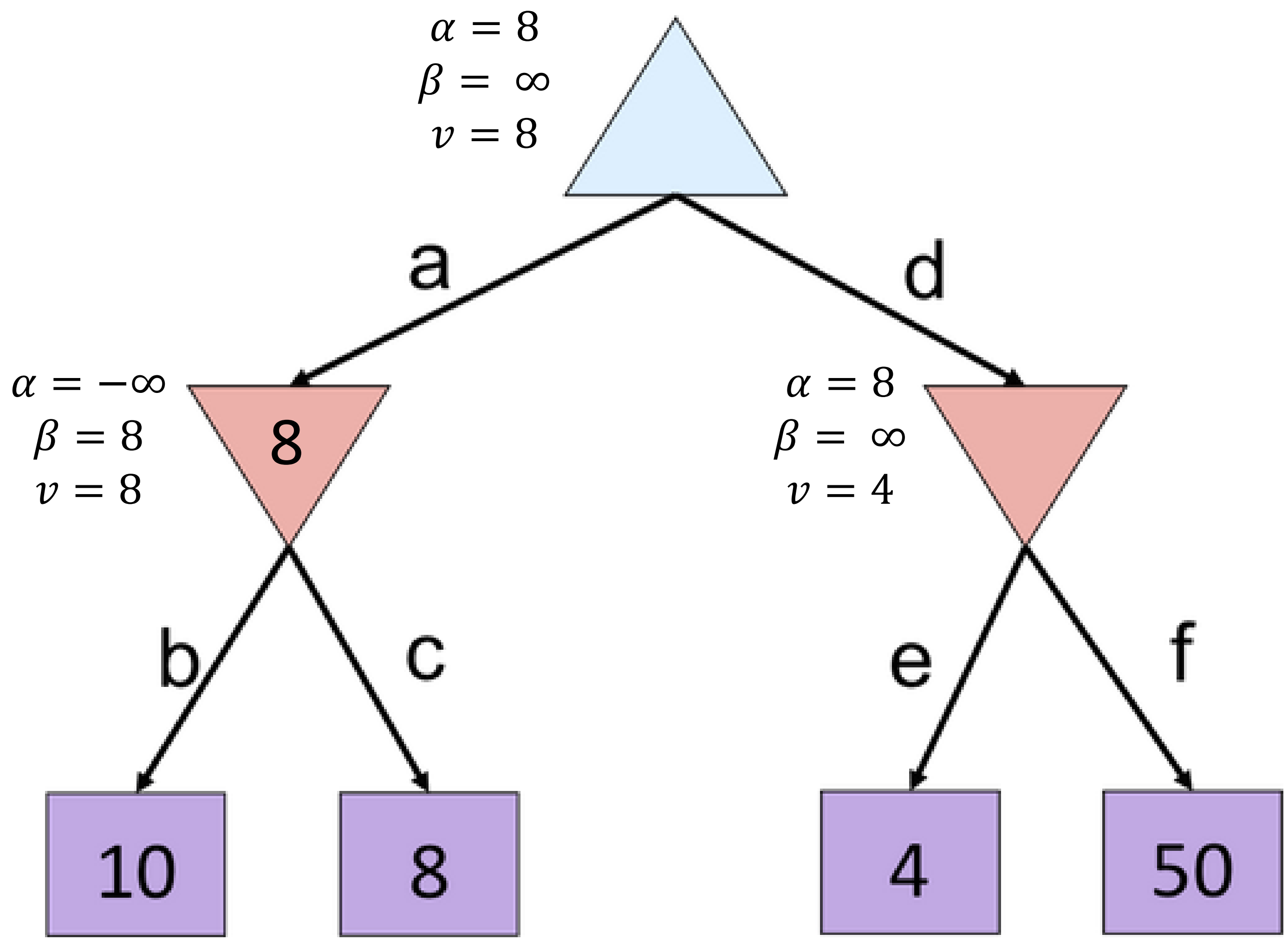


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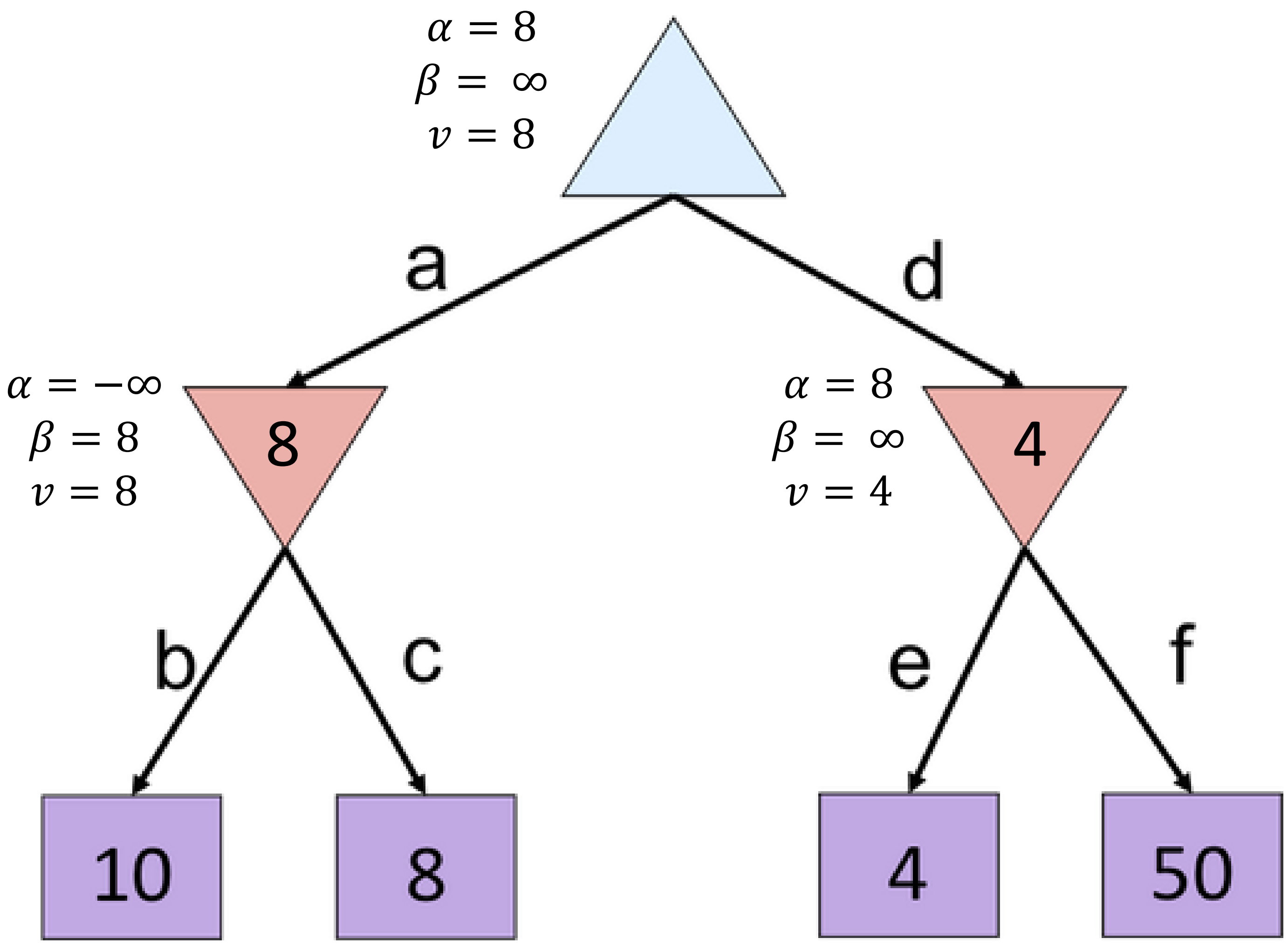


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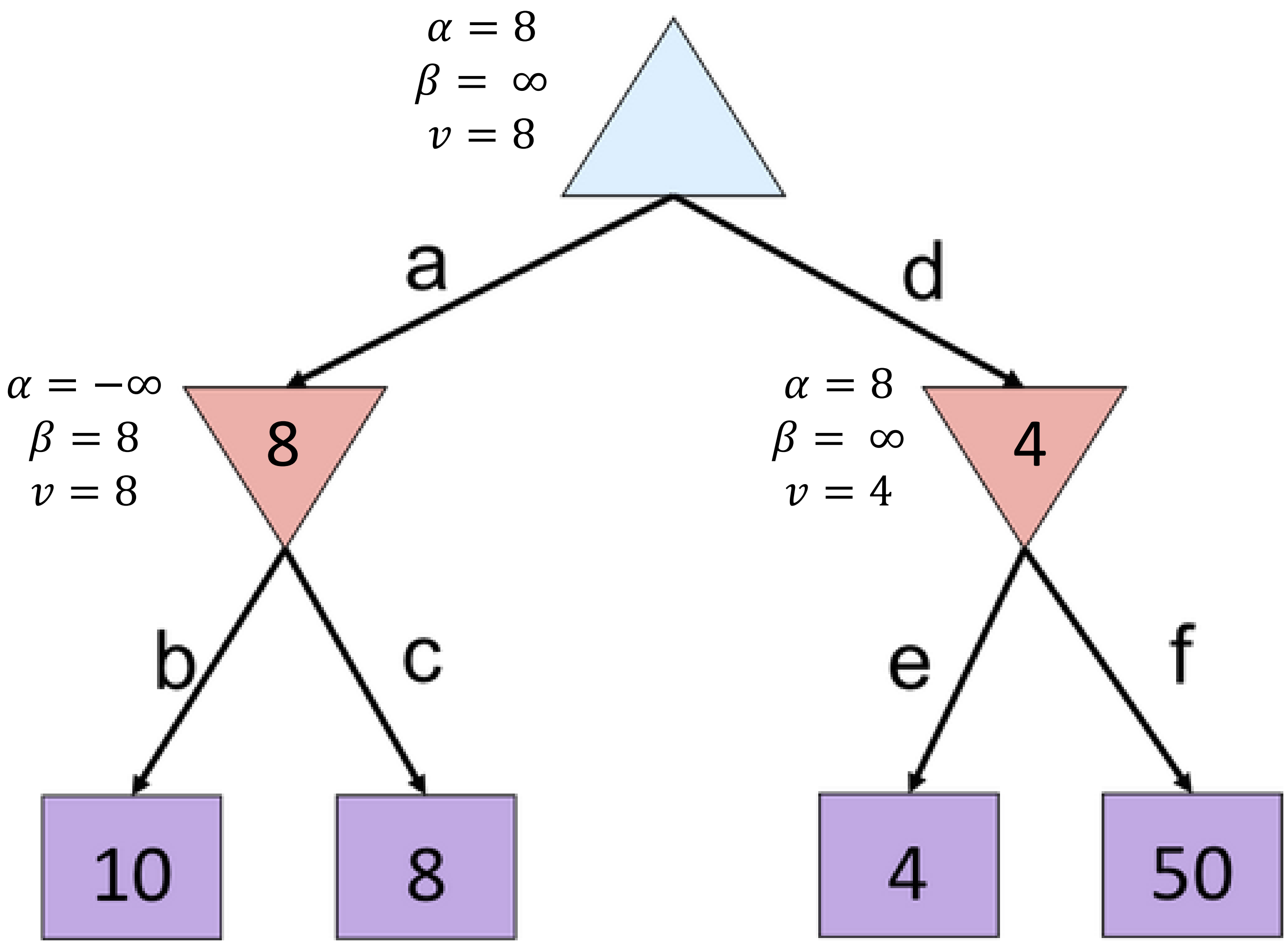


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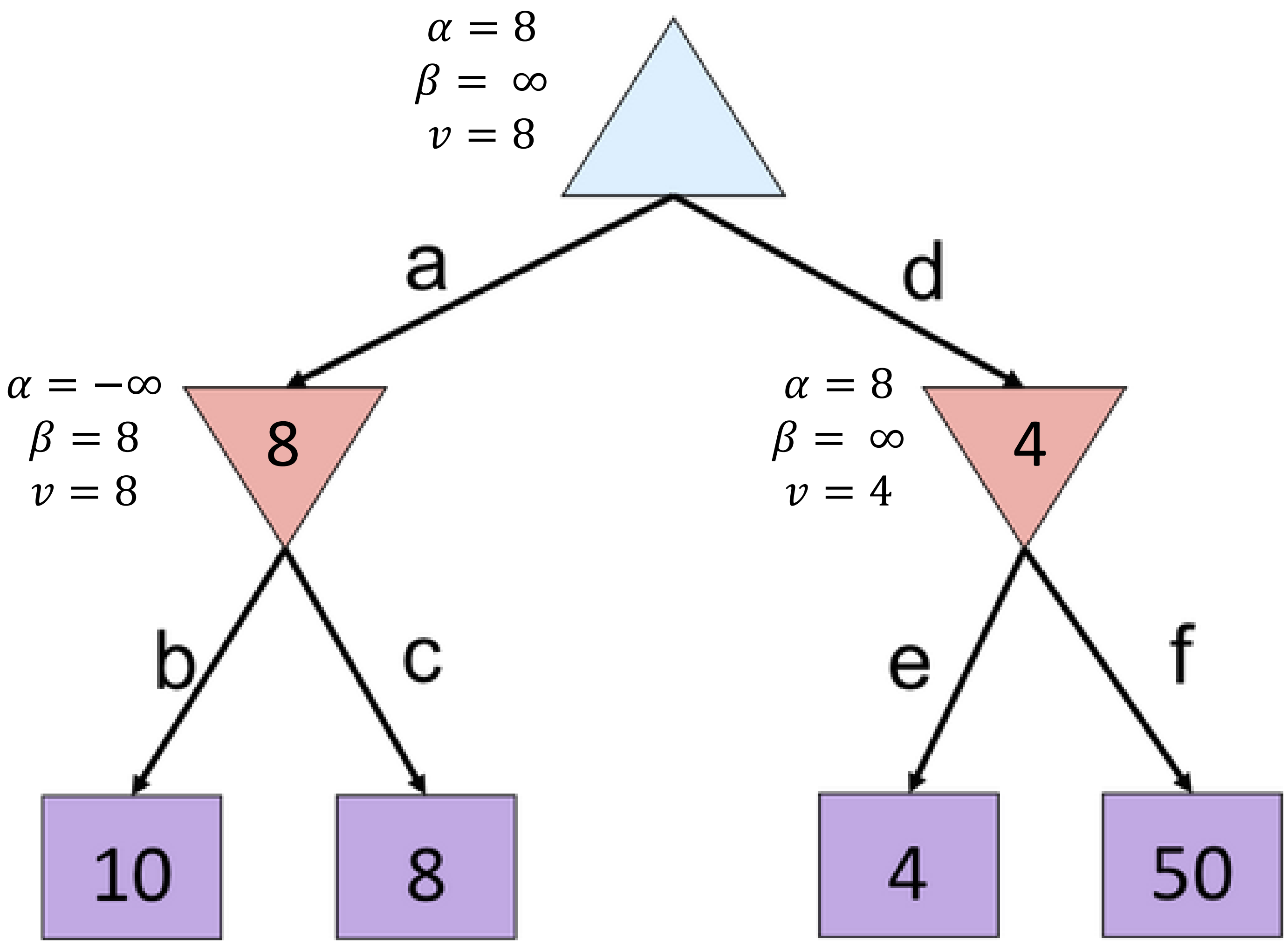


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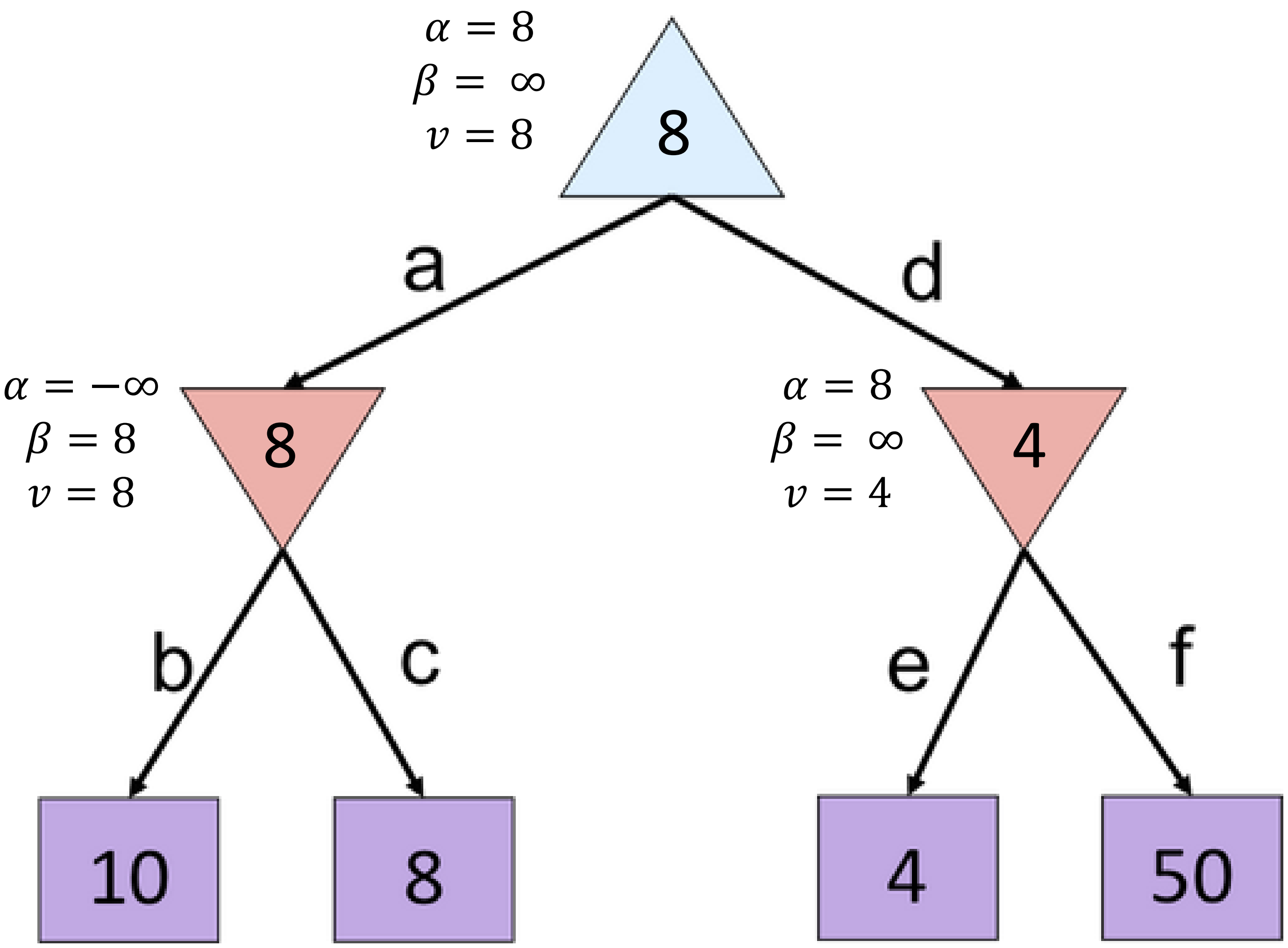


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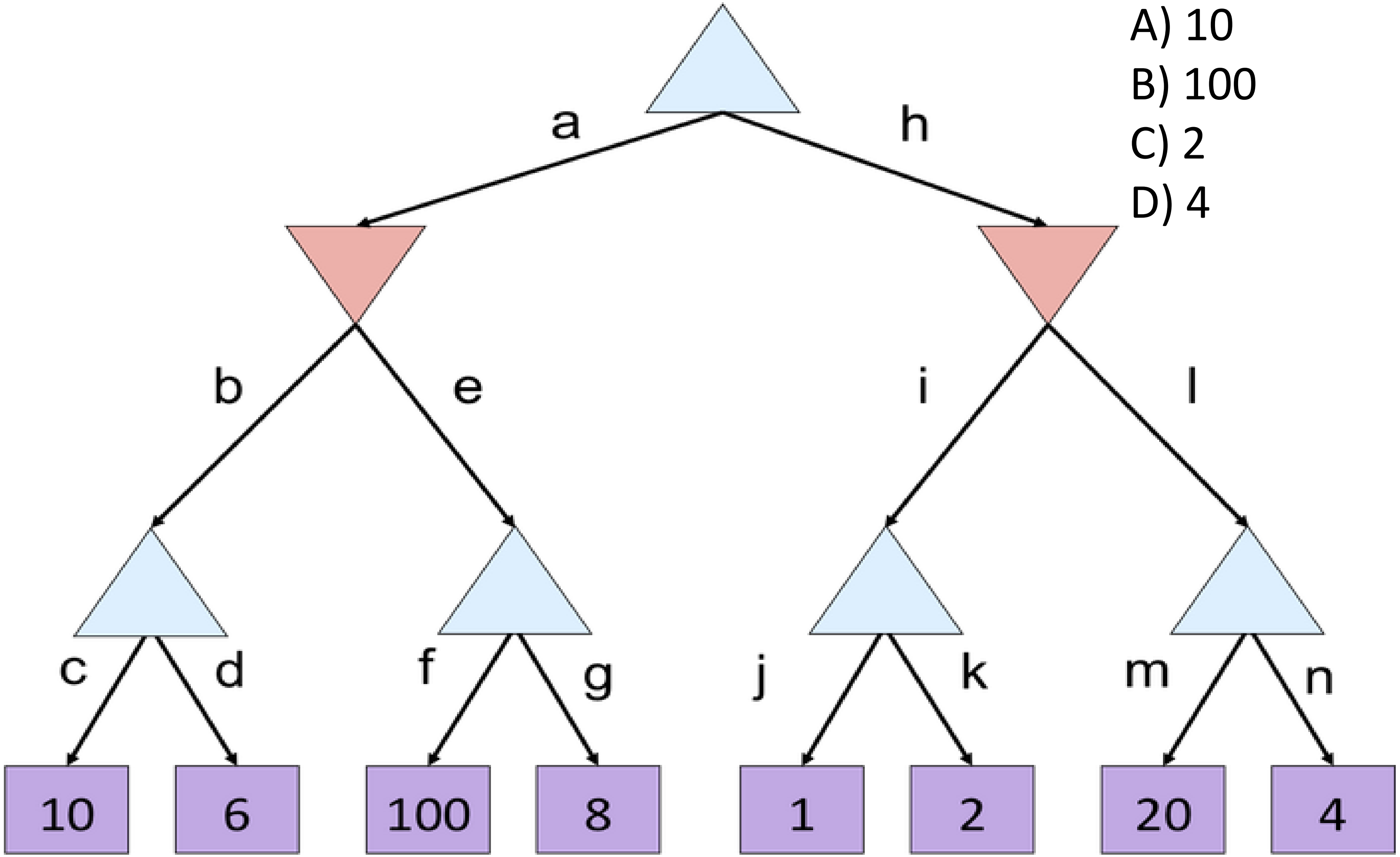
Minimax Quiz



What is the value of the top node?

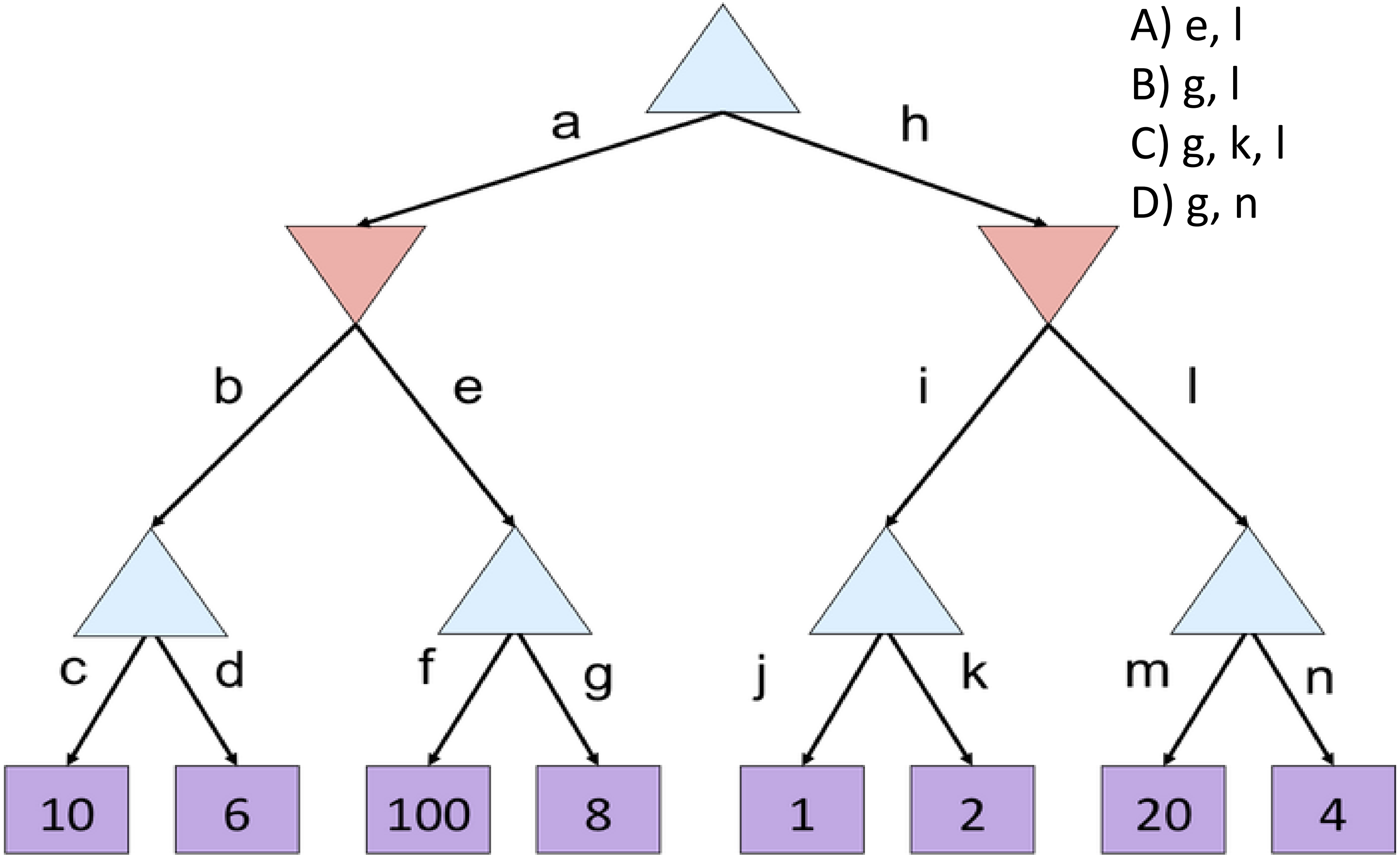
A) 10

B) 100

C) 2

D) 4

Alpha Beta Quiz



Which branches are pruned?

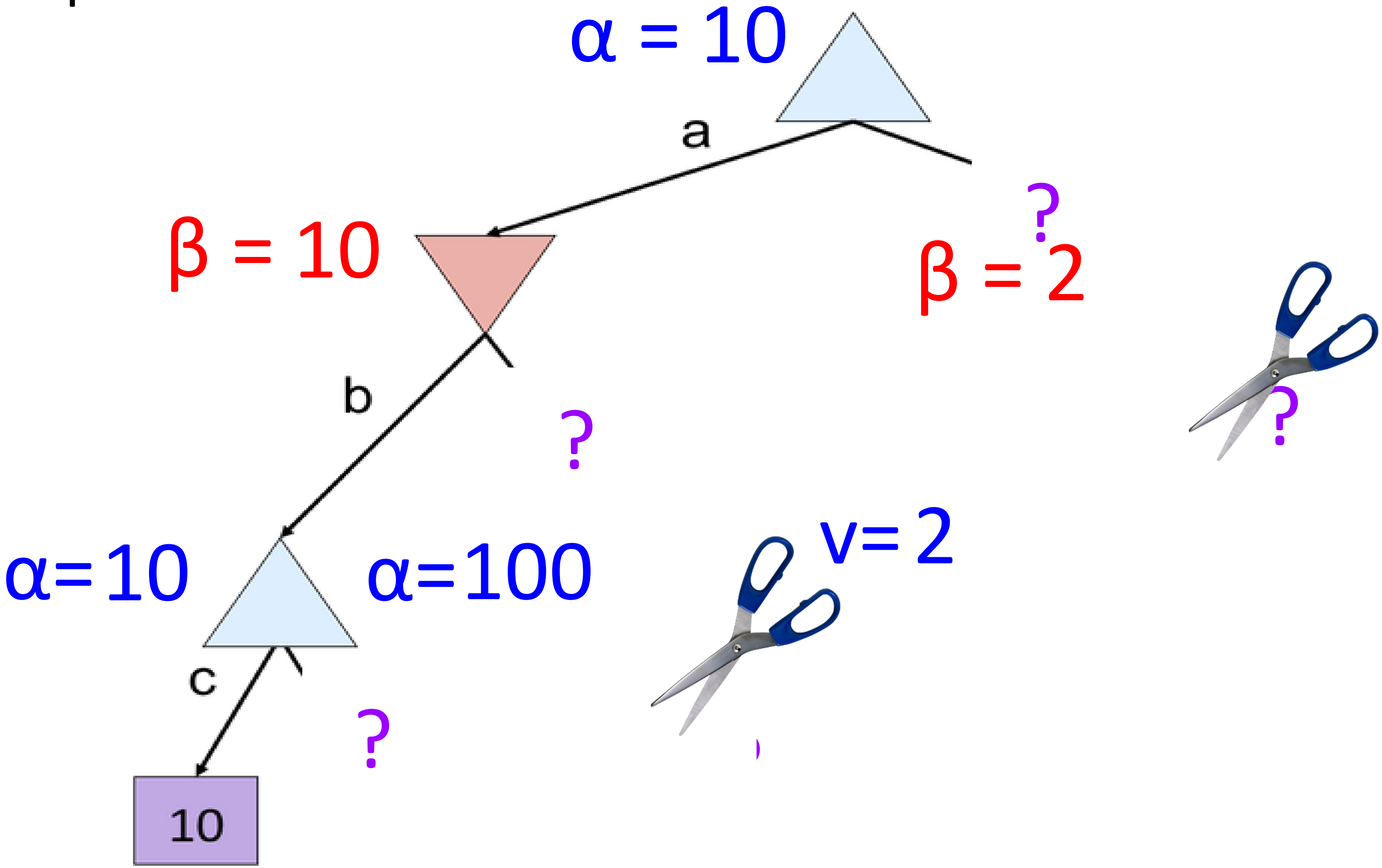
A) e, l

B) g, l

C) g, k, l

D) g, n

Alpha-Beta Quiz 2

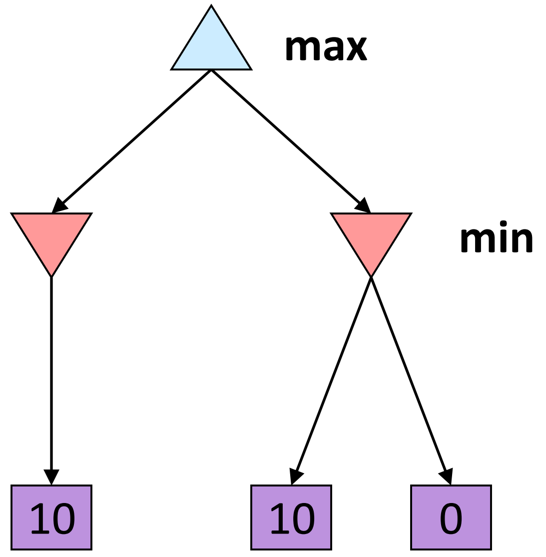


Alpha-Beta Pruning Properties

- Theorem: This pruning has ***no effect*** on minimax value computed for the root!

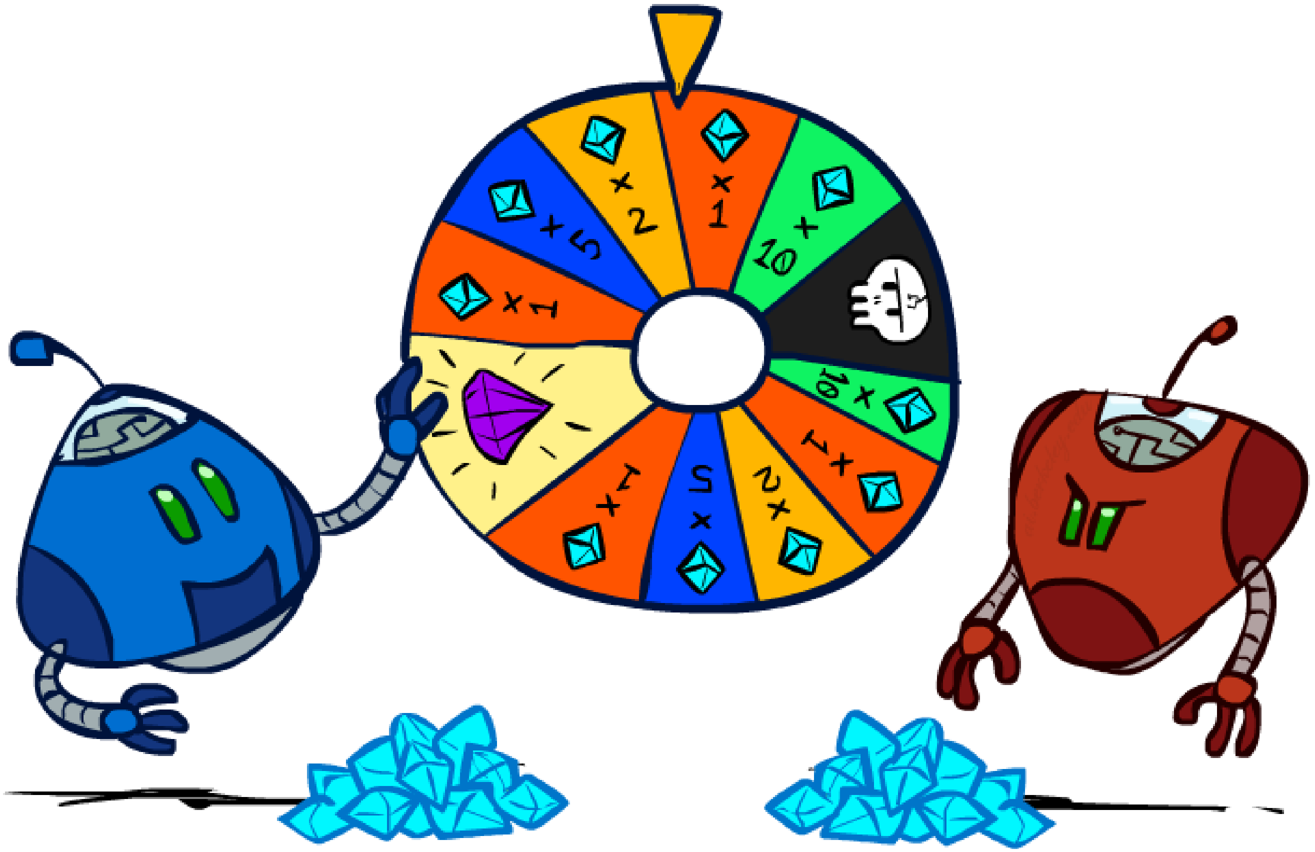
- Good child ordering improves effectiveness of pruning
  - Iterative deepening helps with this

- With "perfect ordering":
  - Time complexity drops to $O(b^{m/2})$
  - Doubles solvable depth!
  - 1M nodes/move => depth=8, respectable

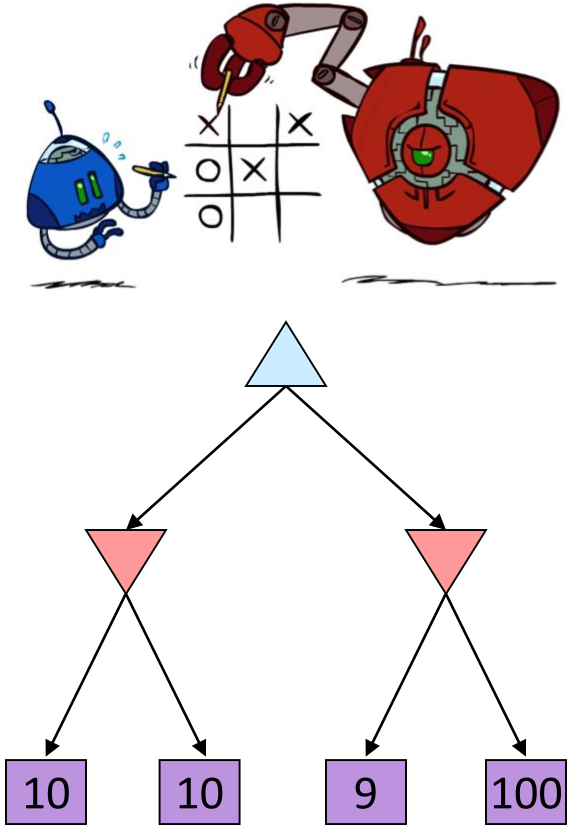


- This is a simple example of metareasoning (computing about what to compute)

Games with uncertain outcomes

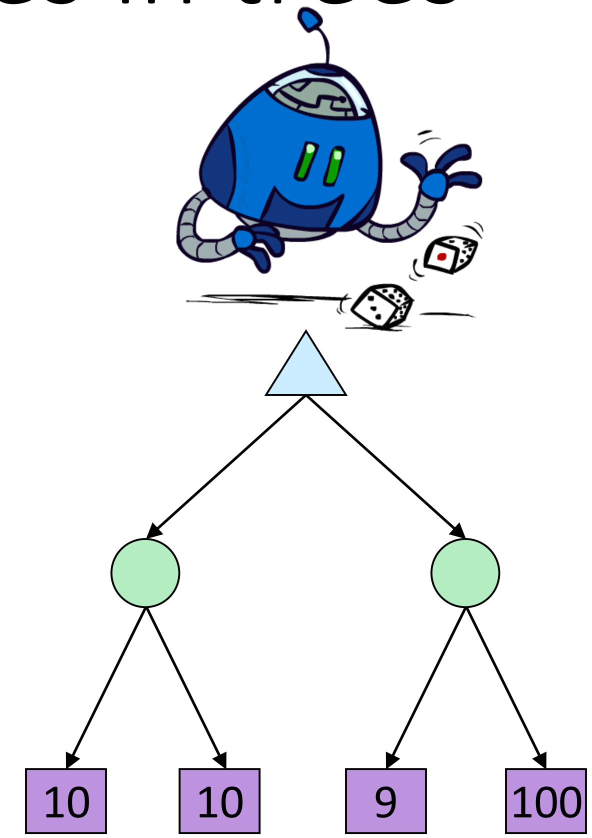


Chance outcomes in trees



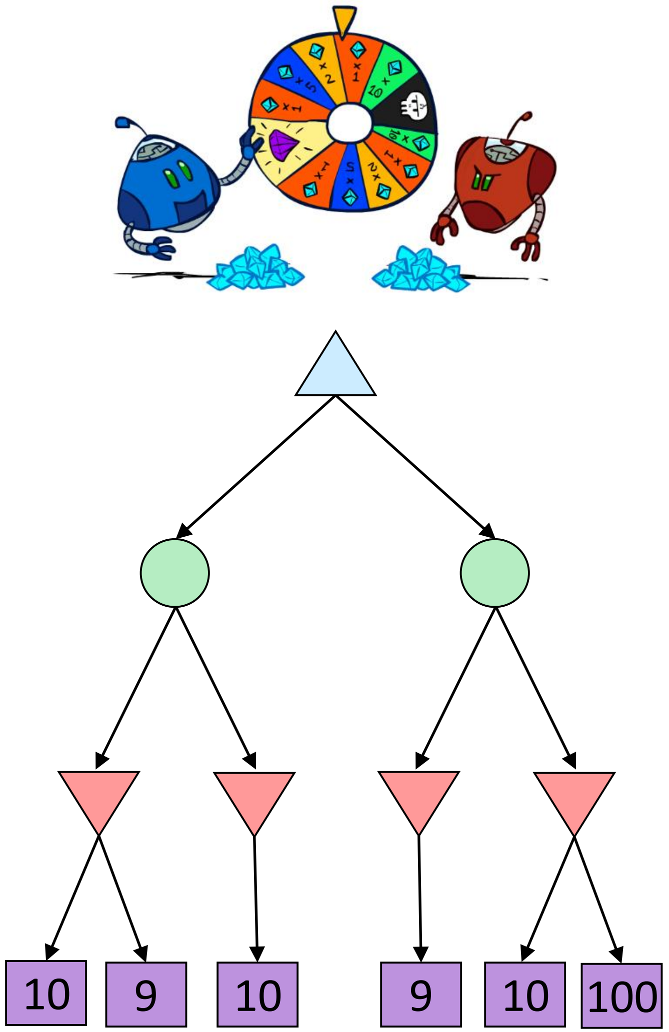
Tictactoe, chess

Minimax



Tetris, investing

Expectimax



Backgammon, Monopoly

Expectiminimax

Minimax

function decision(s) returns an action

return the action a in Actions(s) with the highest value(Result(s,a))

function value(s) returns a value

if Terminal-Test(s) then return Utility(s)

if Player(s) = MAX then return $\max_{a \text{ in Actions}(s)}$ value(Result(s,a))

if Player(s) = MIN then return $\min_{a \text{ in Actions}(s)}$ value(Result(s,a))

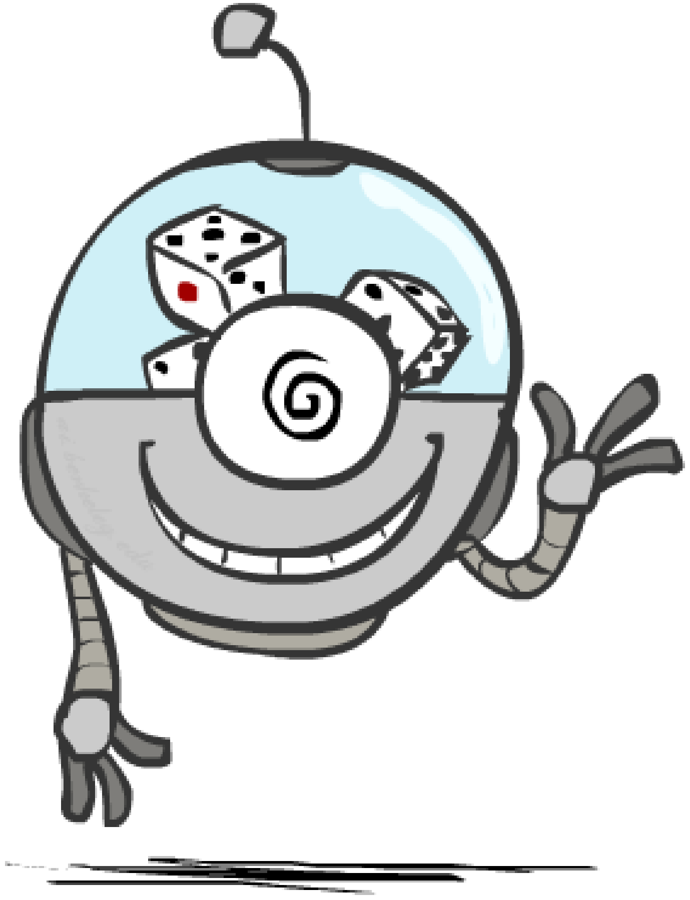
Expectiminimax

```
function decision(s) returns an action
    return the action a in Actions(s) with the highest
    value(Result(s,a))
```

$\Updownarrow$

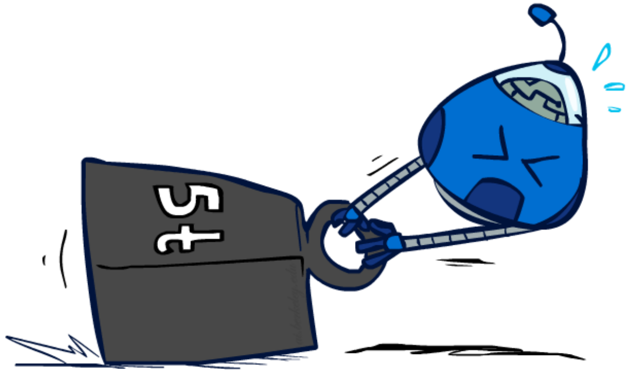
```
function value(s) returns a value
    if Terminal-Test(s) then return Utility(s)
    if Player(s) = MAX      then return max_{a in Actions(s)} value(Result(s,a))
    if Player(s) = MIN      then return min_{a in Actions(s)} value(Result(s,a))
    if Player(s) = CHANCE then return sum_{a in Actions(s)} Pr(a) * value(Result(s,a))
```

Probabilities



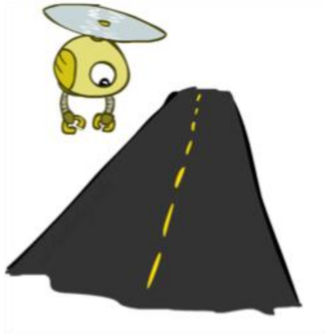
Reminder: Expectations

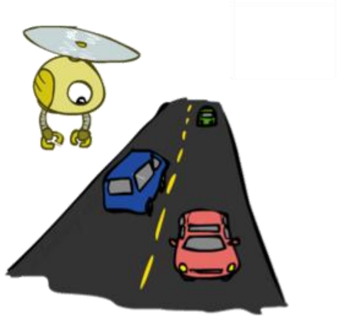
- The expected value of a random variable is the average, weighted by the probability distribution over outcomes



- Example: How long to get to the airport?

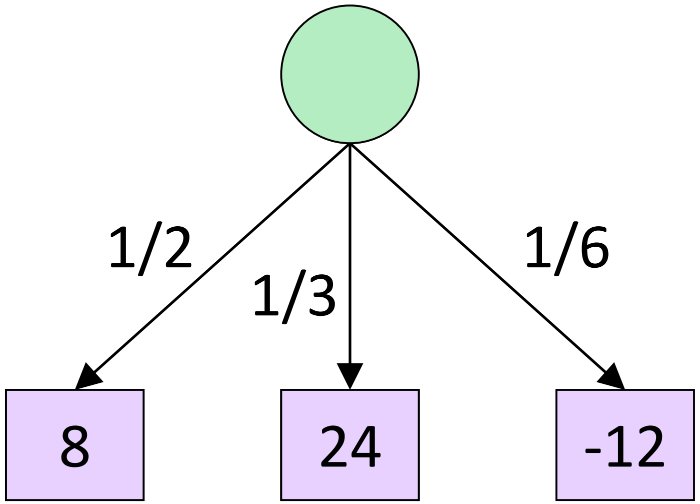
Time:	20 min		30 min		60 min		
	x	+	x	+	x	→	35 min
Probability:	0.25		0.50		0.25		





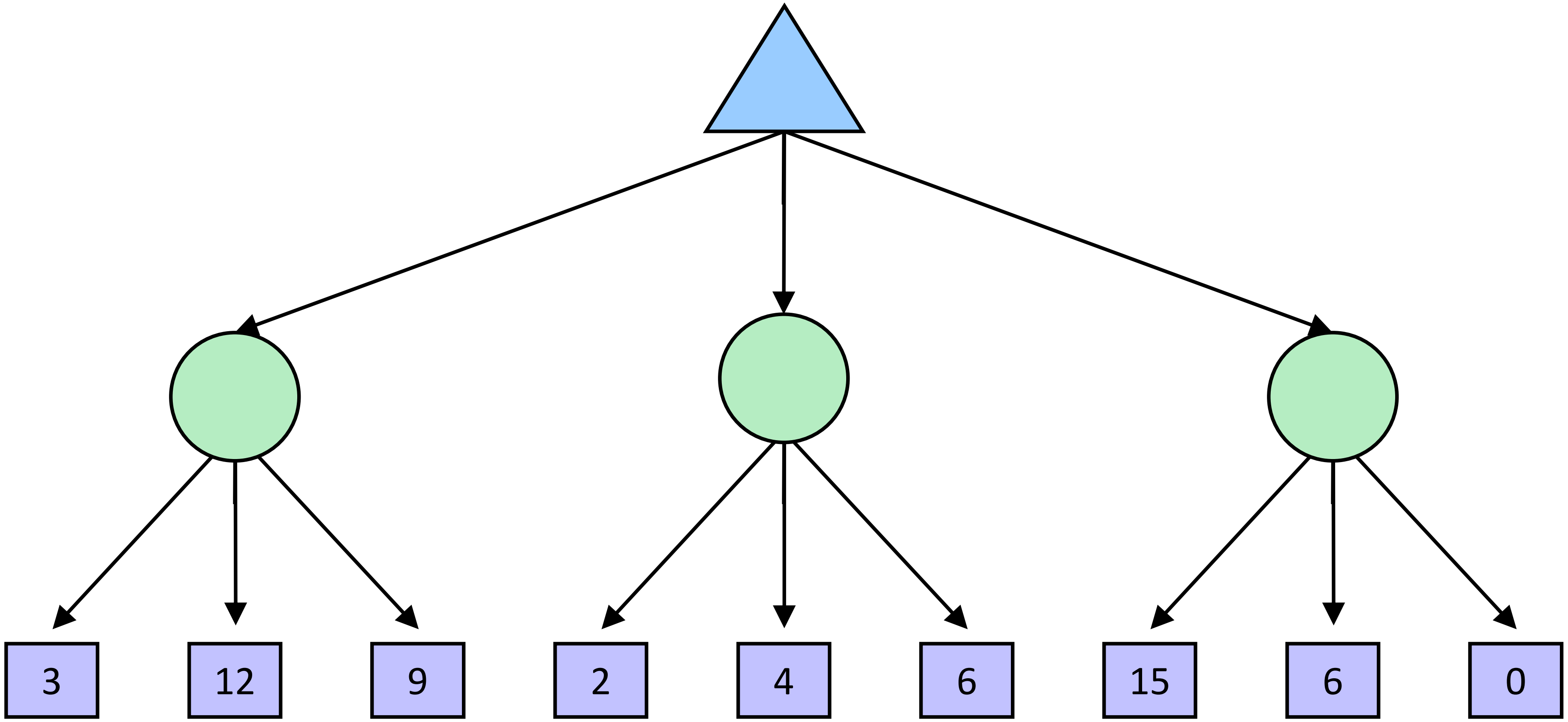
Expectimax Pseudocode

$\text{sum}_{a \text{ in Action}(s)} \Pr(a) * \text{value}(\text{Result}(s,a))$

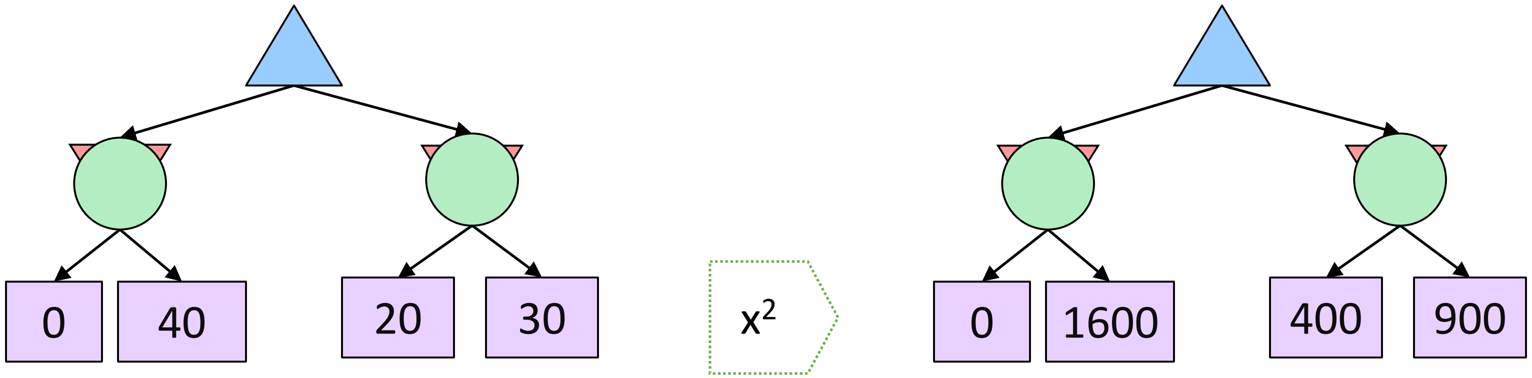


v = (1/2) (8) + (1/3) (24) + (1/6) (-12) = 10

Expectimax Example



What Values to Use?



x>y => f(x)>f(y)

f(x) = Ax+B where A>0

- For worst-case minimax reasoning, evaluation func n scale doesn't ma
  - We just want better states to have higher evalua ons (get the ord g right)
  - Minimax decisions are ***invariant with respect to monotonic tran ormations on values***

- Expectiminimax decisions are ***invariant with respect to positive affine transformations***
- Expectiminimax evaluation functions have to be aligned with actual win probabilities!

Summary

- Multi-agent problems can require more space or deeper trees to search
- Games require decisions when optimality is impossible
  - Bounded-depth search and approximate evaluation functions
- Games force efficient use of computation
  - Alpha-beta pruning
- Game playing has produced important research ideas
  - Reinforcement learning (checkers)
  - Iterative deepening (chess)
  - Rational metareasoning (Othello)
  - Monte Carlo tree search (Go)
  - Solution methods for partial-information games in economics (poker)
- Video games present much greater challenges – lots to do!
  - $b = 10^{500}$, $|S| = 10^{4000}$, $m = 10{,}000$