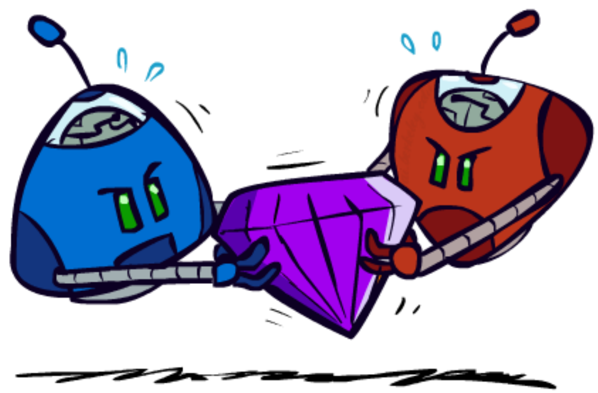
# AI: Representation and Problem Solving

## Game Theory



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: Ariel Procaccia, Fei Fang

# Announcements

- Written homework due 4/25
- Electronic assignment due next week
- Programming assignment due 5/2

- Final exam 5/9 1-4pm (Rashid Auditorium)

- You're doing great!!!

# Autonomous Agents 15-482

- Agent Architectures
- Task scheduling
- Reasoning under uncertainty
- Error monitoring
- Explanation

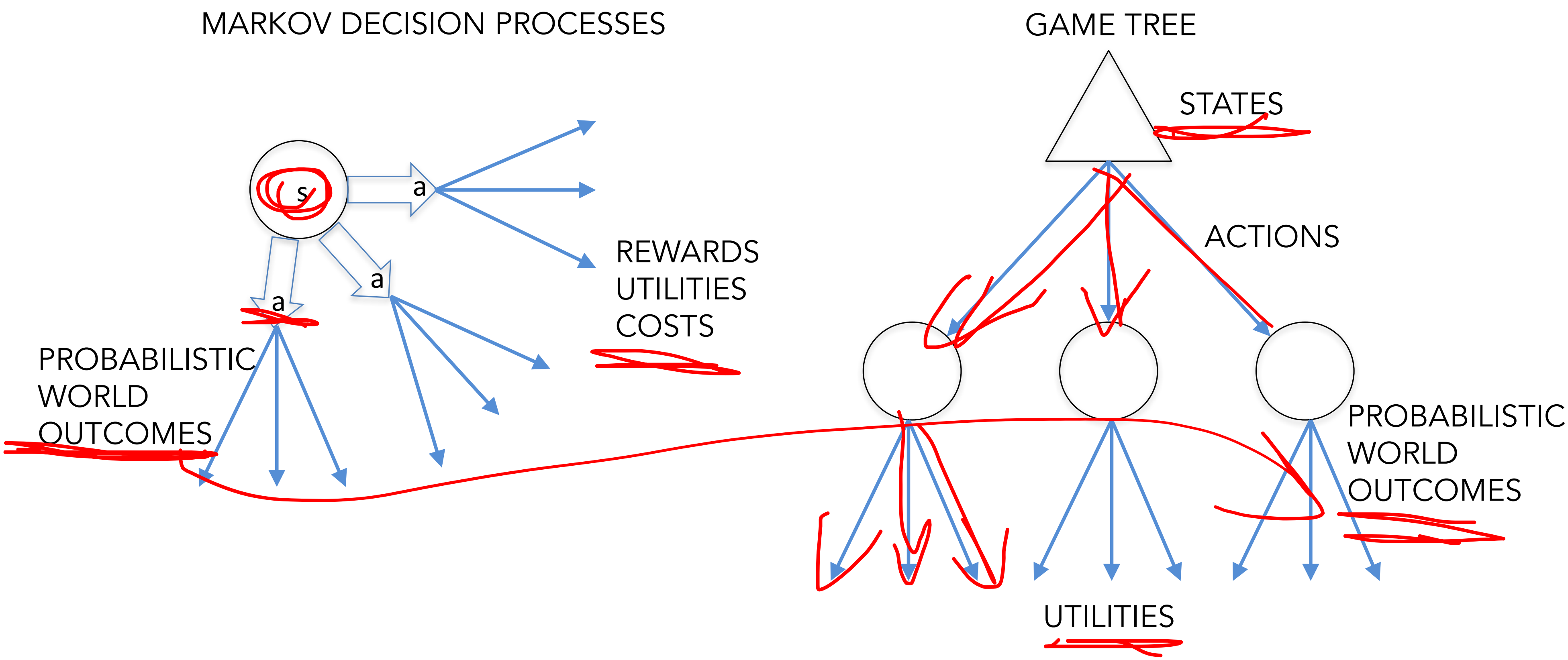
- Robotanist Project – automated greenhouses

# Nils Nilsson (1933-2019)

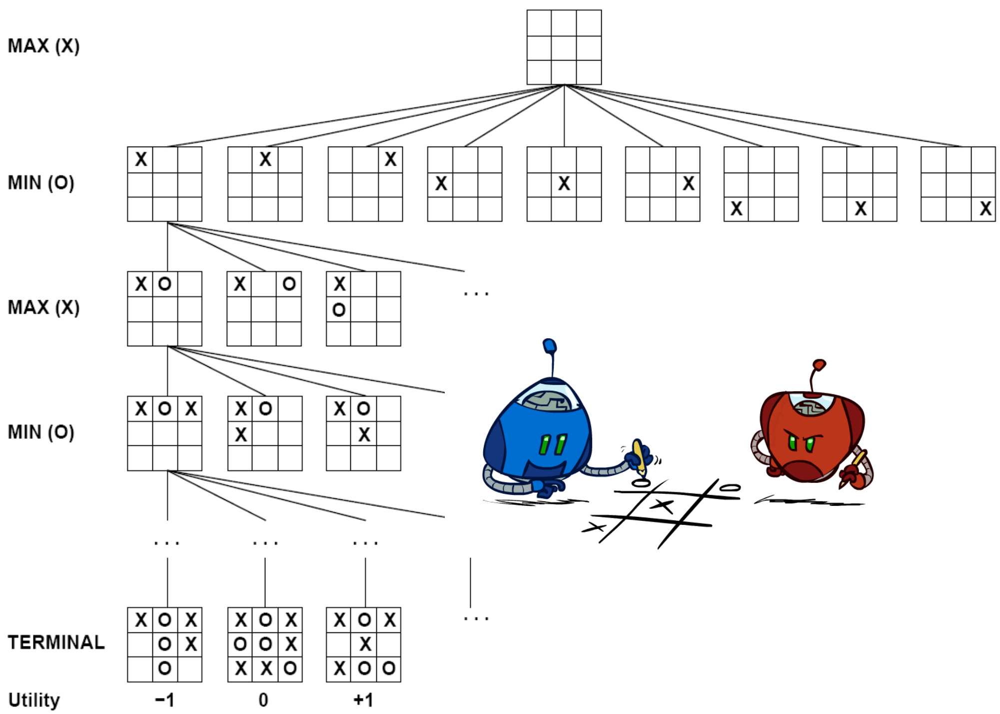
- Stanford Research Institute and Stanford University
- Inventor of A*
- Inventor of automated temporal planning
- Inventor of STRIPS classical planning framework

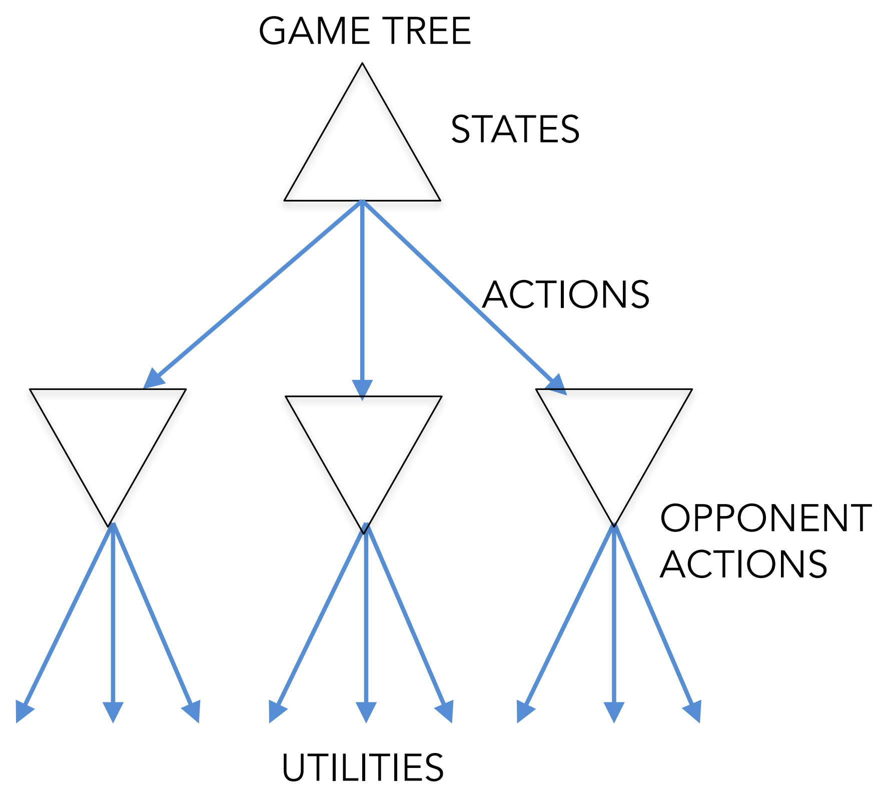
- Research interests in search, planning, knowledge representation, robotics, and more…

# Representing Actions in the World



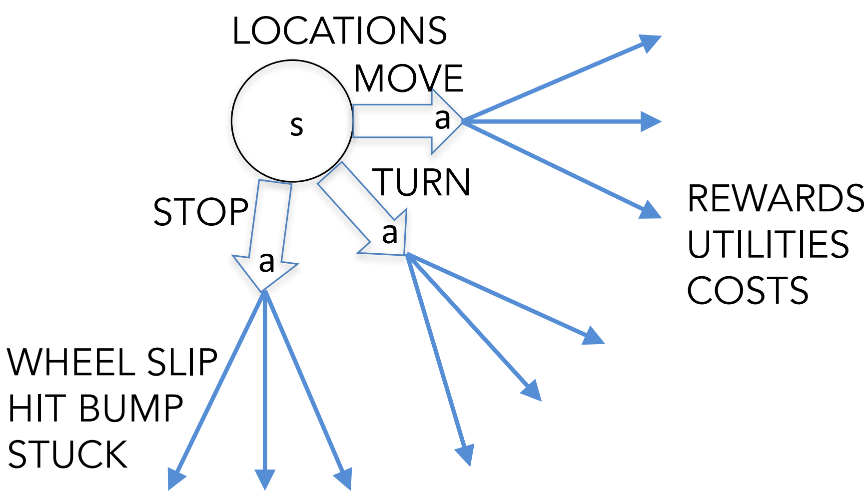
# Tic Tac Toe


MAX (X)
MIN (O)
MAX (X)
MIN (O)
TERMINAL
Utility
−1
0
+1


GAME TREE
STATES
ACTIONS
OPPONENT ACTIONS
UTILITIES

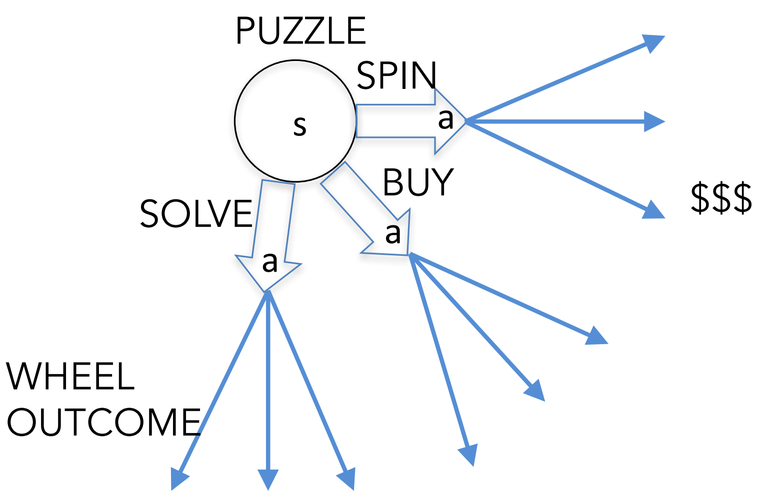
# Navigating in GHC

MARKOV DECISION PROCESSES

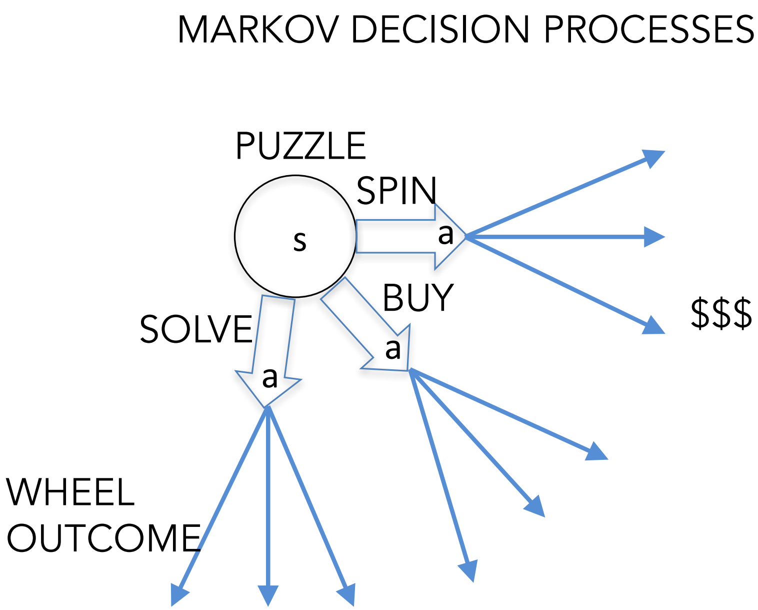


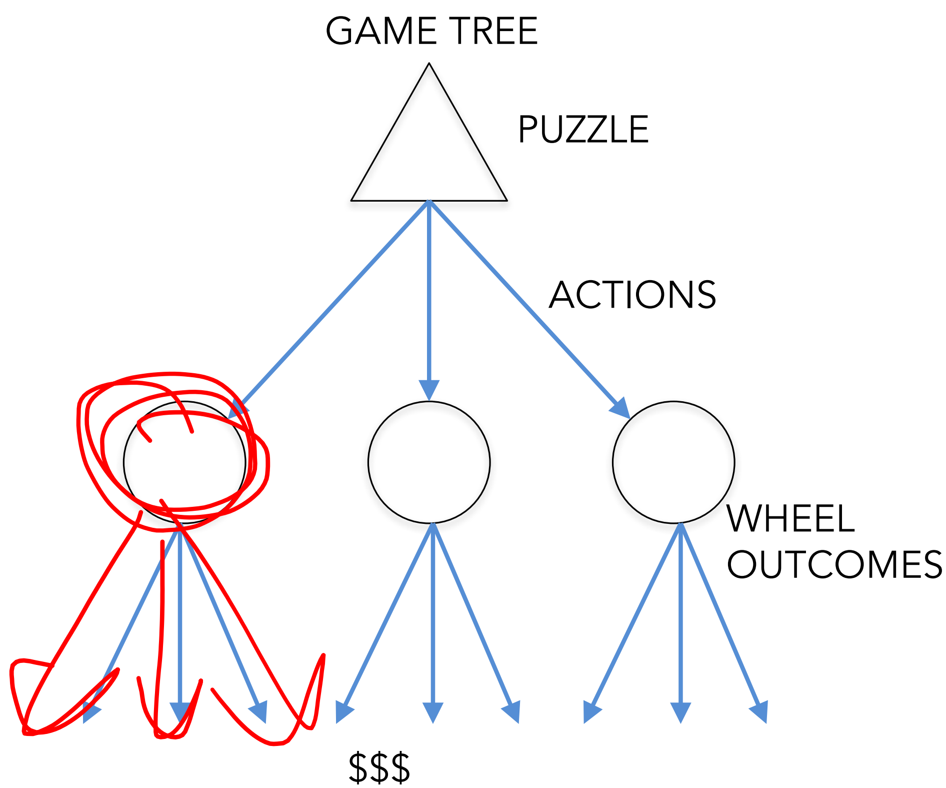

# Wheel Of Fortune

MARKOV DECISION PROCESSES

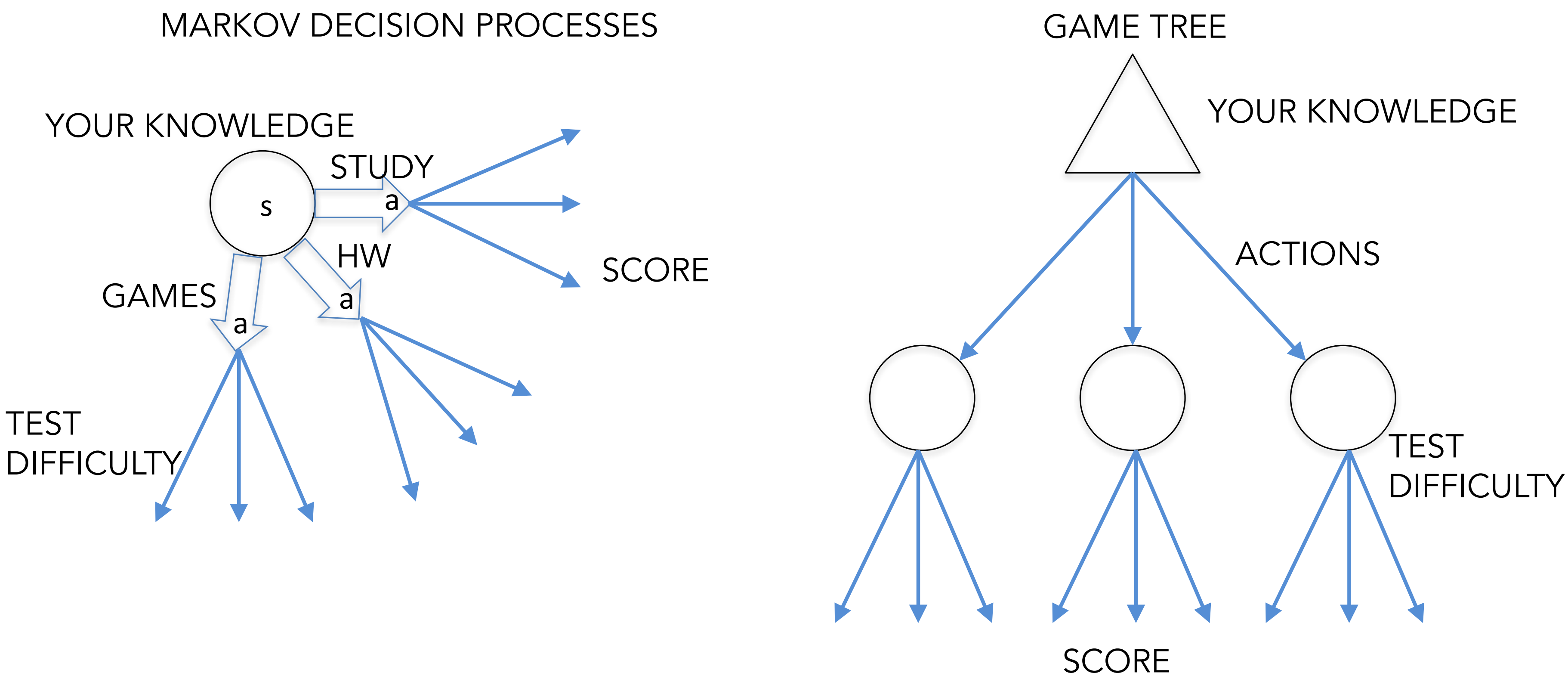



# Wheel Of Fortune



# 15-381 Exam


MARKOV DECISION PROCESSES
YOUR KNOWLEDGE
s
STUDY
a
HW
a
GAMES
a
SCORE
TEST
DIFFICULTY
GAME TREE
YOUR KNOWLEDGE
ACTIONS
TEST
DIFFICULTY
SCORE

# Decision Theory and Game Theory

**Decision Theory:** pick a strategy to maximize utility **given** world outcomes

**Game Theory:** pick a strategy for player that maximizes ***his*** utility **given** the strategies of the other players

Models are essentially the same

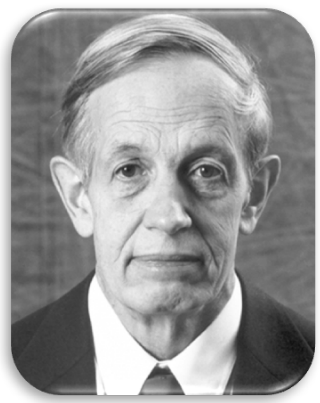
- Imagine the world is a player in the game!

# History of Game Theory

- Game theory is the study of strategic decision making (of more than one player)
- Used in economics, political science etc.

John von Neumann

John Nash



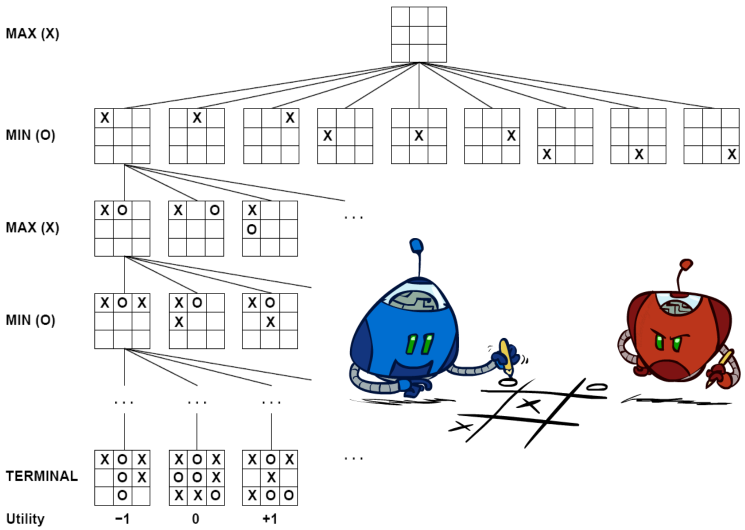
Heinrich Freiherr von Stackelberg

Winners of Nobel Memorial Prize in Economic Sciences

# Games: Extensive Form

Represent:

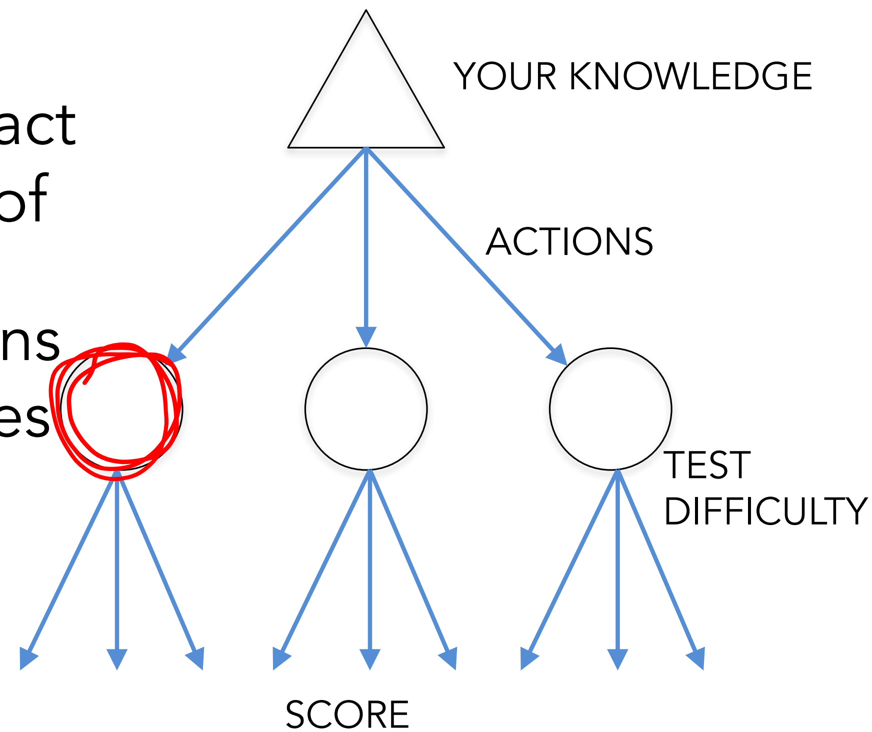
1) the players of a game
2) for every player, every opportunity they have to move
3) what each player can do at each of their moves
4) what each player knows when making every move
5) the payoffs received by everyone for all possible combo of moves



# Decisions: Extensive Form

Represent:

1) the player(s) of a game
2) every opportunity they have to act
3) what the player can do at each of their turns
4) the uncertain outcomes of actions
5) what each player knows/observes for every turn
6) the payoffs received for all possible combo of actions



# Alternative: Normal Form

- Approximate games as single shot
- Represent only actions and utilities
- Easier to determine particular properties of games

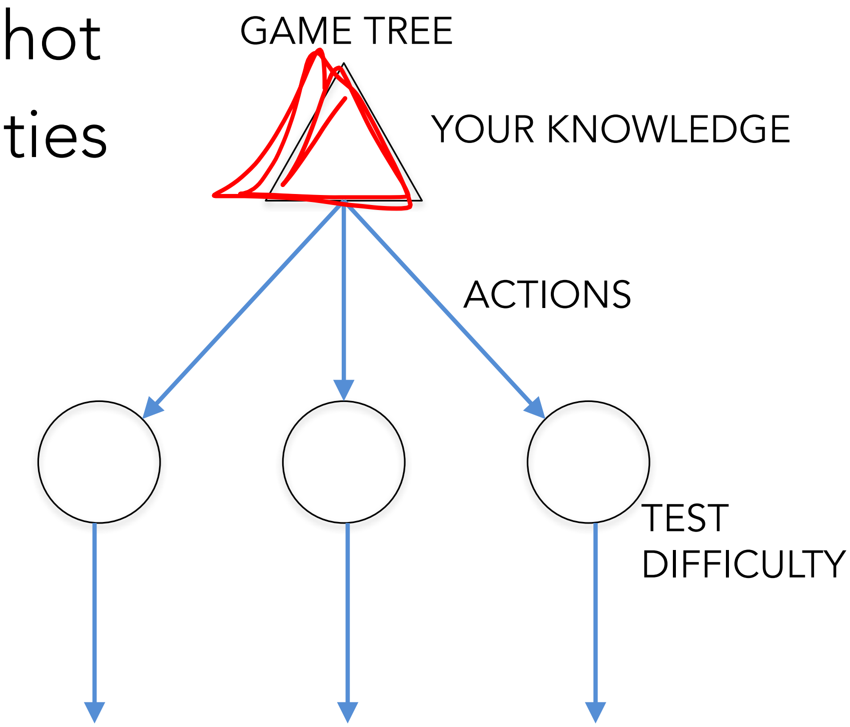
# Studying – Normal Form Game

- Approximate games as single shot
- Represent only actions and utilities

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| GRADE | 98 | 100 | 85 |

UTILITIES



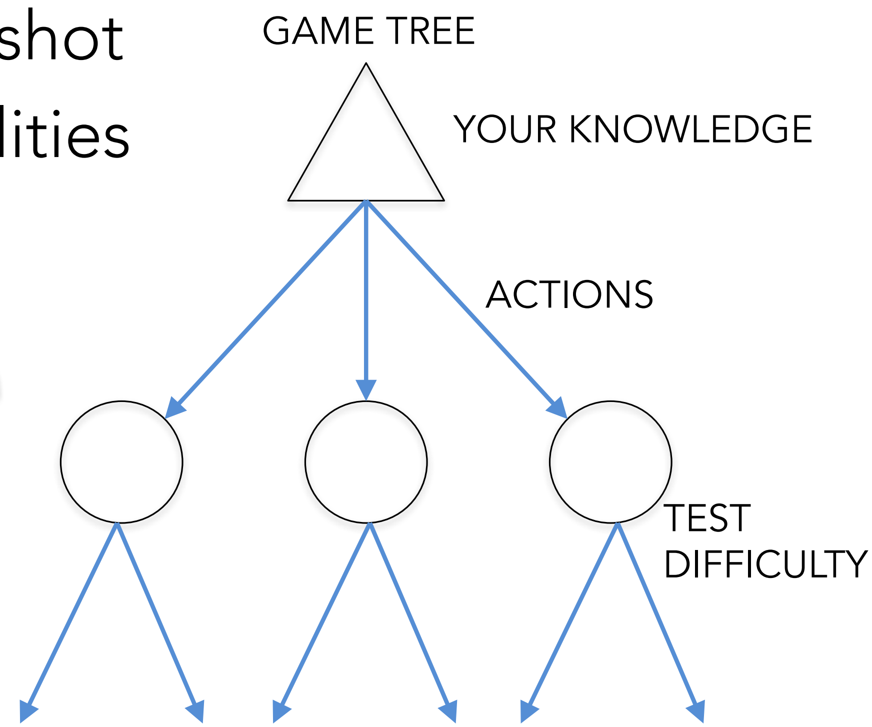
# Studying – Normal Form Game

- Approximate games as single shot
- Represent only actions and utilities

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES



# Studying - Actions and Utilities

World outcomes

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Studying - Actions and Utilities

The world acts at the same time as you choose an action

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Action/Utility Notation

Actions k for player = $a_k \in A$

Utility $u(a_k, world)$

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Questions you can ask…

What action should you take?

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Questions you can ask…

What action should you take?

- Maximize the expected utility based on world probabilities

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Questions you can ask…

What strategy (probability distribution over actions) should you use?

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Strategies

Strategy for player = $s_k \in S$: probability distribution over actions

ACTIONS

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

Strategy 1: Always cram
Strategy 2: Always do HW
Strategy 3: Always play games
Strategy 4: ½ cram, ½ do HW
…

# Pure vs Mixed Strategies

Strategy for player = $s_k \in S$: probability distribution over actions

Pure Strategy: deterministic (p=1) selection of actions

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

Strategy 1: Always cram
Strategy 2: Always do HW
Strategy 3: Always play games
Strategy 4: ½ cram, ½ do HW
…

Mixed Strategy: randomized selection

# Strategies

Strategy for player = $s_k \in S$: probability distribution over actions

Goal: Pick a strategy that maximizes utility given exam probability

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

Strategy 1: Always cram
Strategy 2: Always do HW
Strategy 3: Always play games
Strategy 4: ½ cram, ½ do HW
…

# Calculating Utilities of Pure Strategies

What is the utility of pure strategy: cram?

$$E(\text{cram}) = p(e)u(c,e) + p(h)u(c,h)$$

$$.2(98) + .8(97)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Pure Strategies

What is the utility of pure strategy: study?

$$u(cram) = u(cram, easy) * p(easy) + u(cram, hard) * p(hard)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Pure Strategies

General formula:

$$u(action) = \sum_{world} p(world) * u(action, world)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Pure Strategies

What is the utility of pure strategy: do hw?

$$(100)(.2) + 90(.8) = 92$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Pure Strategies

What is the utility of pure strategy: play game?

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Mixed Strategies

What is the utility of mixed strategy: ½ cram, ½ do hw?

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Mixed Strategies

What is the utility of mixed strategy: ½ cram, ½ do hw?

$$u(s) = \sum_{a \in s} \sum_{world} p(a, world) * u(a, world)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Mixed Strategies

What is the utility of mixed strategy: ½ cram, ½ do hw?

$$u(s) = \sum_{a \in s} p(a) * \sum_{world} p(world) * u(a, world)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Calculating Utilities of Mixed Strategies

What is the utility of mixed strategy: ½ cram, ½ do hw?

$$\left[p(cram)\sum_{world} p(world) * u(cram, world)\right] + \left[p(hw)\sum_{world} p(world) * u(hw, world)\right]$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(EASY) = .2
P(HARD) = .8

# Grocery Shopping Transportation Decision

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE |
|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 |
| RAIN | -2 | -4 | -1 | 0 |

# Polls 1 and 2

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE |
|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 |
| RAIN | -2 | -4 | -1 | 0 |

How many pure strategies to do you have?

A) 1 B) 2 C) 3 D) 4 E) Infinite

How many mixed strategies do you have?

A) 4 B) 8 C) 16 D) 64 E) Infinite

# Poll 3

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE |
|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 |
| RAIN | -2 | -4 | -1 | 0 |

What is your best pure strategy?

A) bike B) walk C) bus D) drive E) it depends

# Poll 4

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE | |
|---|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 | P=.5 |
| RAIN | -2 | -4 | -1 | 0 | P=.5 |

What is your best pure strategy?

A) bike B) walk C) bus D) drive E) it depends

# Poll 5

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE | |
|---|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 | P=.5 |
| RAIN | -2 | -4 | -1 | 0 | P=.5 |

What is the utility of a ¼ walk, ¼ bike, and ½ drive strategy?

A) -1/8 B) -1/4 C) -1/2 D) 1/8 E) 1/2

# Game Theory

# Game: Rock, Paper, Scissors

Each player simultaneously picks rock, paper, or scissors
Rock beats scissors, scissors beats paper, paper beats rock

# Game: Rock, Paper, Scissors

Each player simultaneously picks rock, paper, or scissors
Rock beats scissors, scissors beats paper, paper beats rock

P1's Actions

$A_1 = \{rock, paper, scissors\}$

P2's Actions

$A_2 = \{rock, paper, scissors\}$

# Joint Utilities

When both players choose their actions, they receive a utility based on both of their choices

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | | | |
| | PAPER | | | |
| | SCISSORS | | | |

JOINT UTILITIES

# Joint Utilities

When both players choose their actions, they receive a utility based on both of their choices

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Poll 5

What is P1's utility of P1 picking rock and P2 picking scissors?

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Normal Form Notation

Players 1…M
Pure Strategies $S_i = \{s_{i,1}, s_{i,2}, \dots s_{i,n}\}$ for player i
Utility functions $u_i(s_1, s_2, \dots, s_m)$ that maps a strategy per player to a reward for player i

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Normal Form Notation

Players 1…M

Pure Strategies $S_i = \{s_{i,1}, s_{i,2}, \dots s_{i,n}\}$ for player i

Utility functions $u_i(s_1, s_2, \dots, s_m)$ that maps a strategy per player to a reward for player i (not necessarily pure strategies)

Notation Alert!

We can write a strategy profile of one strategy per player as

$\vec{s} = (s_1, s_2, \dots, s_m)$

and therefore i's utility as $u_i(\vec{s})$

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Strategies for Games

Goal: pick a strategy for player $i$ that maximizes ***his*** utility **given** the strategies of the other players

Pure Strategies:

P2 always picks rock
P1 should _____________

P2 always picks paper
P1 should _____________

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Strategies for Games

Goal: pick a strategy for player $i$ that maximizes ***his*** utility **given** the strategies of the other players

Mixed Strategies:

P2 randomly chooses between 50% rock and 50% paper
P1 should ____________

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Zero-Sum Games

If each cell in the table sums to 0, the game is zero-sum

$$\forall \vec{s} \sum_i u_i(\vec{s}) = 0$$

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Zero-Sum Games

If each cell in the table sums to 0, the game is zero-sum

$$\forall \vec{s} \sum_i u_i(\vec{s}) = 0$$

Is Rock, Paper, Scissors zero-sum?

Is TicTacToe zero-sum?

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Dominant Strategies

A strategy for player i $s_{i,k}$ is strictly dominant if it is better than all other strategies for player i no matter the opponent j's strategy

$$\forall j, \forall n \neq k, u_i(s_{i,k}, s_j) > u_i(s_{i,n}, s_j)$$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

A strategy for player i $s_{i,k}$ is weakly dominant if it is better than all other strategies for player i no matter the opponent j's strategy

$$\forall j, \forall n \neq k, u_i(s_{i,k}, s_j) \geq u_i(s_{i,n}, s_j)$$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

For player Alphabet, strategy A's utilities are the highest compared to B,C,D,E for all of RomanNum's strategies

$$\forall j \in \{i, ii, iii, iv\}, \forall n \neq A, u_{Alphabet}(s_A, s_j) > u_i(s_n, s_j)$$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

For player Alphabet, strategy A's utilities are the highest compared to B,C,D,E for all of RomanNum's strategies

Alphabet should always play A!

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

For player RomanNum, strategy iv's utilities are the highest compared to i,ii,iii for all of Alphabet's strategies

$$\forall j \in \{A, B, C, D, E\}, \forall n \neq iv, u_{RomanNum}(s_{iv}, s_j) > u_i(s_n, s_j)$$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

For player RomanNum, strategy iv's utilities are the highest compared to i,ii,iii for all of Alphabet's strategies

RomanNum should always play iv!

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Poll 6: Is there always a dominant strategy?

Yes or No?

# Is there always a dominant strategy?

No! There is no dominant strategy in Tic Tac Toe, for example.

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Prisoner's Dilemma

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma: Normal Form

2 Players {1,2}
Each as 2 actions {Cooperate,Defect}
Utilities in table:

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma Poll

Is there a dominant strategy?

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Is there a dominant strategy? Yes!

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Is there a dominant strategy? Yes!

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Is there a dominant strategy? Yes!

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

What is the best joint strategy for both prisoners?

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Best joint strategy: prisoners cooperate

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Measure of Social Welfare

The sum of the utilities of the players is the social welfare

SW(C,C) = -2

SW(C,D) = -6

SW(D,D) = -6

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Goal: pick a strategy for player $i$ that maximizes ***his*** utility **given** the strategies of the other players

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Each prisoner would profit by switching to defection assuming that the other prisoner continues to cooperate

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

Each prisoner would profit by switching to defection assuming that the other prisoner continues to cooperate

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

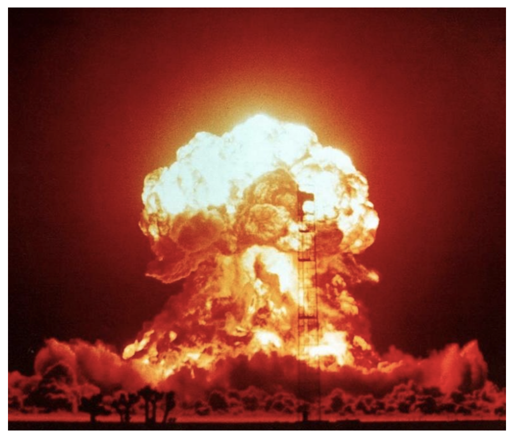
If they both trust that the other prisoner will cooperate, each should defect. But both defecting results in lower scores!

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

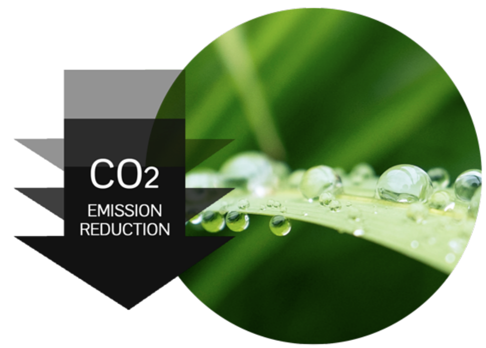
# Tragedy of the Commons

Individuals act in their own self-interest contrary to the common good


Political Ads



Nuclear Arms Race




CO2 Emissions

# Nash Equilibrium

Nash Equilibria are strategy profiles $\vec{s}$ where none of the participants benefit from unilaterally changing their decision.

$$\forall i\ u_i(\vec{s}) \geq u_i(neighbor(\vec{s}))$$

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Nash Equilibrium

Nash Equilibria are strategy profiles $\vec{s}$ where none of the participants benefit from unilaterally changing their decision.

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Nash Equilibrium

NOT A NASH EQUILIBRIUM - participants benefit from unilaterally changing their decision.

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Strict Nash Equilibrium

Strict Nash Equilibria are Nash Equilibria where the "neighbor" strategy profiles have strictly less utility (<u)

$$\forall i \; u_i(\vec{s}) > u_i(neighbor(\vec{s}))$$

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Professor's Dilemma!

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Poll 7: What is/are the Nash Equilibrium?

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Poll 7: Nash Equilibrium Example

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Poll 7.5: Which are Strict Nash Equilibria?

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Strict Nash Equilibria?

Effort/Study is a Strict NE, Slack/Games is not!

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Finding a Pure Nash Equilibrium

Pure Nash Equilibria have a pure strategy

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

# Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

# Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
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| U | 10,3 | 1,5 | 5,4 |
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| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

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- Option 1: Examine each state and determine if it fits the criteria
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| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

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- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

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- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

# Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

→

| | C |
|---|---|
| U | 1,5 |
| M | 2,4 |

# Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a *dominating* strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly *dominated* strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

→

| | C |
|---|---|
| U | 1,5 |
| M | 2,4 |

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- Option 1: Examine each state and determine if it fits the criteria
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| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

| | C |
|---|---|
| U | 1,5 |
| M | 2,4 |

| | C |
|---|---|
| M | 2,4 |

# Finding Nash Equilibrium Example 1

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 1

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | **2,4** | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | **9,1** | 3,0 |
| iv | 6,7 | 9,5 | **5,5** | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | **2,4** | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | **9,1** | 3,0 |
| iv | 6,7 | 9,5 | **5,5** | 8,5 | 4,5 |

No longer strict dominant strategies!

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 8,5 | 4,5 |

>

D is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 8,5 | 4,5 |

D is weakly dominated by B

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | | 4,5 |

<

iii is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | | 4,5 |

<

i is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | | 4,5 |

>

E is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | | 4,5 |

>

C is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | | 4,5 |

>

B is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | | | | | |
|---|---|---|---|---|---|
| | A | | | | |
| | | | | | |
| ii | 3,8 | | | | |
| | | | | | |
| iv | 6,7 | | | | |

ii is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| iv | 6,7 | | | | |

# Rock, Paper, Scissors – Nash Equlibrium?

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Rock, Paper, Scissors – Not with pure strategies!

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Nash Equilibria always exist in finite games

Nash 1950
If there are a finite number of players and each player has a finite number of actions, there always exists a Nash Equilibrium

The NE may be pure or it may be a mixed strategy

# Calculating Utilities of Mixed Strategies

Decision Theory Version:

$$u(s) = \sum_{a \in s} \sum_{world} p(a, world) * u(a, world)$$

# Calculating Utilities of Mixed Strategies

Game Theory Version:

$$u(\vec{s}) = \sum_{(s_1, s_2, \ldots)} p(\vec{s}) * u(\vec{s}) = \sum_{s} u(\vec{s}) \prod_{player\ i} p_i(s_i)$$

# P1 Utility of P1=(½, ½, 0), P2=(0, ½, ½)

RR p(RR) + RP p(RP) + RS p(RS)

+ PR p(PR) + . . . .

0 · 0 + -1 · 1/4

. . . .

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Poll 8: U1? P1=(1/3, 1/3, 1/3), P2=(1/3, 1/3, 1/3)

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Poll 8: U1? P1=(1/3, 1/3, 1/3), P2=(1/3, 1/3, 1/3)

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Is this a mixed strategy equilibrium?

P1=(½, ½, 0), P2=(0, ½, ½)

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK<br>U(s) = 0 | 0,0 | -1,1 | 1,-1 |
| | PAPER<br>U(s) = -.25 | 1,-1 | 0,0 | -1,1 |
| | SCISSORS<br>U(s) = 0 | -1,1 | 1,-1 | 0,0 |

# Is this a mixed strategy equilibrium?

P1=(½, ½, 0), P2=(0, ½, ½)

No! P1 doesn't want to play paper because their utility is lower

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK<br>U(s) = 0 | 0,0 | -1,1 | 1,-1 |
| | PAPER<br>U(s) = -.25 | 1,-1 | 0,0 | -1,1 |
| | SCISSORS<br>U(s) = 0 | -1,1 | 1,-1 | 0,0 |

# Finding the Mixed Strategy Nash Equilibrium

What features of a mixed strategy qualify it as a NE?

There is no reason for either player to deviate from their strategy, which occurs when the utilities of the weighted actions are equal!

# Finding the Mixed Strategy Nash Equilibrium

What features of a mixed strategy qualify it as a NE?

There is no reason for either player to deviate from their strategy, which occurs when the utilities of the weighted actions are equal!

# Finding the Mixed Strategy Nash Equilibrium

P1

P2

$q_1 = q_2 = 1/3$

$p_1 = p_2 = 1/3$

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Another Mixed Strategy NE

P1 F: $u(FF)q + u(FB)(1-q) =$

B: $u(BF)q + u(BB)(1-q)$

$q = .4$

P2 $u(FF)p + u(FB)(1-p) =$

$u(BF)p + u(BB)(1-p)$

$p = .3$

| TENNIS | | Receiver | |
|---|---|---|---|
| | | F | B |
| Server | F ($p$) | 90,10 | 20,80 |
| | B ($1-p$) | 30,70 | 60,40 |

# Other Properties of Strategies

Correlated Equilibrium

Pareto Optimal/Dominated

# Correlated Equilibrium

Suppose a mediator computes the best combined strategy (s1,s2) for p1 and p2, and shares s1 with p1 and s2 with p2

The strategy is a CE if $\forall s_1' \in S_1$

$$\sum_{a_1 \in s_1} \sum_{a_2 \in s_2} p(a_1, a_2) u(a_1, a_2) \geq \sum_{a'_1 \in s'_1} \sum_{a_2 \in s_2} p(a'_1, a_2) u(a'_1, a_2)$$

And the same for s2.

# Solving for Correlated Equilibrium

We can solve for CE's using linear programs

Find (s1,s2) s.t.

$$\forall s_1, s_2, s'_1, s'_2 \sum_{a_1 \in s_1} \sum_{a_2 \in s_2} p(a_1, a_2) u(a_1, a_2) \geq \sum_{a'_1 \in s'_1} \sum_{a_2 \in s_2} p(a'_1, a_2) u(a'_1, a_2),$$

$$\sum_{a_1 \in s_1} \sum_{a_2 \in s_2} p(a_1, a_2) u(a_1, a_2) \geq \sum_{a'_1 \in s'_1} \sum_{a_2 \in s_2} p(a_1, a'_2) u(a_1, a'_2),$$

$$\sum_{a_1, a_2} p(a_1, a_2) = 1, \forall a_1, a_2 \; p(a_1, a_2) \in [0,1]$$

# Pareto Optimal and Pareto Dominated

- An outcome $u(\mathbf{s}) = \langle u_1(\mathbf{s}), \dots, u_n(\mathbf{s}) \rangle$ is Pareto optimal if there is no other outcome that all players would prefer, i.e., each player gets higher utility
  - At least one player would be disappointed in changing strategy
- An outcome $u(\mathbf{s}) = \langle u_1(\mathbf{s}), \dots, u_n(\mathbf{s}) \rangle$ is Pareto dominated by another outcome if all the players would prefer the other outcome

# Summary

Vocabulary

- Pure/Mixed Strategies (and calculating them)
- Zero-Sum Games
- Dominant vs Dominated Strategies
- Strict/Weak Nash Equilibrium
- Tragedy of the Commons
- Correlated Equilibrium
- Pareto Optimal/Dominated
- Social Welfare