# Announcements

## Assignments:

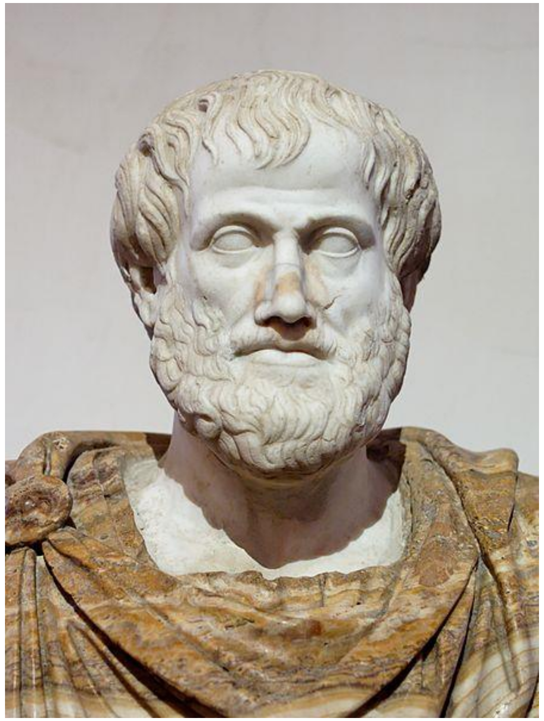
- HW5
  - Due Tue 2/26, 10 pm
- HW6 and P3
  - Coming soon

## Travel

- Pat out Wed 2/27, back for Mon 3/4
  - SIGCSE 2019, Minneapolis

# AI: Representation and Problem Solving

## First-Order Logic



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, http://aima.eecs.berkeley.edu

# Outline

1. Need for first-order logic
2. Syntax and semantics
3. Planning with FOL
4. Inference with FOL

# Pros and Cons of Propositional Logic

- Propositional logic is **declarative**: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- Propositional logic is **compositional**: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends oncontext)
- Propositional logic has very limited expressive power (unlike natural language)

  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

# Pros and Cons of Propositional Logic

## Rules of chess:

- 100,000 pages in propositional logic
- 1 page in first-order logic

## Rules of pacman:

- $\forall x,y,t\ At(x,y,t) \Leftrightarrow [At(x,y,t\text{-}1) \wedge \neg\exists\ u,v\ Reachable(x,y,u,v,Action(t\text{-}1))]\ \vee$
  $[\exists\ u,v\ At(u,v,t\text{-}1) \wedge Reachable(x,y,u,v,Action(t\text{-}1))]$

# First-Order Logic (First-Order Predicate Calculus)

Whereas propositional logic assumes world contains facts,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries, …
- Relations: red, round, bogus, prime, multistoried …,
brother of, bigger than, inside, part of, has color, occurred after, owns, …
- Functions: father of, best friend, third inning of, one more than, end of, …

# Logics in General

| Language | What exists in the world | What an agent believes about facts |
|---|---|---|
| Propositional logic | Facts | true / false / unknown |
| First-order logic | facts, objects, relations | true / false / unknown |
| Probability theory | facts | degree of belief |
| Fuzzy logic | facts + degree of truth | known interval value |

# Syntax of FOL

## Basic Elements

| | |
|---|---|
| Constants | $KingJohn, 2, CMU, \ldots$ |
| Predicates | $Brother, >, \ldots$ |
| Functions | $Sqrt, LeftLegOf, \ldots$ |
| Variables | $x, y, a, b, \ldots$ |
| Connectives | $\wedge \vee \neg \Rightarrow \Leftrightarrow$ |
| Equality | $=$ |
| Quantifiers | $\forall \exists$ |

# Syntax of FOL

Atomic sentence = $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term = $function(term_1, \ldots, term_n)$
or $constant$
or $variable$

Examples

$Brother(KingJohn, RichardTheLionheart)$

$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

# Syntax of FOL

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

Examples

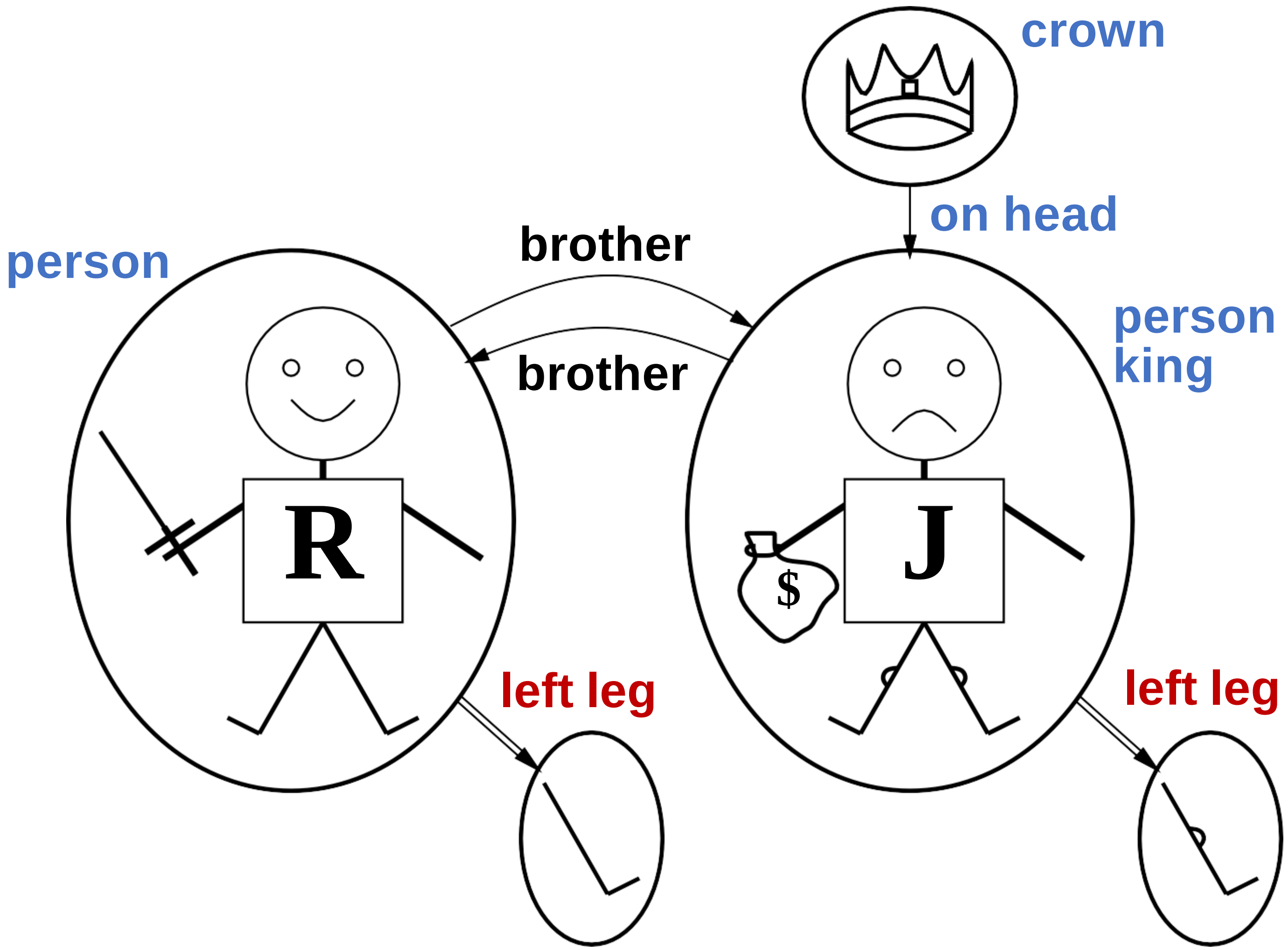
$$Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$$

$$>(1, 2) \vee \leq(1,2)$$

$$>(1, 2) \wedge \neg >(1,2)$$

# Models for FOL

## Example

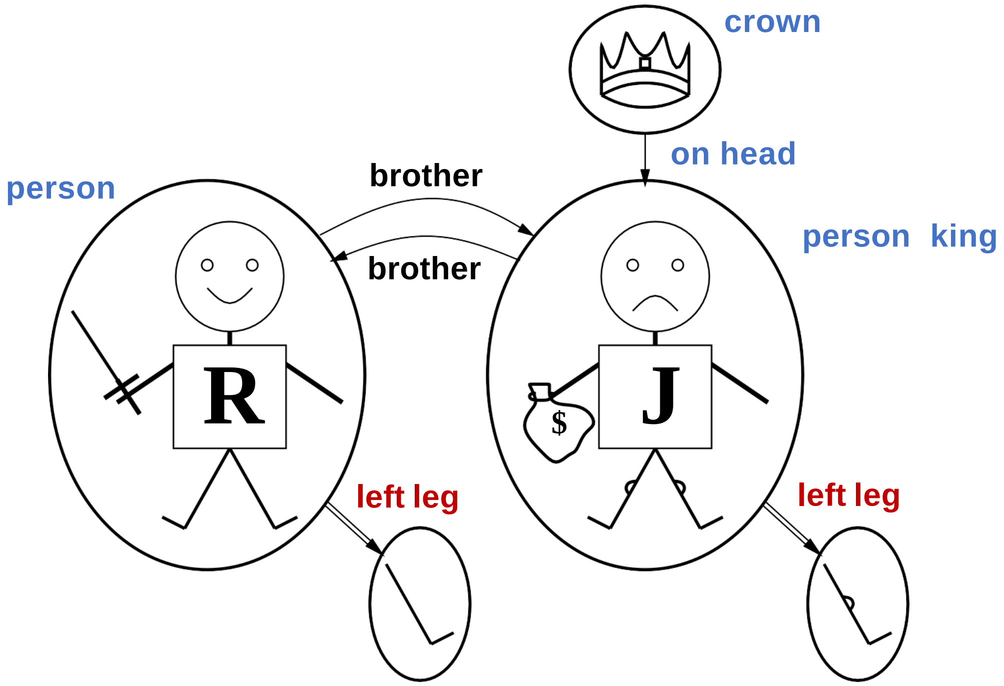


# Models for FOL

*Brother*(*Richard*, *John*)

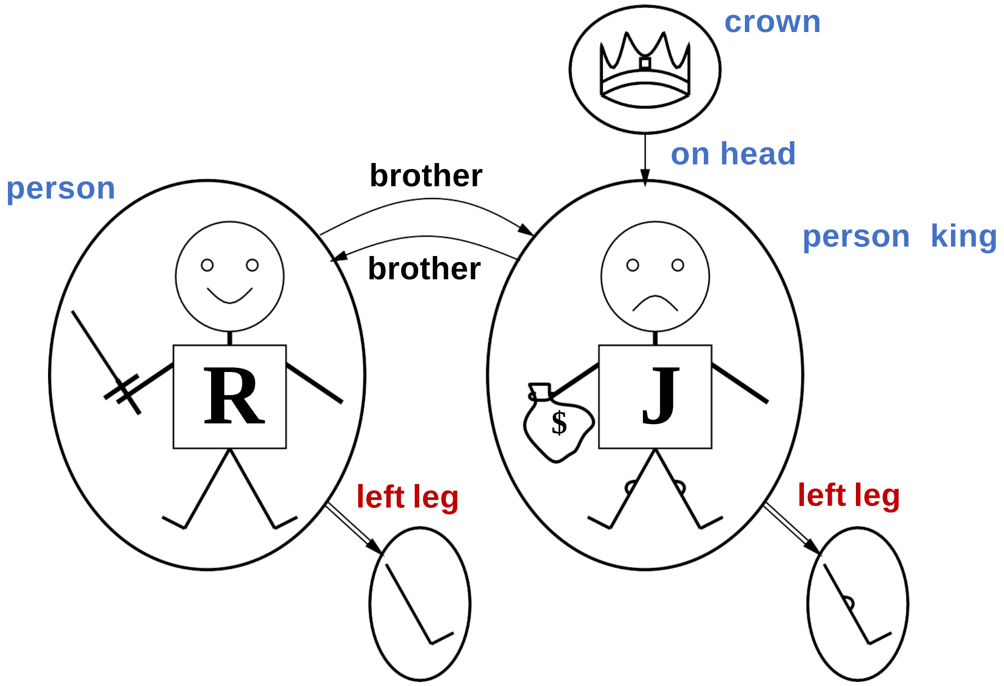
Consider the interpretation in which:

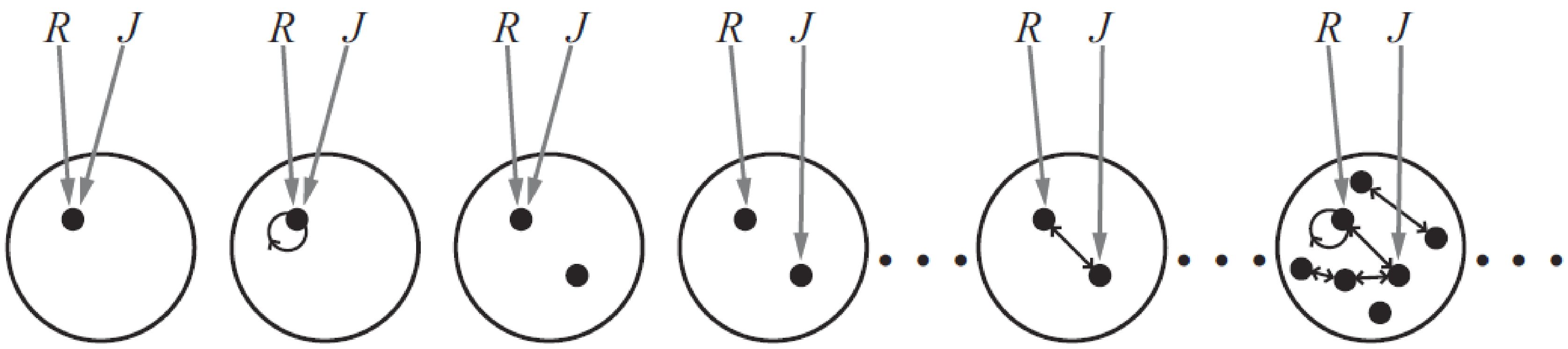
*Richard* → Richard the Lionheart
*John* → the evil King John
*Brother* → the brotherhood relation



# Model for FOL

Lots of models!



# Model for FOL

Lots of models!

Entailment in propositional logic can be computed by enumerating models

We can enumerate the FOL models for a given KB vocabulary:

For each number of domain elements $n$ from 1 to $\infty$
  For each $k$-ary predicate $P_k$ in the vocabulary
    For each possible $k$-ary relation on $n$ objects
      For each constant symbol $C$ in the vocabulary
        For each choice of referent for $C$ from $n$ objects . . .

Computing entailment by enumerating FOL models is not easy!

# Truth in First-Order Logic

Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for

- constant symbols → objects
- predicate symbols → relations
- function symbols → functional relations

model

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true:

iff the objects referred to by $term_1, \ldots, term_n$

are in the relation referred to by $predicate$

# Models for FOL

Consider the interpretation in which:

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*John* → the evil King John
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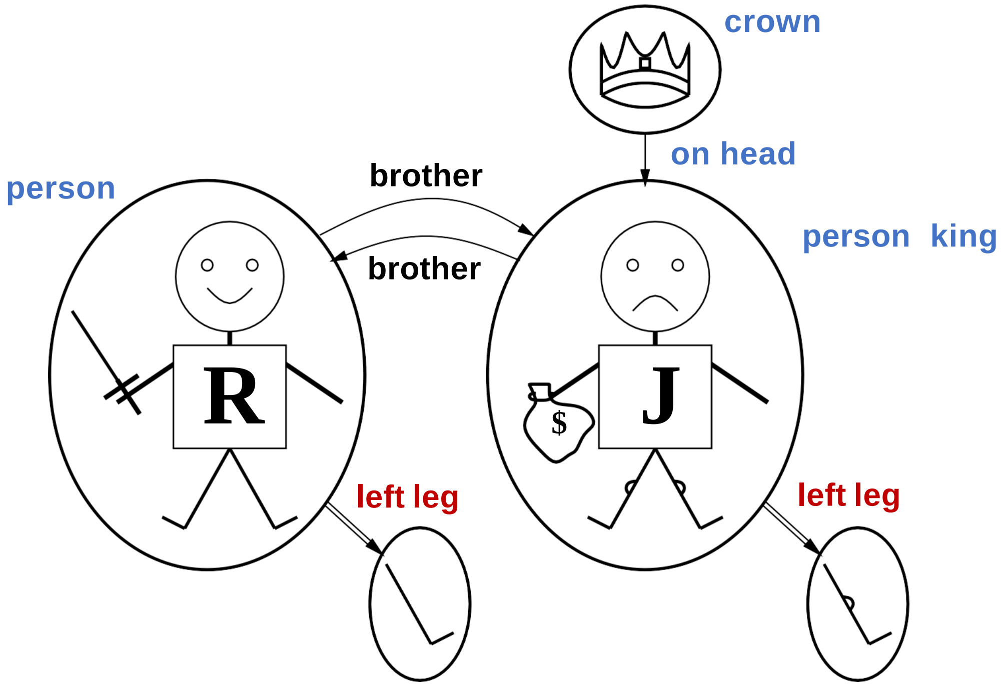



Under this interpretation, *Brother*(*Richard*, *John*) is true just in the case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

# Universal Quantification

$\forall \langle variables \rangle \quad \langle sentence \rangle$

Everyone at the banquet is hungry:

$\forall x \quad At(x, Banquet) \Rightarrow Hungry(x)$

$\forall x \quad P$ is true in a model $m$ iff $P$ is true with $x$ being **each** possible object in the model

**Roughly** speaking, equivalent to the conjunction of instantiations of $P$

$$(At(KingJohn, Banquet) \Rightarrow Hungry(KingJohn))$$
$$\wedge \ (At(Richard, Banquet) \Rightarrow Hungry(Richard))$$
$$\wedge \ (At(Banquet, Banquet) \Rightarrow Hungry(Banquet))$$
$$\wedge \ \ldots$$

# Universal Quantification

Common mistake

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x\ At(x,\ Banquet) \wedge Hungry(x)$$

means "Everyone is at the banquet and everyone is hungry"

# Existential Quantification

$\exists \langle variables \rangle \quad \langle sentence \rangle$

Someone at the tournament is hungry:
$\exists x At(x, Tournament) \land Hungry(x)$

$\exists x P$ is true in a model $m$ iff $P$ is true with $x$ being **some** possible object in the model

**Roughly** speaking, equivalent to the disjunction of instantiations of $P$

$$
\begin{aligned}
& (At(KingJohn, Tournament) \land Hungry(KingJohn)) \\
\lor\ & (At(Richard, Tournament) \land Hungry(Richard)) \\
\lor\ & (At(Tournament, Tournament) \land Hungry(Tournament)) \\
\lor\ & \ldots
\end{aligned}
$$

# Existential Quantification

Common mistake

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists\, x At(x, Tournament) \Rightarrow Hungry(x)$$

is true if there is anyone who is not at the tournament!

# Properties of Quantifiers

$\forall x \;\; \forall y$ is the same as $\forall y \;\; \forall x$

$\exists x \;\; \exists y$ is the same as $\exists y \;\; \exists x$

$\exists x \;\; \forall y$ is not the same as $\forall y \; \exists x$

$\exists x \, \forall y \, Loves(x, y)$

"There is a person who loves everyone in the world"

Pred(x, y)

$\forall y \, \exists x \, Loves(x, y)$

"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \, Likes(x, IceCream) \qquad \neg \exists x \, \neg Likes(x, IceCream)$

$\exists x \, Likes(x, Broccoli) \qquad \neg \forall x \, \neg Likes(x, Broccoli)$

# Fun with Sentences

Brothers are siblings

$$\forall x, y Brother(x, y) \quad \Rightarrow \quad Sibling(x, y).$$

"Sibling" is symmetric

$$\forall x, y Sibling(x, y) \quad \Leftrightarrow \quad Sibling(y, x).$$

A first cousin is a child of a parent's sibling

$$\forall x, y \left[ FirstCousin(x, y) \quad \Leftrightarrow \exists p, ps \quad Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y) \right]$$

# Equality

$term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \ \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable

$2 = 2$ is valid

E.g., definition of (full) *Sibling* in terms of *Parent*:

$$\forall x, y \ \ Sibling(x, y) \Leftrightarrow$$
$$[\neg(x = y) \ \wedge \ \exists m, f \ \neg(m = f) \ \wedge$$
$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

# Piazza Poll 1

Given the following two FOL sentences:

$$\gamma: \quad \forall\, x \quad Hungry(x)$$

$$\delta: \quad \exists\, x \quad Hungry(x)$$

Which of these is true?

A) $\gamma \vDash \delta$
B) $\delta \vDash \gamma$
C) Both
D) Neither

# Piazza Poll 1

Given the following two FOL sentences:

$$\gamma: \quad \forall\, x \quad Hungry(x)$$

$$\delta: \quad \exists\, x \quad Hungry(x)$$

Which of these is true?

A) $\gamma \vDash \delta$

B) $\delta \vDash \gamma$

C) Both

D) Neither

# Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists\, a\ Action(a, 5))$

i.e., does $KB$ entail any particular actions at $t = 5$?

Answer: $Yes, \{a/Shoot\}$ ← substitution (binding list)

Notation Alert!

Given a sentence $S$ and a substitution $\sigma$,
$S\sigma$ denotes the result of plugging $\sigma$ into $S$; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/EVE,\ y/WALL\text{-}E\}$
$S\sigma = Smarter(EVE,\ WALL\text{-}E)$

Notation Alert!

$Ask(KB, S)$ returns some/all $\sigma$ such that $KB \models S\sigma$

# Knowledge Base for Wumpus World

"Perception"

$\forall b, g, t \quad Percept([Smell, b, g], t) \Rightarrow Smelt(t)$

$\forall s, b, t \quad Percept([s, b, Glitter], t) \Rightarrow AtGold(t)$

Reflex: $\forall t \quad AtGold(t) \Rightarrow Action(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall t \quad AtGold(t) \wedge \neg Holding(Gold, t) \Rightarrow Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

# Deducing Hidden Properties

Properties of locations:

$$\forall x, t \quad At(Agent, x, t) \wedge Smelt(t) \Rightarrow Smelly(x)$$
$$\forall x, t \quad At(Agent, x, t) \wedge Breeze(t) \Rightarrow Breezy(x)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall y \quad Breezy(y) \Rightarrow \exists x \quad Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y \quad Pit(x) \wedge Adjacent(x, y) \Rightarrow Breezy(y)$$

Neither of these is complete — e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the *Breezy* predicate:

$$\forall y \quad Breezy(y) \Leftrightarrow [\exists x \quad Pit(x) \wedge Adjacent(x, y)]$$

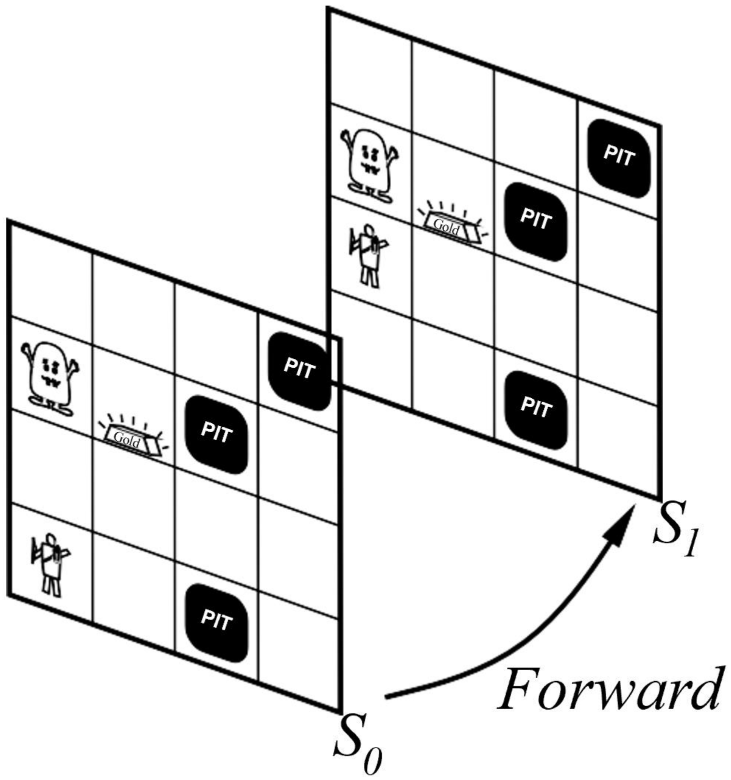
# Keeping Track of Change

Facts hold in situations, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:

> Adds a situation argument to each non-eternal predicate E.g., $Now$ in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing $a$ in $s$



# Describing Actions

"Effect" axiom—describe changes due to action

$\forall s \;\; AtGold(s) \Rightarrow \;\; Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe non-changes due to action

$\forall s \;\; HaveArrow(s) \Rightarrow \;\; HaveArrow(Result(Grab, s))$

Successor-state axioms solve the representational frame problem

Each axiom is "about" a predicate (not an action per se):

P true afterwards $\Leftrightarrow$ [an action made P true
$\lor$ P true already and no action made P false]

For holding the gold:

$$\forall a, s \;\; Holding(Gold, Result(a, s)) \Leftrightarrow [(a = Grab \land AtGold(s)) \lor (Holding(Gold, s) \land \neg(a = Release))]$$

# Describing Actions

Initial condition in KB:

$At(Agent, [1,1], S_0)$

$At(Gold, [1,2], S_0)$

Query: $Ask(KB, \exists\, s\ Holding(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at $S_0$ and that $S_0$ is the only situation described in the KB

# Making Plans

Represent **plans** as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing $p$ in $s$

Then the query $Ask(KB, \exists p \;\; Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$$\forall s \quad PlanResult([\,], s) = s$$
$$\forall a, p, s \quad PlanResult([a, p], s) = PlanResult(p, Result(a, s))$$

# Outline

1. Need for first-order logic
2. Syntax and semantics
3. Planning with FOL
4. Inference with FOL

# Inference in First-Order Logic

## A) Reducing first-order inference to propositional inference

- Removing ∀
- Removing ∃
- Unification

## B) *Lifting* propositional inference to first-order inference

- Generalized Modus Ponens
- FOL forward chaining

# Universal Instantiation

Every instantiation of a universally quantified sentence is entailed by it:

$$\forall v \; a$$

$$\text{Subst}(\{v/g\}, a)$$

for any variable $v$ and ground term $g$

E.g., $\forall x \quad King(x) \wedge Greedy(x) \Rightarrow Evil(x)$ yields

$King(John) \wedge Greedy(John) \Rightarrow Evil(John)$ $King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$
$King(Father(John)) \wedge Greedy(Father(John)) \Rightarrow Evil(Father(John))$

# Existential Instantiation

For any sentence $a$, variable $v$, and constant symbol $k$
that does not appear elsewhere in the knowledge base:

$$\frac{\exists v \quad a}{\text{Subst}(\{v/k\}, a)}$$

E.g., $\exists x \quad Crown(x) \wedge OnHead(x, John)$ yields

$Crown(C_1) \wedge OnHead(C_1, John)$

provided $C_1$ is a new constant symbol, called a Skolem constant

# Reduction to Propositional Inference

Suppose the KB contains just the following:

$$\forall x \ \ King(x) \wedge Greedy(x) \ \Rightarrow \ Evil(x)$$

$King(John)$
$Greedy(John)$
$Brother(Richard, John)$

Instantiating the universal sentence in *all possible* ways, we have

$King(John) \wedge Greedy(John) \ \Rightarrow \ Evil(John)$

$King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$

$King(John)$
$Greedy(John)$
$Brother(Richard, John)$

The new KB is propositionalized: proposition symbols are

$King(John),\ Greedy(John),\ Evil(John),\ King(Richard)$ etc.

# Reduction to Propositional Inference

Claim: a ground sentence* is entailed by new KB iff entailed by original KB

Claim: every FOL KB can be propositionalized so as to preserve entailment

Idea: propositionalize KB and query, apply resolution, return result Problem: with function symbols,

there are infinitely many ground terms,

e.g., $Father(Father(Father(John)))$

Theorem: Herbrand (1930). If a sentence $\alpha$ is entailed by an FOL KB, it is entailed by a finite subset of the propositional KB

Idea: For $n = 0$ to $\infty$ do

create a propositional KB by instantiating with depth-$n$ terms see if $\alpha$ is entailed by this KB

Problem: works if $\alpha$ is entailed, loops if $\alpha$ is not entailed

Theorem: Turing (1936), Church (1936), entailment in FOL is semidecidable

# Problems with Propositionalization

Propositionalization seems to generate lots of irrelevant sentences. E.g., from

$$\forall x\, King(x) \wedge Greedy(x) \Rightarrow Evil(x)$$

$$King(John)$$

$$\forall y\, Greedy(y)$$

$$Brother(Richard, John)$$

it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $Greedy(Richard)$ that are irrelevant

# Unification

We can get the inference immediately if we can find a substitution $\theta$ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\text{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

| $p$ | $q$ | $\theta$ |
|---|---|---|
| $Knows(John, x)$ | $Knows(John, Jane)$ | $\{x/Jane\}$ |
| $Knows(John, x)$ | $Knows(y, OJ)$ | $\{x/OJ, y/John\}$ |
| $Knows(John, x)$ | $Knows(y, Mother(y))$ | $\{y/John, x/Mother(John)\}$ |
| $Knows(John, x)$ | $Knows(x, OJ)$ | $fail$ |

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

# Generalized Modus Ponens (GMP)

$$\frac{p_1{}^t,\ p_2{}^t, \ldots, p_n{}^t,\ (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta}$$

where $p_i{}^t\theta = p_i\theta$ for all $i$

$p_1{}^t$ is $King(John)$ | $p_1$ is $King(x)$
$p_2{}^t$ is $Greedy(y)$ | $p_2$ is $Greedy(x)$
$\theta$ is $\{x/John, y/John\}$ | $q$ is $Evil(x)$
$q\theta$ is $Evil(John)$

GMP used with KB of definite clauses (exactly one positive literal)
All variables assumed universally quantified

# FOL Forward Chaining

**function** FOL-FC-Ask($KB$, $\alpha$) **returns** a substitution or *false*

  **repeat until** *new* is empty
    *new* ← { }
    **for each** sentence $r$ **in** $KB$ **do**
      $(p_1 \wedge \ldots \wedge p_n \Rightarrow q) \leftarrow$ Standardize-Apart($r$)
      **for each** $\theta$ such that $(p_1 \wedge \ldots \wedge p_n)\theta = (p'_1 \wedge \ldots \wedge p'_n)\theta$
          for some $p'_1, \ldots, p'_n$ in $KB$
        $q' \leftarrow$ Subst($\theta$, $q$)
        **if** $q'$ is not a renaming of a sentence already in $KB$ or *new* **then do**
          add $q'$ to *new*
          $\varphi \leftarrow$ Unify($q'$, $\alpha$)
          **if** $\varphi$ is not *fail* **then return** $\varphi$
    add *new* to $KB$
  **return** *false*