Lec 19: Partial Differential Equations I

15-369/669/769: Numerical Computing

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Motivation

Overview

- PDEs relate partial derivatives of a multivariable function $f : \mathbb{R}^n \to \mathbb{R}^m$.
- Goal: find f satisfying relationships between its spatial derivatives.
- PDEs arise in physics (fluids, electromagnetism), imaging, geometry, and many other fields.
- Contrast with ODEs: derivatives appear in multiple variables, not only time.

Motivation

Differential Operators

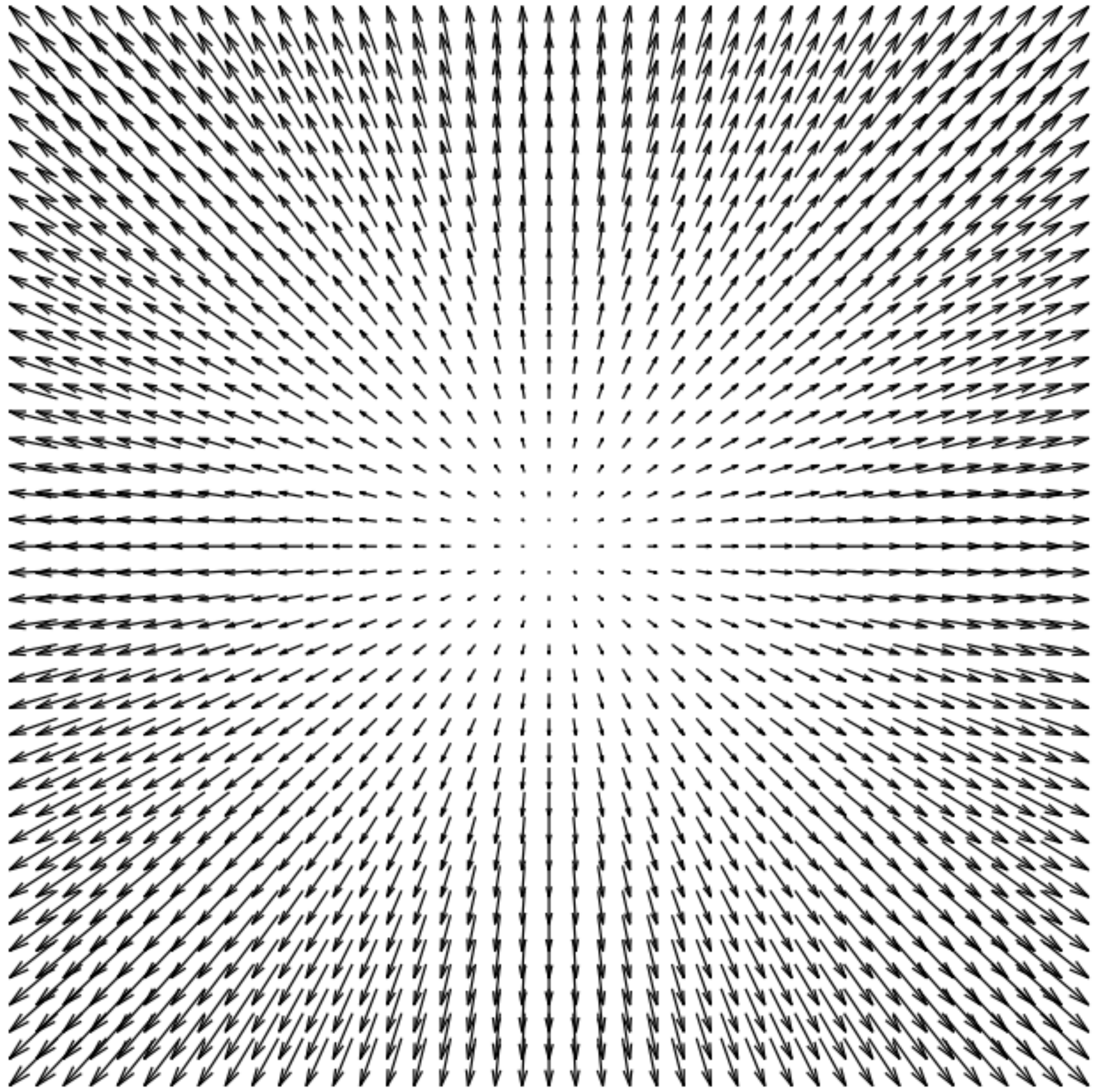
- Common operators for $f : \mathbb{R}^3 \to \mathbb{R}$ and vector fields $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$:
- Gradient: $\nabla f = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ f_{x_3} \end{bmatrix}$
- Divergence: $\nabla \cdot \mathbf{v} = v_{1,x_1} + v_{2,x_2} + v_{3,x_3}$
- Curl: $\nabla \times \mathbf{v} = \begin{bmatrix} v_{3,x_2} - v_{2,x_3} \\ v_{1,x_3} - v_{3,x_1} \\ v_{2,x_1} - v_{1,x_2} \end{bmatrix}$
- Laplacian: $\nabla^2 f = f_{x_1 x_1} + f_{x_2 x_2} + f_{x_3 x_3}$

(Can view $\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}$ as a special vector.)

(Define $\nabla^2 \equiv \nabla \cdot \nabla$ here)

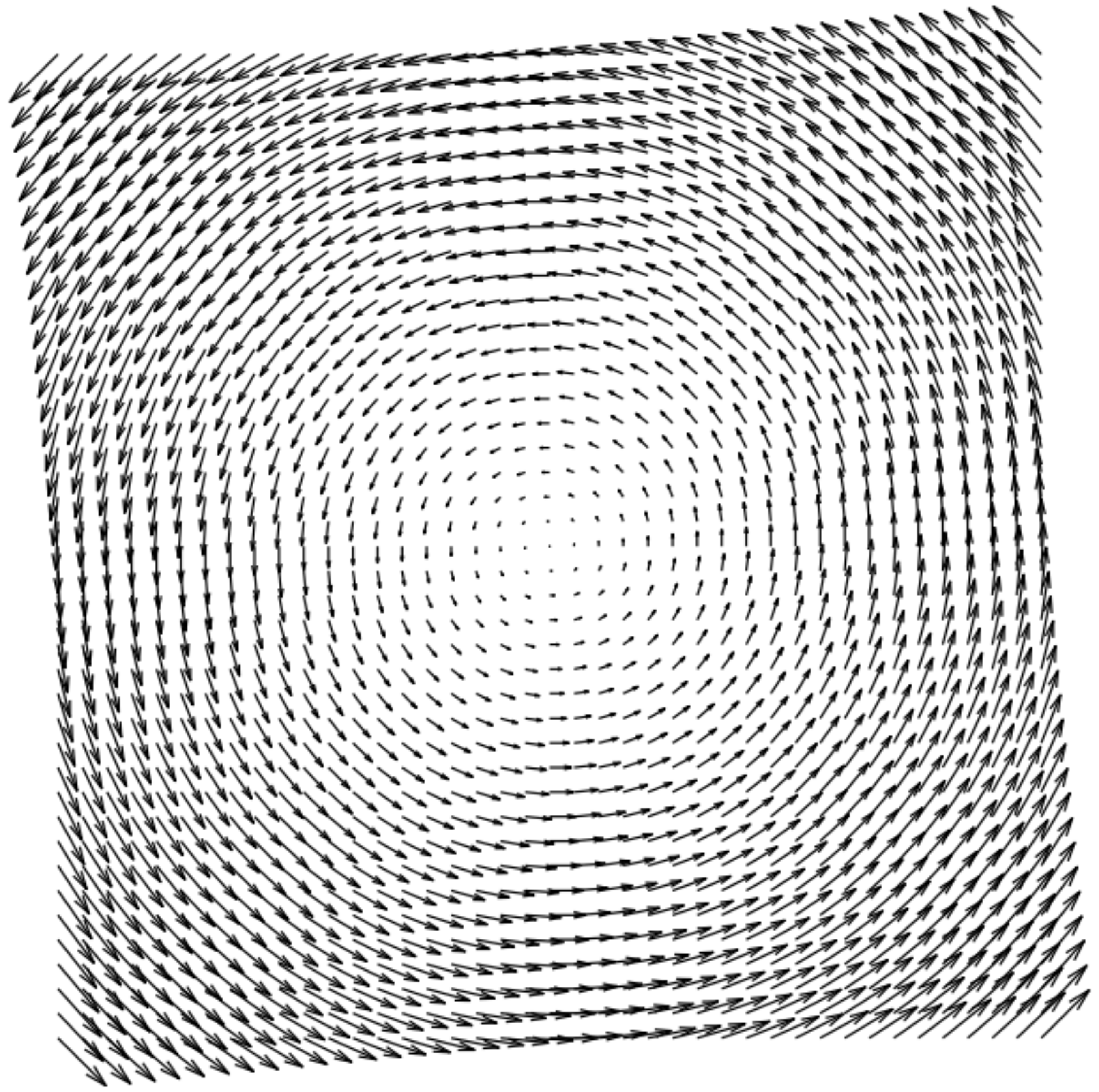
Motivation

Visualization of Differential Operators



$\nabla \cdot \mathbf{v}$ large

$\nabla \times \mathbf{v}$ small



$\nabla \cdot \mathbf{v}$ small

$\nabla \times \mathbf{v}$ large

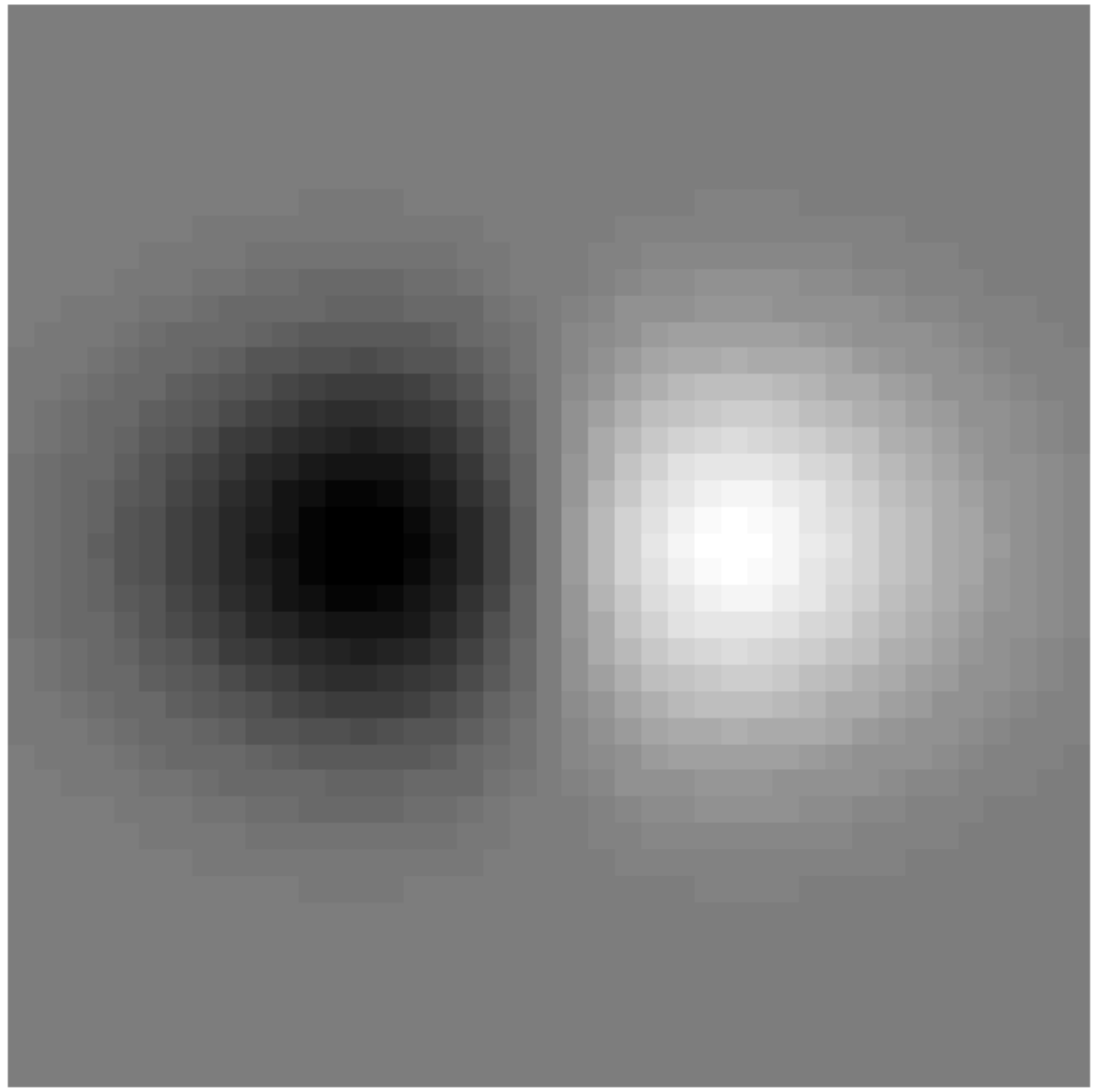
- **Divergence** describes the rate of infinitesimal volume change of a vector field.

- **Curl** describes the infinitesimal circulation of a vector field

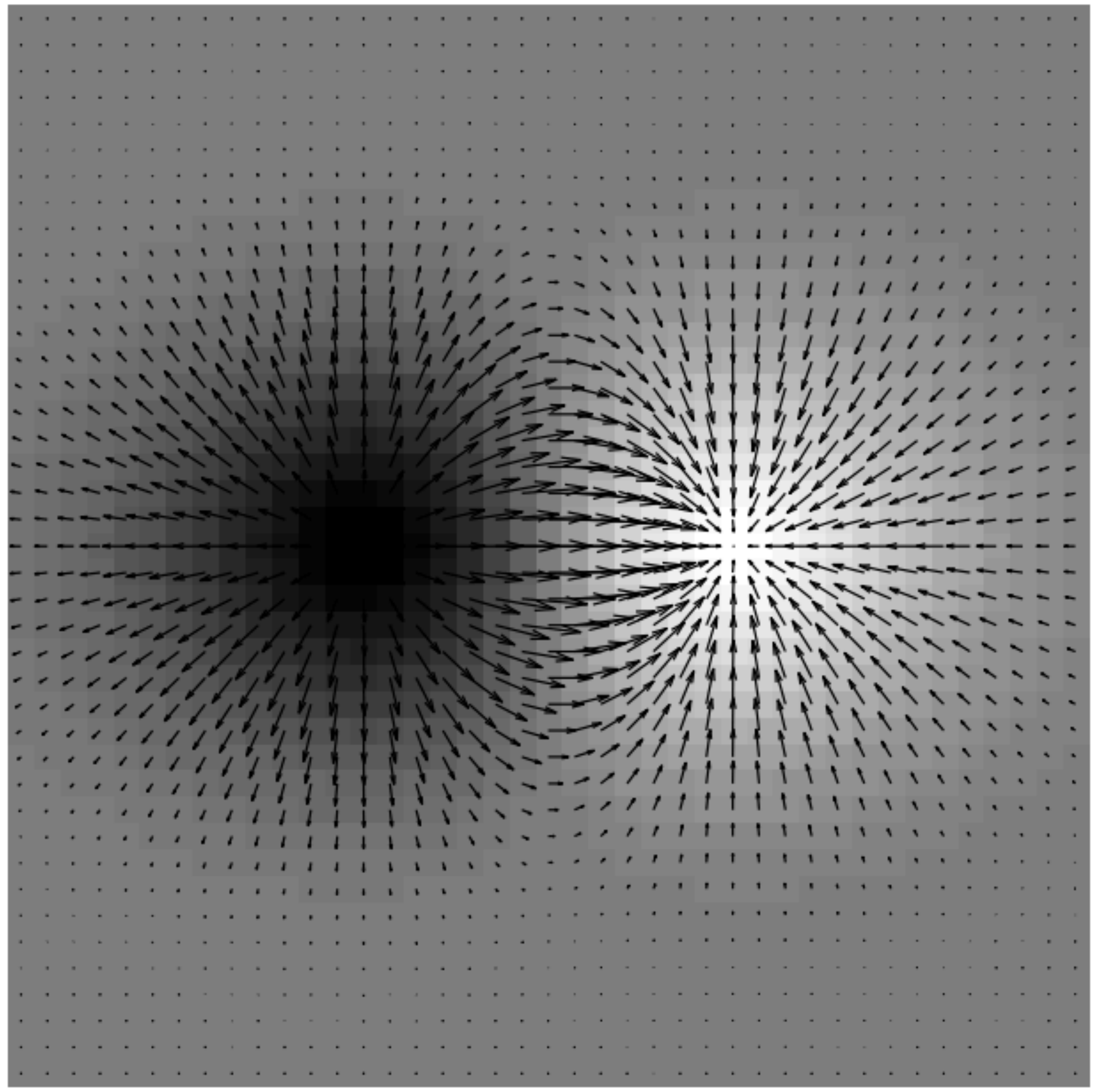
Motivation

Visualization of Differential Operators

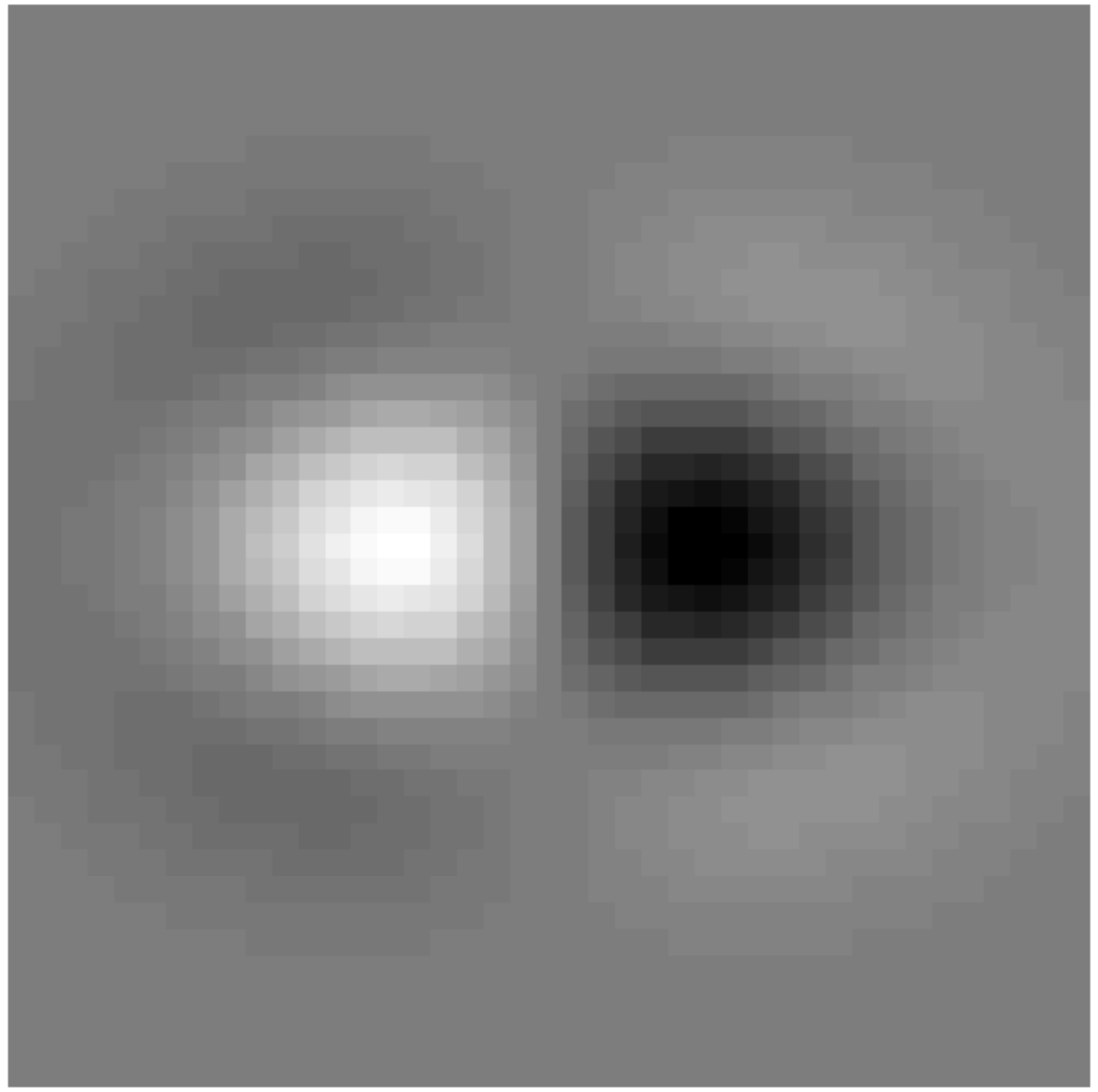
Laplacian measures by how much the
average value of f over small balls
centered at p deviates from $f(p)$.



$f(\mathbf{x})$



$\nabla f(\mathbf{x})$



$\Delta f(\mathbf{x})$

Motivation

Example: Navier–Stokes Equations

- Variables:

$$t \in [0, \infty), \quad \underset{\text{Flow velocity}}{\mathbf{v}(t) : \Omega \to \mathbb{R}^3}, \quad \underset{\text{Pressure}}{p(t) : \Omega \to \mathbb{R}}, \quad \underset{\text{External force, e.g. gravity}}{\mathbf{f}(t) : \Omega \to \mathbb{R}^3}.$$

- Incompressibility condition:

$$\nabla \cdot \mathbf{v} = 0.$$

- Momentum conservation:

$$\rho \overset{\substack{\text{Material derivative} \\ \text{of velocity}}}{\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)} = -\nabla p + \underset{\text{Viscosity}}{\mu \nabla^2 \mathbf{v}} + \mathbf{f}.$$

- PDE couples time derivatives and spatial derivatives.

Motivation

Example: Maxwell's Equations (Vacuum)

- Electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density ρ, current density $\mathbf{J}$.
- In vacuum:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0,$$

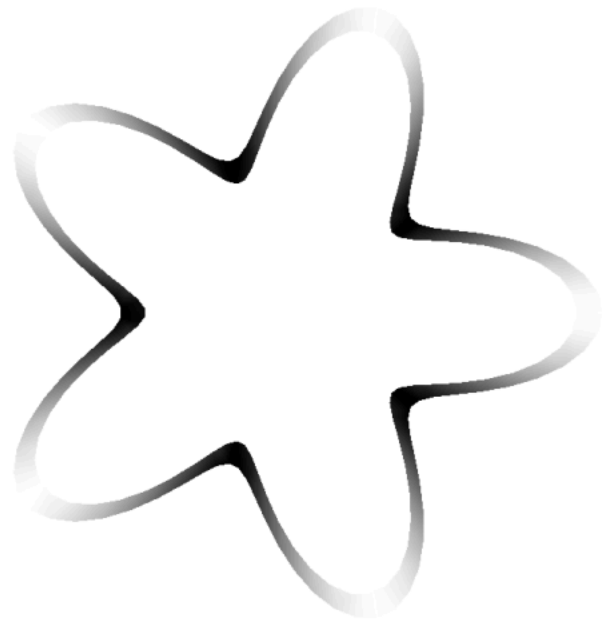
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

Motivation

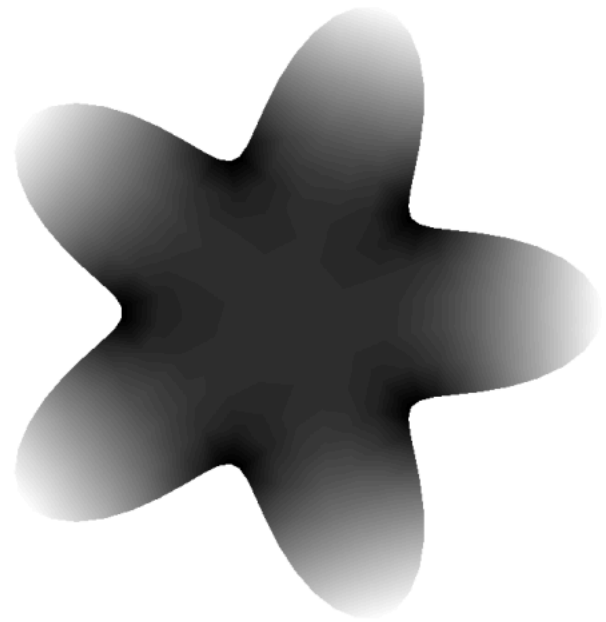
Variational Derivation: Laplace's Equation

- Goal: interpolate boundary data $g : \partial\Omega \to \mathbb{R}$ with a smooth interior function $f : \Omega \to \mathbb{R}$.
- Minimize Dirichlet energy:

$$E[f] = \int_{\Omega} \|\nabla f(\mathbf{x})\|_2^2 \, d\mathbf{x}.$$



Boundary conditions (on $\partial\Omega$)



Laplace solution (on Ω)

- Constrained by $f = g$ on $\partial\Omega$.
- Consider perturbations $f + \varepsilon h$ with $h = 0$ on $\partial\Omega$.
- Expand: $E[f + \varepsilon h] = \int_{\Omega} \|\nabla f + \varepsilon \nabla h\|^2 \, dx = \int_{\Omega} (\|\nabla f\|^2 + 2\varepsilon \nabla f \cdot \nabla h + \varepsilon^2 \|\nabla h\|^2) \, dx$.
- Differentiate: $\frac{d}{d\varepsilon} E[f + \varepsilon h]\Big|_{\varepsilon=0} = 2 \int_{\Omega} \nabla f \cdot \nabla h \, dx$.

Motivation

Deriving Laplace's Equation

- Integration by parts (boundary term vanishes):

$$\int_\Omega \nabla f \cdot \nabla h \, dx = -\int_\Omega h \nabla^2 f \, dx.$$

  - Compute the divergence identity: $\nabla \cdot (h\nabla f) = \nabla h \cdot \nabla f + h\,\Delta f$.
  - Integrate over Ω: $\int_\Omega \nabla h \cdot \nabla f \, dx = \int_\Omega \nabla \cdot (h\nabla f)\, dx - \int_\Omega h\,\Delta f \, dx$.
  - Apply the divergence theorem: $\int_\Omega \nabla \cdot (h\nabla f)\, dx = \int_{\partial\Omega} h \frac{\partial f}{\partial n}\, dS$.
  - Use $h = 0$ on $\partial\Omega$: $\int_{\partial\Omega} h \frac{\partial f}{\partial n}\, dS = 0$, so $\int_\Omega \nabla f \cdot \nabla h \, dx = -\int_\Omega h\,\Delta f \, dx$.

Motivation

Conclusion: Laplace's Equation

- For all h with $h|_{\partial\Omega} = 0$,

$$\int_{\Omega} h(\mathbf{x})\, \nabla^2 f(\mathbf{x})\, d\mathbf{x} = 0.$$

- Therefore:

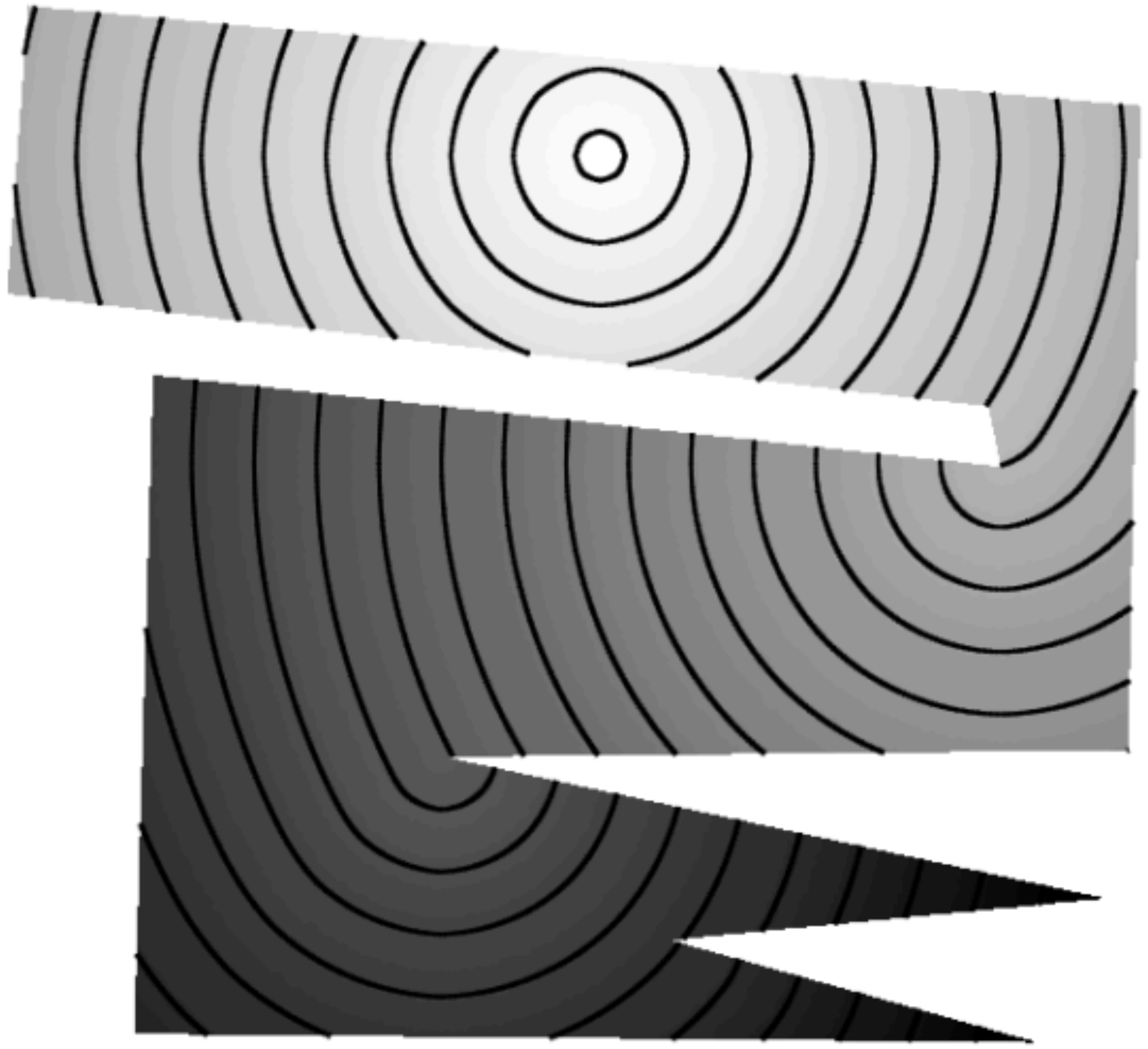
$$\nabla^2 f(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Omega \setminus \partial\Omega.$$

- With boundary condition:

$$f(\mathbf{x}) = g(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega.$$

Motivation

Example: Eikonal Equation



- Distance to a point $\mathbf{x}_0$ inside a region Ω:

$$d(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_0\|_2.$$

- For general Ω, distance satisfies:

$$\|\nabla d(\mathbf{x})\|_2 = 1.$$

- Nonlinear PDE used in robotics, graphics, and geometry processing.

Motivation

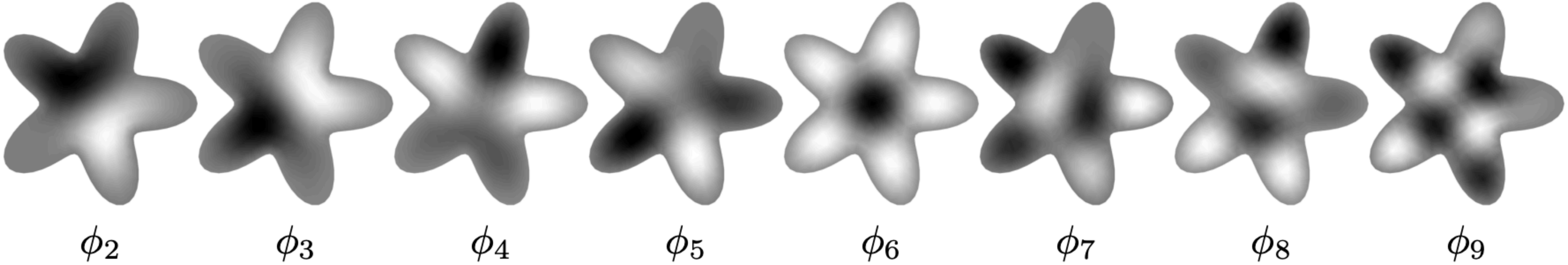
Example: Harmonic Analysis

- Vibrations governed by:

$$\nabla^2\phi = \lambda\phi, \qquad \phi = 0 \text{ on } \partial\Omega.$$

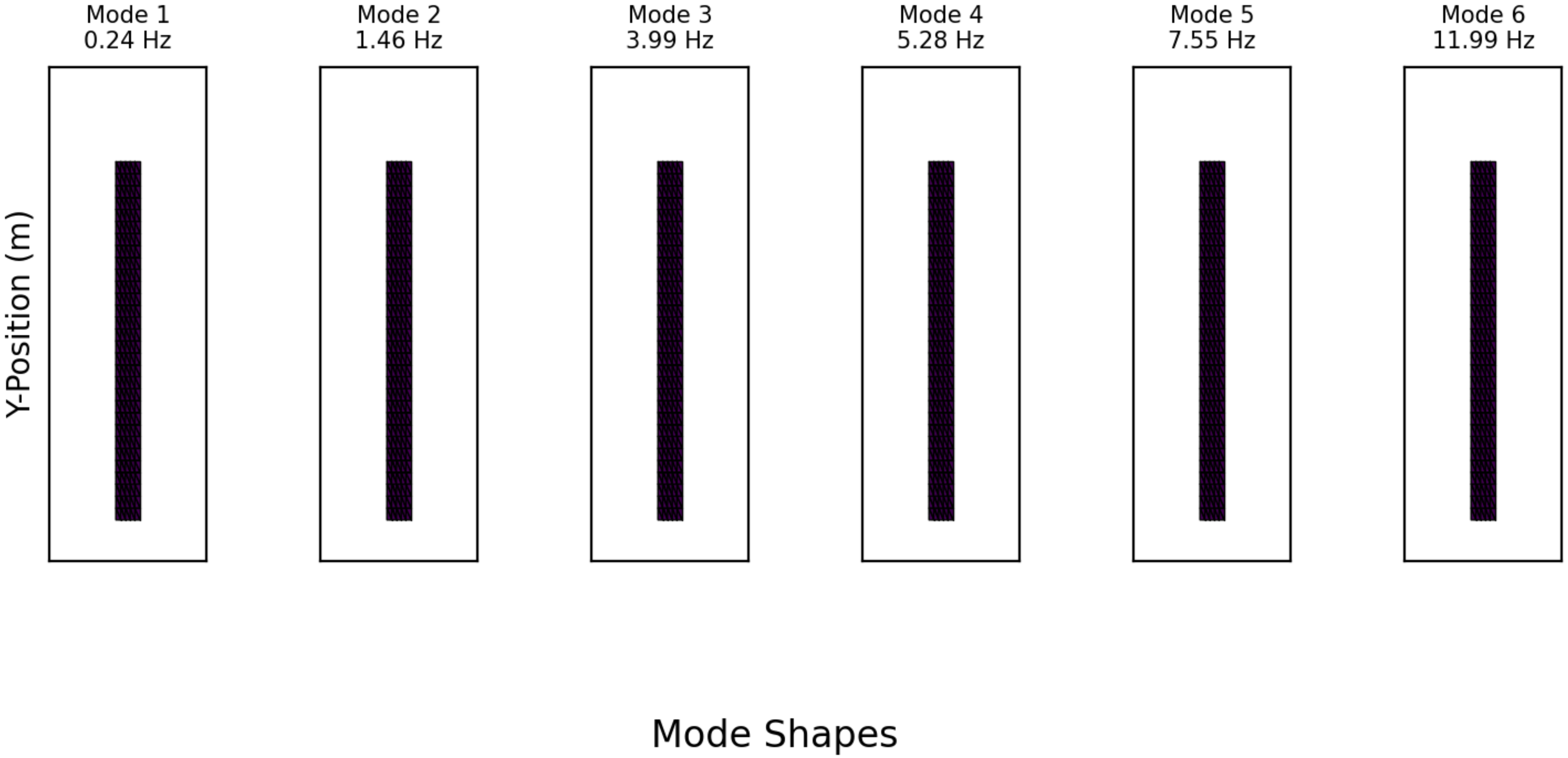
- Eigenfunctions give natural vibration modes.
- 1D example:

$$\phi(x) = \sin(kx), \qquad \lambda = -k^2.$$



Motivation

Vibration Modes of An Elastic Bar



https://phys-sim-book.github.io/lec25.7.2-linear_demo.html

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Statement and Structure of PDEs

Overview

A **partial differential equation (PDE)** relates an unknown function $u(\mathbf{x})$ of several independent variables to its partial derivatives.

Vocabulary for PDEs is extensive; different classes exhibit very different:

- solvability properties,
- theoretical understanding,
- discretization challenges.

We focus on representative examples rather than the full PDE theory.

Statement and Structure of PDEs

Notation for PDEs

We assume $u(\mathbf{x})$ is a scalar-valued function of multiple variables. We use subscript notation for derivatives:

$$u_x := \frac{\partial u}{\partial x}, \quad u_y := \frac{\partial u}{\partial y}, \quad u_{xy} := \frac{\partial^2 u}{\partial x \partial y}.$$

Most examples consider $u = u(x, y)$ but the notation extends directly to higher dimensions.

Statement and Structure of PDEs

Properties of PDEs

Examining the algebraic form of a PDE reveals useful structural properties.

- **Homogeneous**: PDE written using linear combinations of u and its derivatives. Coefficients may depend on independent variables. Example: $x^2 u_{xx} + u_{xy} - u_y + u = 0$.
- **Linear**: PDE may include inhomogeneous right-hand sides but remains linear in u. Example: $u_{xx} - y u_{yy} + u = xy^2$.
- **Quasi-linear**: Linear in highest-order derivatives. Example: $u_{xy} + 2u_{xx} + u_y^2 + u_x^2 = y$.
- **Constant-coefficient**: Coefficients of u and its derivatives are constant. Example: $u_{xx} + 3u_y = u_{zz}$.

Statement and Structure of PDEs

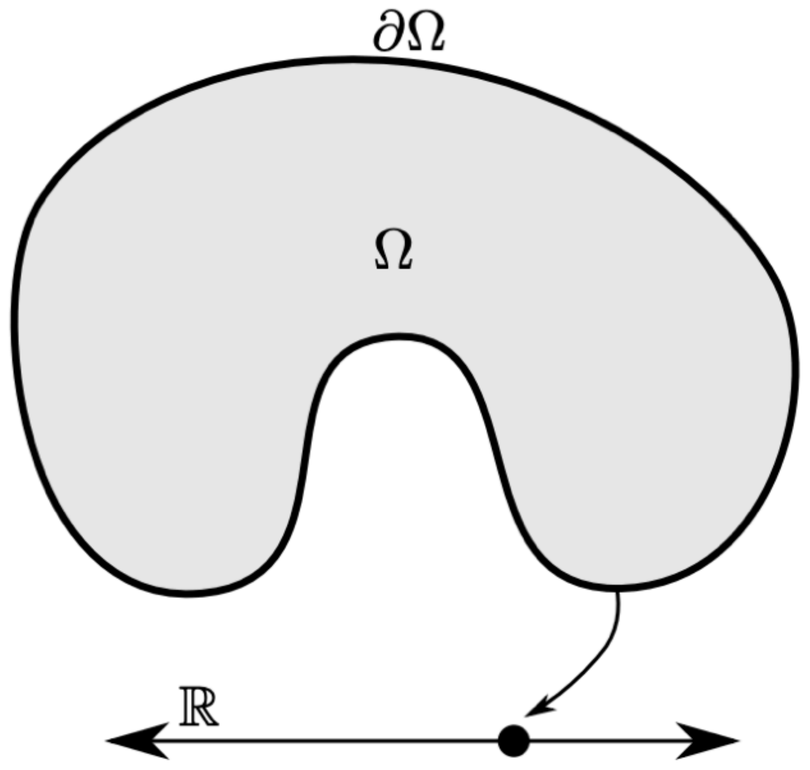
Boundary Conditions

ODEs are usually **initial-value problems**, but PDEs require additional boundary data.

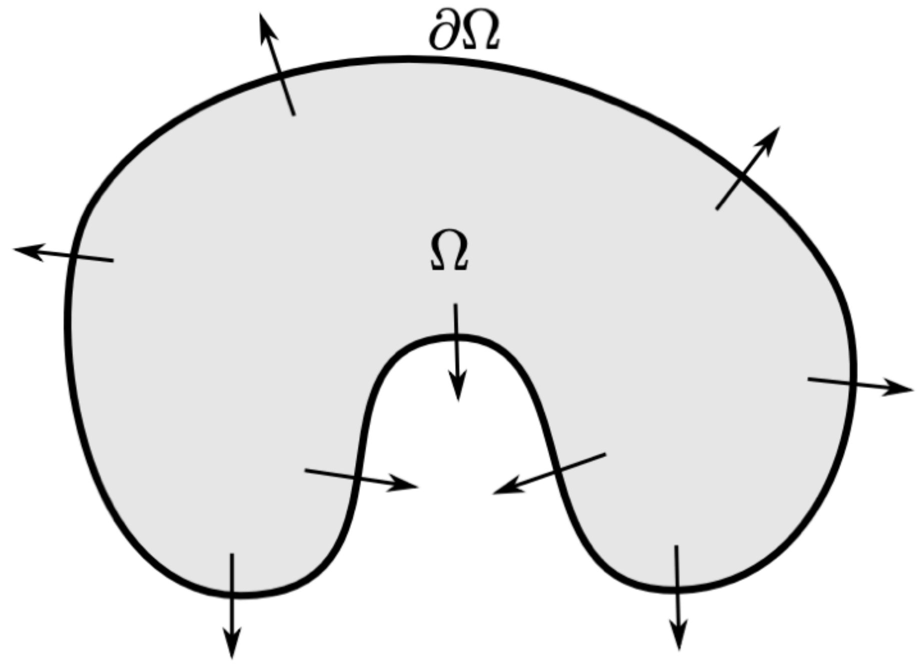
For domain $\Omega \subset \mathbb{R}^n$, boundary conditions include:

- **Dirichlet**: Specify $u(\mathbf{x})$ on $\partial\Omega$.
- **Neumann**: Specify the derivative of u normal to $\partial\Omega$.
- **Mixed/Robin**: Linear combination of Dirichlet and Neumann conditions.

Boundary conditions must be consistent with the PDE; incompatible conditions may make the PDE unsolvable.



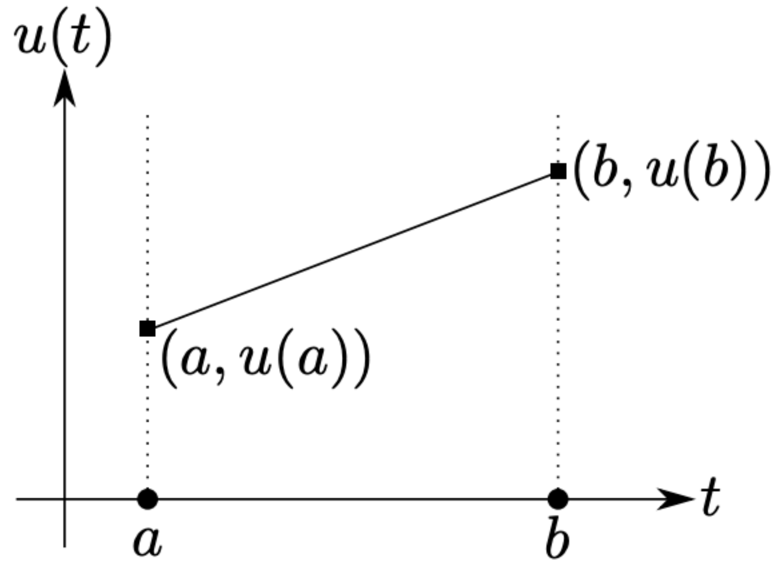
Dirichlet



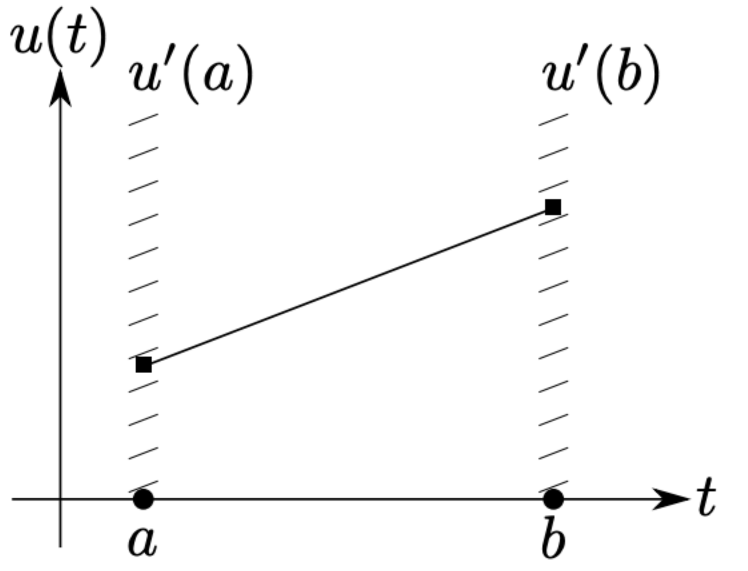
Neumann

Statement and Structure of PDEs

Example: Boundary Conditions in 1D



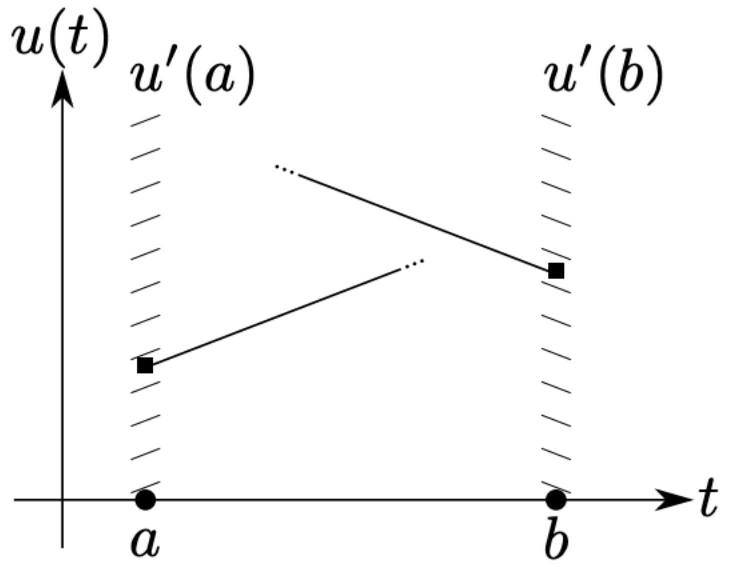
Dirichlet



Neumann (compatible)

Consider the PDE (actually an ODE):

$$u_{tt} = 0, \qquad t \in [a, b].$$



Neumann (incompatible)

General solution:

$$u(t) = \alpha t + \beta.$$

Different boundary conditions yield:

- **Dirichlet at both ends**: $u(a)$ and $u(b)$ determine a unique solution.
- **Neumann at both ends**: $u'(a) = u'(b) = \alpha$ is consistent, but β remains free.
- **Incompatible Neumann**: $u'(a) \neq u'(b)$ gives no solution.

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Model Equations

Model PDEs

We restrict attention to **linear, constant-coefficient, homogeneous** second-order PDEs:

$$\sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \; + \; \sum_i b_i \frac{\partial u}{\partial x_i} \; + \; cu = 0.$$

Introduce gradient operator:

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right).$$

Matrix form:

$$(\nabla^T A \nabla + \nabla \cdot \mathbf{b} + c)u = 0.$$

Since partial derivatives commute, we can assume A is symmetric. Classification depends on definiteness of A.

Model Equations

Classification of PDEs

Properties of matrix A (second-order terms) determine PDE type:

- **Elliptic**: A is positive or negative definite. (e.g., Laplace's equation)
- **Parabolic**: A is positive semidefinite with a nontrivial null space. (e.g., Heat equation)
- **Hyperbolic**: A has eigenvalues of mixed sign. (e.g., Wave equation)
- **Ultrahyperbolic**: None of the above; rare in applications.

Model Equations

Elliptic PDEs

Model elliptic PDE: the **Laplace equation**

$$\nabla^2 u = 0, \qquad \text{in 2D: } u_{xx} + u_{yy} = 0.$$

Key properties:

- Solutions interpolate boundary values into the interior.
- **Elliptic regularity**: solutions are smooth if boundary data is smooth.
- Physically correspond to static equilibrium states (e.g., steady temperature).

Model Equations

Parabolic PDEs

$$A = \begin{array}{c|ccc} & t & x & y \\ \hline t & 0 & 0 & 0 \\ x & 0 & 1 & 0 \\ y & 0 & 0 & 1 \end{array}.$$

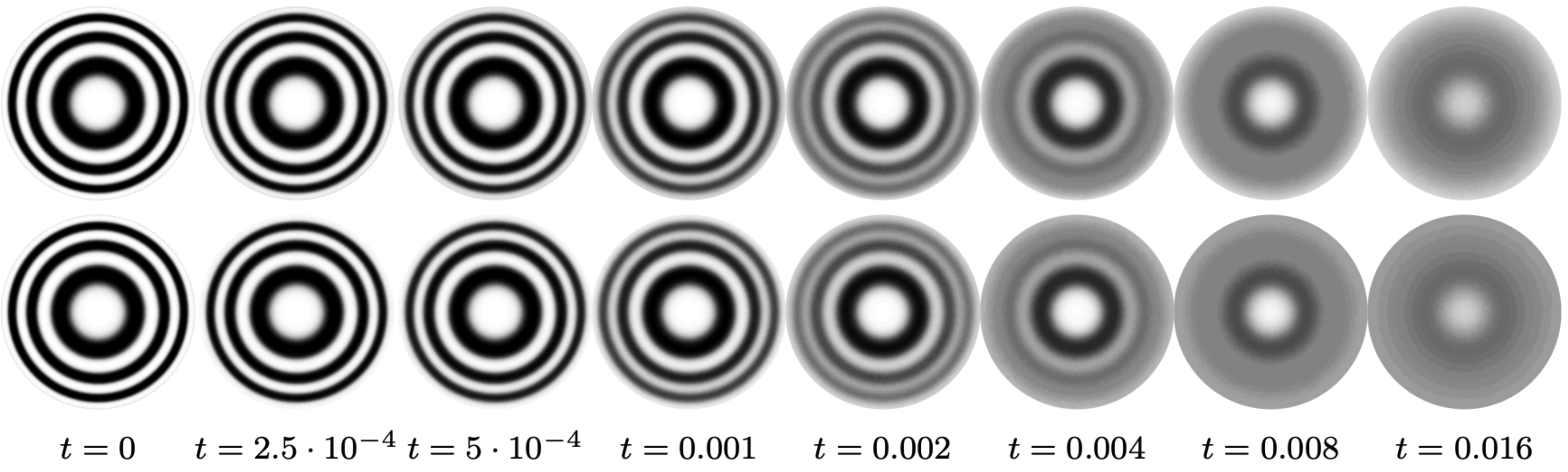


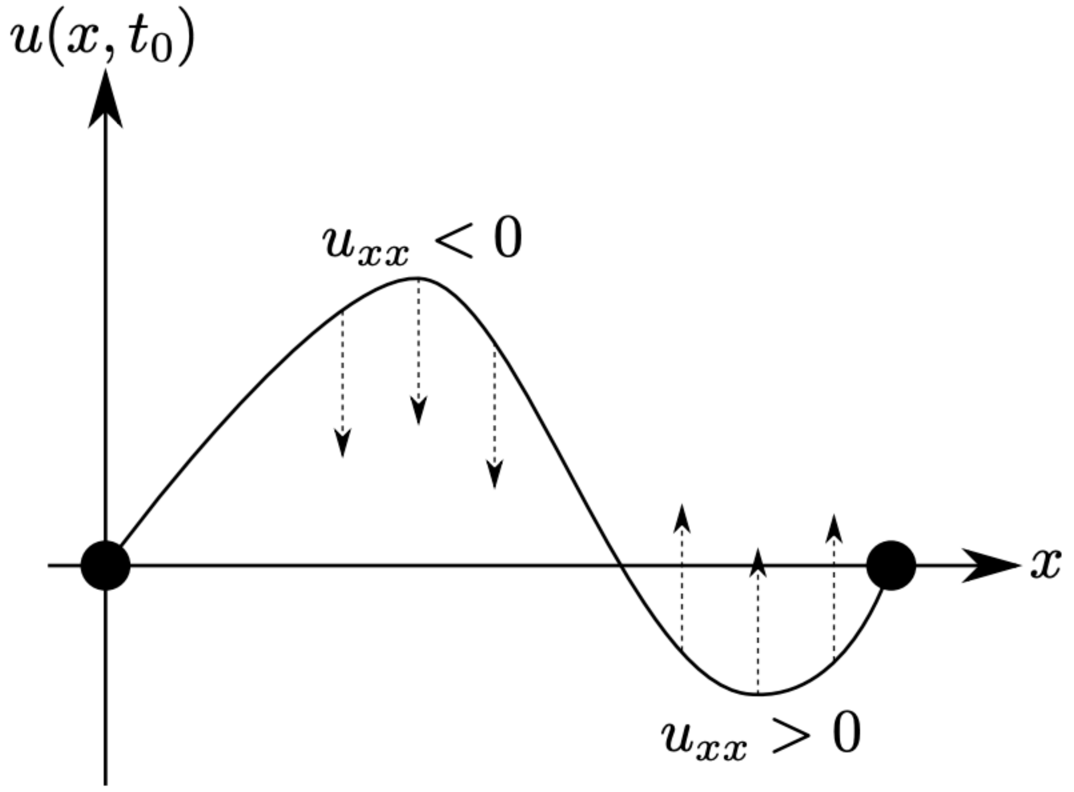
Figure 16.9 Solution to the heat equation $u_t = u_{xx} + u_{yy}$ on the unit circle with Dirichlet (top) and Neumann (bottom) boundary conditions. Solutions are colored from -1 (black) to 1 (white).

Model parabolic PDE: the **heat equation**

$$u_t = \alpha(u_{xx} + u_{yy}) = \alpha \nabla^2 u.$$

Features:

- First-order in time, second-order in space.
- Represents diffusive smoothing; solutions converge to equilibrium.
- A has eigenvalues $(0, 1, 1)$: one zero eigenvalue.



Model Equations
Hyperbolic PDEs

Model hyperbolic PDE: the **wave equation**

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

Characteristics:

- Second derivative in time, spatial second derivatives with opposite sign structure.
- Solutions propagate waves; energy moves without diffusing.
- Boundary and initial conditions determine vibration modes.

Model Equations

Summary: Elliptic vs Parabolic vs Hyperbolic

- **Elliptic** (e.g., Laplace): Steady-state behavior, smooth solutions, governed by spatial equilibrium.
- **Parabolic** (e.g., Heat): Time evolution toward equilibrium, diffusion dominates.
- **Hyperbolic** (e.g., Wave): Time evolution with oscillatory propagation, little or no damping.
- Classification is determined by the eigenvalues of the second-order coefficient matrix A.

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