# 1 Definitions

1. Conditional Probability: $P(A \mid B) =$

   $P(A \mid B) = \frac{P(A,B)}{P(B)}$

2. Product Rule: $P(A, B) =$

   $P(A, B) = P(A|B)P(B)$

3. Bayes' Theorem: $P(A \mid B) =$

   $P(A \mid B) = \frac{P(B|A)P(A)}{P(B)}$

4. Normalization: $P(A \mid B) =$

   $P(A \mid B) = \frac{P(A,B)}{P(B)} = \frac{P(A,B)}{\sum_a P(a,B)}$

5. Chain Rule: $P(A, B, C) =$

   $P(A, B, C) = P(A \mid B, C)P(B \mid C)P(C)$

6. Law of Total Probability: [using only $P(B)$ and $P(A \mid B)$] $P(A) =$

   $P(A) = \sum_{b \in B} P(A \mid b)P(b)$

   For binary $B$:

   $P(A) = P(A \mid b_1)P(b_1) + P(A \mid b_2)P(b_2)$

7. Independence: $A, B$ independent, $P(A, B) =$

   If $A$ and $B$ are independent, then $P(A, B) = P(A)P(B)$

8. Conditional Independence: If $A$ and $B$ are conditionally independent given $C$, then $P(A, B \mid C) =$

   If $A$ and $B$ are conditionally independent given $C$, then $P(A, B \mid C) = P(A \mid C)P(B \mid C)$

# 2 Warm Up

(a) State the two ways to write the chain rule (conditional probability decomposition) for $P(A, B)$

   $P(A)P(B \mid A) = P(B)P(A \mid B)$

(b) Rearrange the above equation to find $P(A \mid B)$

$P(A \mid B) = \frac{P(A)P(B|A)}{P(B)}$

(c) Find $P(a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$ (Hint: these are specific values, answer should include a sum)

$P(a) = \sum_{b \in B} P(a, b)$

(d) Find $P(b \mid a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$

$P(b \mid a) = \frac{P(a,b)}{\sum_{b' \in B} P(a,b')}$

(e) Find $P(b \mid a)$ in terms of the distributions $P(b)$, $P(a \mid b)$, for any $a \in A, b \in B$

$P(b \mid a) = \frac{P(a|b)P(b)}{\sum_{b' \in B} P(a|b')P(b')}$

(f) Assume we had some fixed $a$ and wanted to find each element of $P(b \mid a)$ (i.e. wanted to find $P(B \mid a)$). Would the numerator of the fraction in the previous question change for each value of $b$? What about the denominator? How could you use this to do the calculation with less steps?

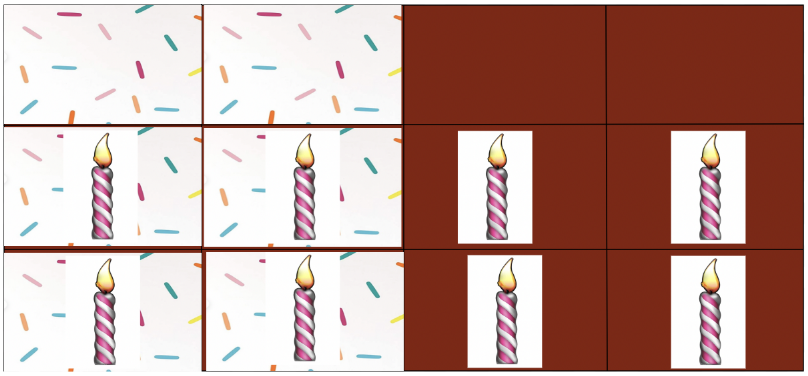
The numerator changes because the value of $b$ changes. The denominator is constant because $P(a)$ will be the same for every value of $b$ that we change. We can calculate all the numerators first, then normalize/equivalently compute the denominator at the end.

(g) Assume $A$ is a random variable that can take 3 values, $B$ is a random variable that can take 2 values, and $C$ is a random variable that can take 1 value. What do the following probability tables sum to?

(a) $P(A \mid b)$
(b) $P(A \mid C)$
(c) $P(C \mid B)$
(d) $P(B \mid a)$
(e) $P(B \mid A)$

$P(A \mid b), P(A \mid C)$, and $P(B \mid a)$ sum to 1. $P(C \mid B)$ sums to 2 because $B$ can take 2 values ($b_1$ and $b_2$). $P(C \mid b_1)$ and $P(C \mid b_2)$ each sum to 1, so if we add them, we get 2. $P(B \mid A)$ sums to 3 because $A$ can take 3 values ($a_1$, $a_2$, and $a_3$). Each of $P(B \mid a_1)$, $P(B \mid a_2)$, and $P(B \mid a_3)$ sum to 1, so the total sums to 3.

# 3 Cake



Consider the above cake with 12 slices. Let $s_1$ indicate a slice with no sprinkles and $s_2$ be a slice with sprinkles. Let $c_1$ indicate a slice with no candles and $c_2$ be a slice with candles. Let $S$ be a random variable indicating sprinkles and $C$ be a random variable indicating candles. Calculate the following probabilities.

1. $P(C = c_1)$

   By counting the number of slices that don't have candles, we can see that the probability of getting a slice with no candles is 4/12.

2. $P(S = s_1, C = c_2)$

   By counting the number of slices that don't have sprinkles but have candles, we can see that the probability of getting a slice with candles and no sprinkles is 4/12.

3. $P(C = c_2 \mid S = s_1)$

   We first constrain our world to only include the slices that contain no sprinkles which is 6 slices. 4 of those slices contain candles, so this probability becomes 4/6. Another way to calculate this probability is by using the definition of conditional probability, $P(C = c_2 \mid S = s_1) = \frac{P(C=c_2,S=s_1)}{P(S=s_1)} = \frac{4/12}{6/12} = \frac{4}{6}$

4. $\sum_{s\in\{s_1,s_2\}} \sum_{c\in\{c_1,c_2\}} P(s, c)$

   $P(s_1, c_1) + P(s_1, c_2) + P(s_2, c_1) + P(s_2, c_2) = 1$. Because we are summing up all possible disjoint combinations of the given sample space, the answer is 1.

5. $\sum_{c\in\{c_1,c_2\}} \sum_{s\in\{s_1,s_2\}} P(s \mid c)$

   $P(s_1 \mid c_1)+P(s_2 \mid c_1)+P(s_1 \mid c_2)+P(s_2 \mid c_2) = 2/4+2/4+4/8+4/8 = 2$. Intuitively, we are summing up two different complete probability distributions, $P(S \mid c_1)$ and $P(S \mid c_2)$: one world where there are no candles, and another world where there are definitely candles.

6. $\sum_{s\in\{s_1,s_2\}} \sum_{c\in\{c_1,c_2\}} P(c \mid s)$

   $P(c_1 \mid s_1)+P(c_2 \mid s_1)+P(c_1 \mid s_2)+P(c_2 \mid s_2) = 2/6+4/6+2/6+4/6 = 2$. Intuitively, we are summing up two different complete probability distributions, $P(C \mid s_1)$ and $P(C \mid s_2)$: one world where there is no sprinkles, and another world where there is definitely sprinkles.

# 4 Queries on a Large Joint Distribution

Consider binary (two outcomes) random variables $A, B, C, D$, and the following joint distribution table of all four variables.

| $A$ | $B$ | $C$ | $D$ | $P(A, B, C, D)$ |
|---|---|---|---|---|
| $+a$ | $+b$ | $+c$ | $+d$ | 12/64 |
| $+a$ | $+b$ | $+c$ | $-d$ | 4/64 |
| $+a$ | $+b$ | $-c$ | $+d$ | 2/64 |
| $+a$ | $+b$ | $-c$ | $-d$ | 2/64 |
| $+a$ | $-b$ | $+c$ | $+d$ | 8/64 |
| $+a$ | $-b$ | $+c$ | $-d$ | 4/64 |
| $+a$ | $-b$ | $-c$ | $+d$ | 2/64 |
| $+a$ | $-b$ | $-c$ | $-d$ | 4/64 |
| $-a$ | $+b$ | $+c$ | $+d$ | 6/64 |
| $-a$ | $+b$ | $+c$ | $-d$ | 3/64 |
| $-a$ | $+b$ | $-c$ | $+d$ | 4/64 |
| $-a$ | $+b$ | $-c$ | $-d$ | 6/64 |
| $-a$ | $-b$ | $+c$ | $+d$ | 2/64 |
| $-a$ | $-b$ | $+c$ | $-d$ | 1/64 |
| $-a$ | $-b$ | $-c$ | $+d$ | 3/64 |
| $-a$ | $-b$ | $-c$ | $-d$ | 1/64 |

1. Calculate the following probabilities:

   (a) $P(+c)$

   Sum all the entries that contain $+c$ to get: $P(+c) = \sum_a \sum_b \sum_d P(a, b, +c, d) = 40/64$

   (b) $P(+a, -b)$

   Sum all the entries that contain both $+a$ and $-b$ to get: $P(+a, -b) = \sum_c \sum_d P(+a, -b, c, d) = 18/64$

   (c) $P(-b \mid +a)$

   1) Sum all entries with both $+a$ and $-b$ to get: $P(+a, -b) = \sum_c \sum_d P(+a, -b, c, d) = 18/64$

   2) Sum all entries with $+a$ to get: $P(+a) = \sum_b \sum_c \sum_d P(+a, b, c, d) = 38/64$

   3) Use definition of conditional probability to compute: $P(-b \mid +a) = \frac{P(+a,-b)}{P(+a)} = 18/38$

   (d) $P(-a, +b, -d)$

   Sum all entries with $-a$, $+b$, and $-d$ to get: $P(-a, +b, -d) = \sum_c P(-a, +b, c, -d) == P(-a, +b, +c, -d) + P(-a, +b, -c, -d) = 9/64$

   (e) $P(+c \mid -a, +b, -d)$

   1) Find entry with $-a, +b, +c, -d$ to get: $P(-a, +b, +c, -d) = 3/64$

   2) Sum all entries with $-a, +b, -d$ to get: $P(-a, +b, -d) = \sum_c P(-a, +b, c, -d) = 9/64$

   3) Use definition of conditional probability to compute: $P(+c \mid -a, +b, -d) = \frac{P(-a,+b,+c,-d)}{P(-a,+b,-d)} = 3/9$

   (f) $P(+c, +d \mid +a, +b)$

1) Find entry with $+a, +b, +c, +d$ to get: $P(+a, +b, +c, +d) = 12/64$

2) Sum all entries with $+a, +b$ to get: $P(+a, +b) = \sum_c \sum_d P(+a, +b, c, d) = 20/64$

3) Use definition of conditional probability to compute: $P(+c, +d \mid +a, +b) = \frac{P(+a,+b,+c,+d)}{P(+a,+b)} = 12/20$

2. What value do the following probability tables sum to?

   (a) $P(B)$

   The short answer is that we are considering the entries for all the possible values of $B$, so this should sum to 1. You could calculate both entries in this table to convince yourself, $P(+b)$ and $P(-b)$.

   (b) $P(+b \mid C, +d)$

   Sadly, there is no shortcut here. The two entries in this table come from two different worlds that are unrelated: one world where $+c$ and $+d$ are given; and another world where $-c$ and $+d$ are given.

   Important note: There is no real reason to add these numbers together in this strange probability table. This is primarily a counterexample to show that these do *not* sum to one.

   We compute these two values similar to the methods in the previous questions, and then add them together.

   $P(+b \mid +c, +d) = \frac{P(+b,+c,+d)}{P(+c,+d)} = 9/14$

   $P(+b \mid -c, +d) = \frac{P(+b,-c,+d)}{P(-c,+d)} = 6/11$ When we add these two values together, we get 1.1883.

   (c) $P(C, D \mid +a, +b)$

   The short answer is that we are considering all possible entries for a single world where $+a$ and $+b$ are given, so this should sum to 1. You could calculate all four entries in this table to convince yourself, $P(+c, +d \mid +a, +b)$, $P(+c, -d \mid +a, +b)$, $P(-c, +d \mid +a, +b)$, $P(-c, -d \mid +a, +b)$.