1 Tracking the Jabberwock

Lewis' Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers $z = (x, y) \in \mathbb{Z}^2 = \{\dots, -2, -1, 0, 1, 2, \dots\} \times \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. At each time step $t = 1, 2, 3, \dots$, the Jabberwock is in some cell $Z_t = z \in \mathbb{Z}^2$, and it moves to cell Z_{t+1} randomly as follows: with probability 1/2, it stays where it is; otherwise, it chooses one of its four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write a function for the transition probability $P(Z_{t+1} = (x', y') | Z_t = (x, y))$.

We will use the particle filtering algorithm to track the Jabberwock. As a source of randomness use values in order from the following sequence $\{a_i\}_{1 \le i \le 14}$. Use these values to sample from any discrete distribution of the form $P(X)$ where X takes values in $\{1, 2, \dots, N\}$. Given $a_i \sim U[0, 1]$, return j such that $\sum_{k=1}^{j-1} P(X = k) \le a_i < \sum_{k=1}^{j} P(X = k)$. You have to fix an ordering of the elements for this procedure to make sense.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
0.142	0.522	0.916	0.792	0.703	0.231	0.036	0.859	0.677	0.221	0.156	0.249

At each time step t you get an observation of the x coordinate R_t in which the Jabberwock sits, but it is a noisy observation. Given the true position $Z_t = (x, y)$, you observe the correct value according to the following probability:

$$P(R_t = r | Z_t = (x, y)) \propto (0.5)^{|x-r|}$$

(b) Suppose that you know that half of the time, the Jabberwock starts at $z_1 = (0, 0)$, and the other half, at $z_1 = (1, 1)$. You get the following observations: $R_1 = 1, R_2 = 0, R_3 = 1$. Fill out the table for each time step using a particle filter with 2 particles to compute an approximation to $P(Z_1, Z_2, Z_3 | r_1, r_2, r_3)$. Sample transitions from the table below using the a_i's as our source of randomness. The a_i's you should use for each step habe been indicated in the last row of each table. Note that going "left" decrements the x-coordinate by 1, and going "down" decrements the y-coordinate by 1.

[0; 0.5)	Stay
[0.5; 0.625)	Up
[0.625; 0.75)	Left
[0.75; 0.875)	Right
[0.875; 1)	Down

Initial	Belief $\hat{P}(z_1)$	Weights $P(r_1\|z_1)$	Unnormalized $\hat{P}(z_1, r_1)$	Normalized $\hat{P}(z_1\|r_1)$	Resampling
$p_1 =$ (0,0) $p_2 =$ (1,1) a_1, a_2	1/2 1/2				$p_1 =$ (,) $p_2 =$ (,) a_3, a_4

Transition $P(z_2\|z_1)$	Belief $\hat{P}(z_2\|r_1)$	Weights $P(r_2\|z_2)$	Unnormalized $\hat{P}(z_2, r_2\|r_1)$	Normalized $\hat{P}(z_2\|r_1, r_2)$	Resampling
$p_2 =$ (,) $p_2 =$ (,) a_5, a_6					$p_1 =$ (,) $p_2 =$ (,) a_7, a_8

Transition $P(z_3\|z_2)$	Belief $\hat{P}(z_3\|r_1, r_2)$	Weights $P(r_3\|z_3)$	Unnormalized $\hat{P}(z_3, r_3\|r_1, r_2)$	Normalized $\hat{P}(z_3\|r_1, r_2, r_3)$	Resampling
$p_1 =$ (,) $p_2 =$ (,) a_9, a_{10}					$p_1 =$ (,) $p_2 =$ (,) a_{11}, a_{12}

(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of Z_3 is different than the column of Z_3, i.e. $X_3 \neq Y_3$.

(e) What is the problem of using the elimination algorithm instead of a particle filter for tracking Jabberwock?

2 Game Theory: Equilibrium Warm-Up

(a) What is a Nash Equilibrium?

(b) Does a Nash Equilibrium always exist?

(c) What is another example of a solution concept that might be useful? What is the difference between this concept and Nash Equilibrium?

(d) What are some drawbacks to following the strategy of Nash equilibrium as a solution concept?

(e) What is a strategy? What is the difference between a pure and a mixed strategy?

(f) What is the difference between a weakly dominant and strictly dominant strategy?

(g) Consider rock paper scissors where Player 1's strategy is to always play rock, and Player 2's strategy is to play scissors or paper with equal probability. Is this a Nash Equilibrium?

(h) Recursively remove dominated strategies to find the Nash Equilibrium of the following game. The order of utilities in each cell is the roman numeral player then the alphabet player.

	A	B	C
i	3,0	0,-5	0,-4
ii	1,-1	3,3	-2,4
iii	2,4	4,1	-1,8

3 Game Theory: Pacman Hunt!

Let us define the Pacman Hunt game for Roy and Ethan (derivative of the well-known Stag Hunt game!). Suppose Roy and Ethan are both hunters, they can choose to hunt Pacman or ghosts. If they hunt a ghost, they will always be successful and gain the modest payoff of 1. If they hunt Pacman, they are successful only if they both chose to hunt Pacman because then they can cooperate. In this case, they gain payoff of 2 each. However, if one hunts Pacman and one hunts ghosts, the one that hunted Pacman gets a payoff of 0. We can formulate the game in the below utility table:

Roy, Ethan	Pacman	Ghost
Pacman	2,2	0,1
Ghost	1,0	1,1

(a) What are the pure Nash Equilibria of this problem?

(b) We will now investigate the possibility of a mixed Nash equilibrium. Recall that in a mixed Nash Equilibrium, the utilities of the weighted actions are equal. Let p be the probability that Ethan picks Pacman.

(i) What is the expected value of action Pacman for Roy?

(ii) What is the expected value of action Ghost for Roy?

(iii) What value of p makes these two expected values the same?

(iv) Since the table is symmetric, the probability that equalizes the value of action Pacman and Ghost for both players is the same. What is the expected utility for both Ethan and Roy if they play according to the mixed Nash Equilibria? How does this utility compare to the equilibria from (a)?

(c) What if the game changed and somehow the Pacmen in the wild got larger or more profitable so the utility table then became

Roy, Ethan	Pacman	Ghost
Pacman	3,3	0,1
Ghost	1,0	1,1

Calculate the mixed Nash Equilibrium for this game. Are the results surprising?

4 More Game Theory

(a) What are the differences between extensive form and normal form games?

(b) We can practice computing utilities using this example that was given in lecture:

CRAM	DO HW	PLAY GAME	
98	100	85	P(EASY) = .2
97	90	65	P(HARD) = .8

- What is the utility of the pure strategy: cram?

- What is the utility of the pure strategy: do HW?

- What is the utility of the mixed strategy: $\frac{1}{2}$ cram, $\frac{1}{2}$ do HW?

(c) Consider a two player game, where each player must simultaneously choose a number from $\{2, 3, ..., 99, 100\}$. Let x_1 represent the value chosen by player 1, and x_2 represent the value chosen by player 2. The rules of the game are such that the utility for a player 1 can be given as:

$$u(p_1) = \begin{cases} x_1 & x_1 = x_2 \\ x_2 - 2 & x_1 > x_2 \\ x_2 + 2 & x_1 < x_2 \end{cases}$$

Because the rules of the game for everyone are the same, the utility function for player 2 is symmetric to $u(p1)$. Does there exist a pure Nash Equilibrium for this game? It may help to try to play a few rounds of this game with someone next to you.