AI: Representation and Problem Solving

Adversarial Search

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Slide credits: CMU AI, http://ai.berkeley.edu
Outline

History / Overview

Zero-Sum Games (Minimax)

Evaluation Functions

Search Efficiency (α-β Pruning)

Games of Chance (Expectimax)
Game Playing State-of-the-Art

**Checkers:**
- 1950: First computer player
- 1959: Samuel’s self-taught program
- 1994: First computer world champion: Chinook ended 40-year-reign of human champion Marion Tinsley using complete 8-piece endgame
- 2007: Checkers solved! Endgame database of 39 trillion states

**Chess:**
- 1960s onward: gradual improvement under “standard model”
- 1997: special-purpose chess machine Deep Blue defeats human champion Gary Kasparov in a six-game match. Deep Blue examined 200M positions per second and extended some lines of search up to 40 ply. Current programs running on a PC rate > 3200 (vs 2870 for Magnus Carlsen)

**Go:**
- 1968: Zobrist’s program plays legal Go, barely (b>300!)
- 2005-2014: Monte Carlo tree search enables rapid advances: current programs beat strong amateurs, and professionals with a 3-4 stone handicap
- 2015: AlphaGo from DeepMind beats best player Lee Sedol

**Poker:**
- 1921: Borel introduces poker as the game theory benchmark
- 1950s: 3-card-deck tiny variant (Kuhn poker) solved by Kuhn, Nash, etc.
- 1950s-1970s: rule-based AIs; not strong
- 1990s: ML-based AIs; not strong
- 2000s-present: Game-theory-based AIs
  - 2008: Superhuman play in 2-player limit Texas hold’em [Bowling et al.]
  - 2015: Near-optimal play in 2-player limit Texas hold’em [Bowling et al.]
  - 2019: Superhuman AI *Pluribus* for 2-player no-limit Texas hold’em [Brown & Sandholm]
Types of Games

Many different types of game!

Axes:
- Deterministic or stochastic?
- Perfect information (fully observable)?
- One, two, or more players?
- Turn-taking or simultaneous?
- Zero sum?

Want algorithms for calculating a contingent plan (a.k.a. strategy or policy) which recommends a move for every possible eventuality
Zero-Sum Games

- Two-Player Zero-Sum Games
  - Agents have **opposite** utilities
  - Pure competition:
    - One *maximizes*, the other *minimizes*

- General Games
  - Agents have *independent* utilities
  - Cooperation, indifference, competition, shifting alliances, and more are all possible
“Standard” Games

Standard games are deterministic, observable, two-player, turn-taking, zero-sum

Game formulation:
- Initial state: $s_0$
- Players: Player(s) indicates whose move it is
- Actions: Actions(s) for player on move
- Transition model: Result(s,a)
- Terminal test: Terminal-Test(s)
- Terminal values: Utility(s,p) for player $p$
  - Or just Utility(s) for player making the decision at root
Adversarial Search
Single-Agent Trees

[Diagram of a single-agent tree with nodes labeled 2, 0, 2, 6, 4, 6, and 8]
Minimax

States

Actions

Values

-8

-5

-10

+8
Minimax

States

Actions

Values
def max_value(state):
    if state.is_leaf:
        return state.value
    # TODO Also handle depth limit
    best_value = -10000000
    for action in state.actions:
        next_state = state.result(action)
        next_value = min_value(next_state)
        if next_value > best_value:
            best_value = next_value
    return best_value

def min_value(state):
Poll 1 (+ worksheet Poll 2 and 3 for Q1a/b)

What is the minimax value at the root?

A) 2
B) 3
C) 6
D) 12
E) 14

```
3  12  8  2  4  6  14  5  2
```
Poll 1

What is the minimax value at the root?

A) 2
B) 3
C) 6
D) 12
E) 14

3

3

2

2

3

12

8

2

4

6

14

5

2
Minimax Notation

\[ V(s) = \max_a V(s'), \]
where \( s' = \text{result}(s, a) \)

```python
def max_value(state):
    if state.is_leaf:
        return state.value  # TODO Also handle depth limit
    best_value = -10000000
    for action in state.actions:
        next_state = state.result(action)
        next_value = min_value(next_state)
        if next_value > best_value:
            best_value = next_value
    return best_value

def min_value(state):
```
Minimax Notation

\[ V(s) = \max_a V(s'), \]
\[ \text{where } s' = \text{result}(s, a) \]

\[ \hat{a} = \arg\max_a V(s'), \]
\[ \text{where } s' = \text{result}(s, a) \]
Generic Game Tree Pseudocode

function minimax_decision( state )
    return argmax \(a\) in state.actions value( state.result(a) )

function value( state )
    if state.is_leaf
        return state.value

    if state.player is MAX
        return max \(a\) in state.actions value( state.result(a) )

    if state.player is MIN
        return min \(a\) in state.actions value( state.result(a) )
Generalized minimax (better name: backward induction)

What if the game is not zero-sum, or has multiple players?

Generalization of minimax:
- Terminals have *utility tuples*
- Node values are also utility tuples
- *Each player maximizes its own component*
- Can give rise to cooperation and competition dynamically…
Minimax Efficiency

How efficient is minimax?
- Just like (exhaustive) DFS
- Time: $O(b^m)$
- Space: $O(bm)$

Example: For chess, $b \approx 35$, $m \approx 100$
- Exact solution is completely infeasible
- Humans can’t do this either, so how do we play chess?
- Bounded rationality – Herbert Simon
Resource Limits
Resource Limits

Problem: In realistic games, cannot search to leaves!

Solution 1: Bounded lookahead
- Search only to a preset *depth limit* or *horizon*
- Use an *evaluation function* for non-terminal positions

Guarantee of optimal play is gone

More plies make a BIG difference

Example:
- Suppose we have 100 seconds, can explore 10K nodes / sec
- So can check 1M nodes per move
- For chess, \( b \approx 35 \) so reaches about depth 4 – not so good
Depth Matters

Evaluation functions are always imperfect

Deeper search => better play (usually)

Or, deeper search gives same quality of play with a less accurate evaluation function

An important example of the tradeoff between complexity of features and complexity of computation
Evaluation Functions
Evaluation Functions

Evaluation functions score non-terminals in depth-limited search

Ideal function: returns the actual minimax value of the position
In practice: typically weighted linear sum of features:

- $\text{EVAL}(s) = w_1 f_1(s) + w_2 f_2(s) + \ldots + w_n f_n(s)$
- E.g., $w_1 = 9$, $f_1(s) = \text{(num white queens} – \text{num black queens)}$, etc.
Evaluation for Pacman
Game Tree Pruning
Minimax Example
Alpha-Beta Example

\[ \alpha = \text{best option so far from any MAX node on this path} \]

The order of generation matters: more pruning is possible if good moves come first.
def min-value(state, α, β):
    initialize v = +∞
    for each successor of state:
        v = min(v, value(successor, α, β))
        if v ≤ α
            return v
    β = min(β, v)
    return v

def max-value(state, α, β):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor, α, β))
        if v ≥ β
            return v
    α = max(α, v)
    return v

α: MAX’s best option on path to root
β: MIN’s best option on path to root
On your own

Which branches are pruned?
(Left to right traversal)
(Select all that apply)
Poll 4

Which branches are pruned? (Left to right traversal)
A) e, l
B) g, l
C) g, k, l
D) g, n
Poll 4

\[ \alpha = 10 \]

\[ \beta = 10 \]

\[ \alpha = 100 \]

\[ \beta = 2 \]
Alpha-Beta Code

\[ \alpha: \text{MAX's best option on path to root} \]
\[ \beta: \text{MIN's best option on path to root} \]

\[
\text{def max-value(state, } \alpha, \beta) : \\
\quad \text{initialize } v = -\infty \\
\quad \text{for each successor of state:} \\
\quad \quad v = \max(v, \text{value(successor, } \alpha, \beta)) \\
\quad \quad \text{if } v \geq \beta \\
\quad \quad \quad \text{return } v \\
\quad \quad \alpha = \max(\alpha, v) \\
\quad \text{return } v
\]
**Alpha-Beta Code**

α: MAX’s best option on path to root  
β: MIN’s best option on path to root  

def min-value(state, α, β):
    initialize v = +∞  
    for each successor of state:
        v = min(v, value(successor, α, β))  
        if v ≤ α  
            return v  
    β = min(β, v)  
    return v  

α = 10  

β = min(β, v)
Alpha-Beta Pruning Properties

Theorem: This pruning has **no effect** on minimax value computed for the root!

Good child ordering improves effectiveness of pruning
- Iterative deepening helps with this

With “perfect ordering”:
- Time complexity drops to $O(b^{m/2})$
- Doubles solvable depth!
- 1M nodes/move => depth=8, respectable

This is a simple example of **metareasoning** (computing about what to compute)
Modeling Assumptions

Know your opponent
Modeling Assumptions

Know your opponent
Modeling Assumptions

Dangerous Pessimism
Assuming the worst case when it’s not likely

Dangerous Optimism
Assuming chance when the world is adversarial
Chance outcomes in trees

Tictactoe, chess
*Minimax*

Tetris, investing
*Expectimax*

Backgammon, Monopoly
*Expectiminimax*
Probabilities
Probabilities

A random variable represents an event whose outcome is unknown.

A probability distribution is an assignment of weights to outcomes.

Example: Traffic on freeway
- Random variable: \( T \) = whether there’s traffic
- Outcomes: \( T \) in \{none, light, heavy\}
- Distribution:
  \[
  P(T=\text{none}) = 0.25, \quad P(T=\text{light}) = 0.50, \quad P(T=\text{heavy}) = 0.25
  \]

Probabilities over all possible outcomes sum to one.
Expected Value

Expected value of a function of a random variable:
Average the values of each outcome, weighted by the probability of that outcome

Example: How long to get to the airport?

<table>
<thead>
<tr>
<th>Time</th>
<th>Probability</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 min</td>
<td>0.25</td>
<td>20 min * 0.25 = 4.00 min</td>
</tr>
<tr>
<td>30 min</td>
<td>0.50</td>
<td>30 min * 0.50 = 15.00 min</td>
</tr>
<tr>
<td>60 min</td>
<td>0.25</td>
<td>60 min * 0.25 = 15.00 min</td>
</tr>
</tbody>
</table>

Total = 35 min

35 min
Expectations

Time: 20 min \times 0.25 + 30 min \times 0.50 + 60 min \times 0.25

Max node notation

\[ V(s) = \max_a V(s'), \quad \text{where } s' = result(s, a) \]

Chance node notation

\[ V(s) = \]
**Expectations**

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\[ V(s) = \max_a V(s'), \quad \text{where } s' = \text{result}(s, a) \]

**Max node notation**

**Chance node notation**

\[ V(s) = \sum_{s'} P(s') V(s') \]
On your own...

Expectimax tree search:
Which action do we choose?

A: Left
B: Center
C: Right
D: Eight

1/4 1/2 1/2 1/3 2/3

12 8 4 8 6 12 6
On your own...

Expectimax tree search:
Which action do we choose?

A: Left
B: Center
C: Right
D: Eight
Expectimax Code

function value( state )
    if state.is_leaf
        return state.value

    if state.player is MAX
        return max a in state.actions value( state.result(a) )

    if state.player is MIN
        return min a in state.actions value( state.result(a) )

    if state.player is CHANCE
        return sum s in state.next_states P( s ) * value( s )
Expectimax Pruning?
Modeling Assumptions

<table>
<thead>
<tr>
<th></th>
<th>Minimax Ghost</th>
<th>Random Ghost</th>
</tr>
</thead>
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<tr>
<td>Minimax Pacman</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expectimax Pacman</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Results from playing 5 games
Summary

**Games require decisions when optimality is impossible**
- Bounded-depth search and approximate evaluation functions

**Games force efficient use of computation**
- E.g., alpha-beta pruning

**Game playing has produced important research ideas**
- Reinforcement learning (checkers)
- Iterative deepening (chess)
- Monte Carlo tree search (Go)
- Solution methods for partial-information games, e.g., in economics (poker)

**Lots to do!**
- E.g., video games present greater challenges: $b = 10^{500}$, $|S| = 10^{4000}$, $m = 10,000$
- See Prof. Sandholm course CS 15-888 Computational Game Solving
$V(s) = \max_a \sum_{s'} P(s') V(s')$
Preview: MDP/Reinforcement Learning Notation

Standard expectimax:
\[ V(s) = \max_a \sum_{s'} P(s'|s, a)V(s') \]

Bellman equations:
\[ V(s) = \max_a \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma V(s')] \]

Value iteration:
\[ V_{k+1}(s) = \max_a \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma V_k(s')], \quad \forall s \]

Q-iteration:
\[ Q_{k+1}(s, a) = \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma \max_{a'} Q_k(s', a')], \quad \forall s, a \]

Policy extraction:
\[ \pi_V(s) = \arg\max_a \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma V(s')], \quad \forall s \]

Policy evaluation:
\[ V_{k+1}^\pi(s) = \sum_{s'} P(s'|s, \pi(s))[R(s, \pi(s), s') + \gamma V_k^\pi(s')], \quad \forall s \]

Policy improvement:
\[ \pi_{new}(s) = \arg\max_a \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma V_{\pi_{old}}(s')], \quad \forall s \]
Preview: MDP/Reinforcement Learning Notation

Standard expectimax:
\[ V(s) = \max_a \sum_{s'} P(s'|s,a)V(s') \]

Bellman equations:
\[ V(s) = \max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V(s')] \]

Value iteration:
\[ V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V_k(s')], \quad \forall s \]

Q-iteration:
\[ Q_{k+1}(s,a) = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma \max_{a'} Q_k(s',a')], \quad \forall s,a \]

Policy extraction:
\[ \pi_V(s) = \arg\max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V(s')], \quad \forall s \]

Policy evaluation:
\[ V_{k+1}^\pi(s) = \sum_{s'} P(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma V_k^\pi(s')], \quad \forall s \]

Policy improvement:
\[ \pi_{\text{new}}(s) = \arg\max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V_{\pi_{\text{old}}}(s')], \quad \forall s \]
Why Expectimax?

Pretty great model for an agent in the world
Choose the action that has the: highest expected value
Bonus Question

Let’s say you know that your opponent is actually running a depth 1 minimax, using the result 80% of the time, and moving randomly otherwise

Question: What tree search should you use?

A: Minimax

B: Expectimax

C: Something completely different