AI: Representation and Problem Solving

Game Theory: Equilibrium

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Slide credits: CMU AI, Fei Fang

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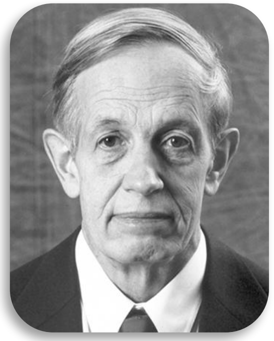
From Games to Game Theory

The study of mathematical models of conflict and cooperation between intelligent decision makers

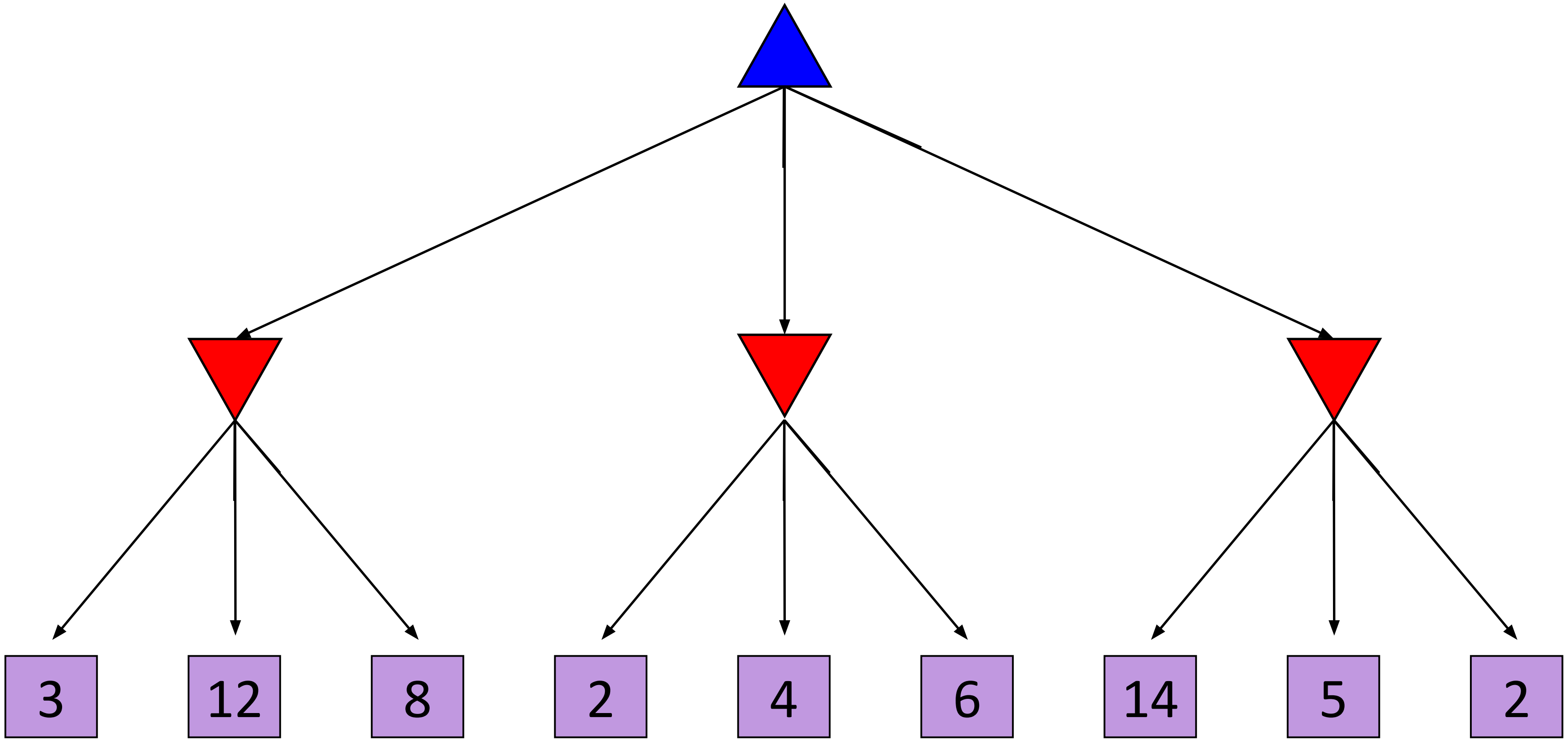
Used in economics, political science, etc



John Nash
Winner of Nobel Memorial Prize in Economic Sciences

Recall: Adversarial Search

Zero-sum, perfect information, two-player games with turn-taking



Payoff Matrices: Simultaneous Choice of Strategies

Rock-Paper-Scissors (RPS)

- Rock beats Scissors
- Scissors beats Paper
- Paper beats Rock

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

2-player normal-form game with finite set of strategies (which are just individual actions in this example) chosen simultaneously

represented in a (bi)matrix

Player 1 is row player (typically first number)

Player 2 is column player (typically second number)

Rock, Paper, Scissors, Lizard, Spock

CBS, Big Bang Theory

https://www.youtube.com/watch?v=iSHPVCBsnLw

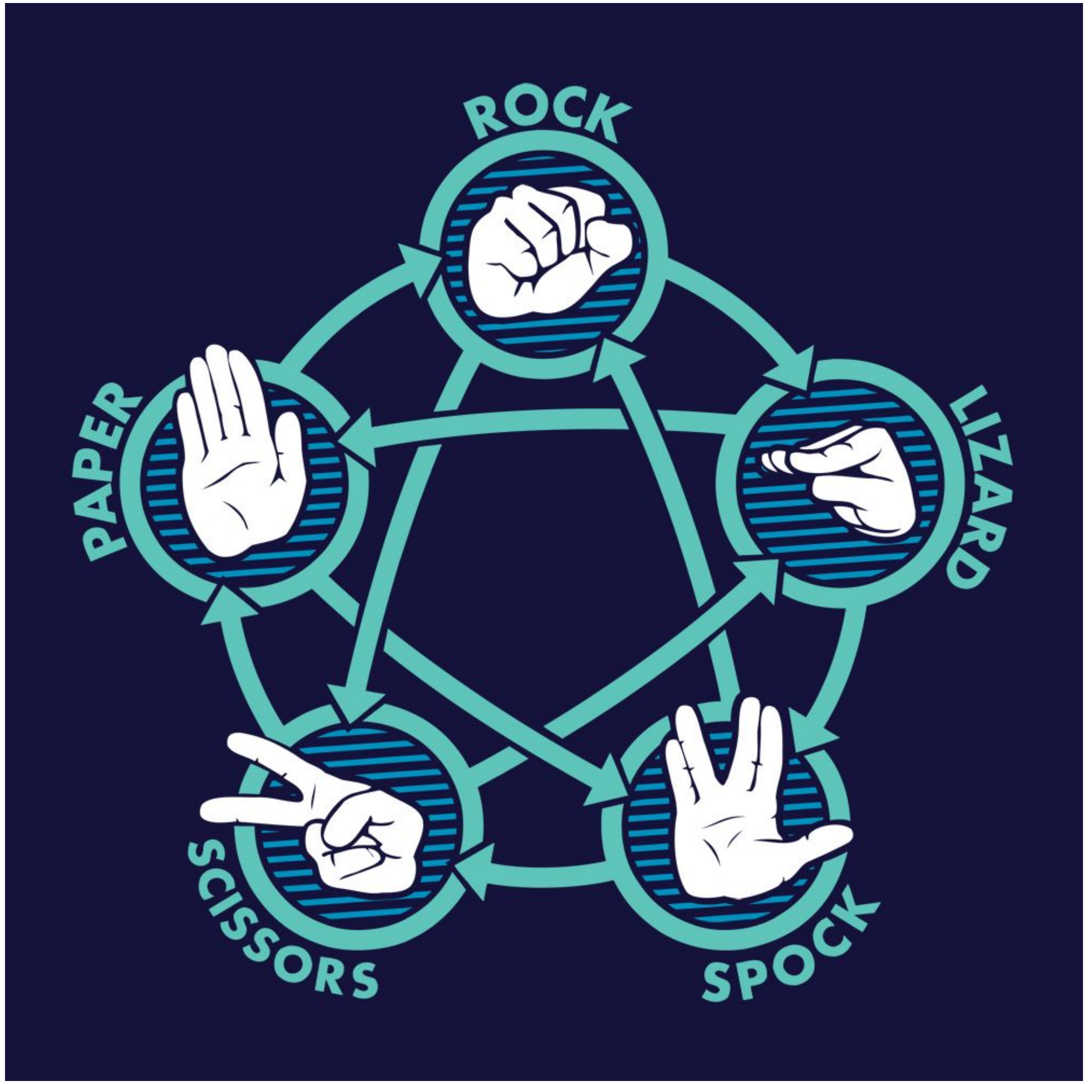


Image credit: https://www.snorgtees.com/rock-paper-scissors-lizard-spock

Classical Games and Payoff Matrices

Prisoner’s Dilemma (PD)

- If both Cooperate: 1 year in jail each
- If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
- If both Defect: 2 years in jail each
- Let’s play!

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Variation: Split or Steal

https://www.youtube.com/watch?v=yM38mRHY150

Classical Games and Payoff Matrices

Football vs Concert (FvsC)

- Historically known as Battle of Sexes
- If football together: Alex ☺️☺️, Berry ☺️
- If concert together: Alex ☺️, Berry ☺️☺️
- If not together: Alex ☹️, Berry ☹️

Fill in the payoff matrix

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Normal-Form Games

A game in normal form consists of the following elements

- Set of players
- Set of actions for each player
- Payoffs / Utility functions
  - Determines the utility for each player given the actions chosen by all players (referred to as action profile)
- Bimatrix game is special case: two players, finite action sets

Players move simultaneously and the game ends immediately afterwards

Strategy

Pure strategy: choose an action deterministically

Mixed strategy: choose actions according to a probability distribution

- Notation: $s = (0.3, 0.7, 0)$
- Support: set of actions chosen with non-zero probability

Notation Alert! We use s to represent strategy here (not states)

Does your AI play a deterministic strategy or a mixed strategy?

What is the support size of your AI's strategy?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Zero-sum vs General-sum

Zero-sum game

- No matter what actions are chosen by the players, the utilities for all the players sum up to zero (or a constant)

General-sum game

- The sum of utilities of all the players is not a constant

Which ones are general-sum games?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Expected Utility

Notation Alert!

Use a, s, u to represent action, strategy, utility of a player

Use $\mathbf{a}, \mathbf{s}, \mathbf{u}$ to represent action, strategy, utility profile

Given the strategies of all players,

Expected Utility for player i u_i=

$$\sum_{\mathbf{a}} \text{Prob(action profile } \mathbf{a}) \times \text{Utility for player } i \text{ in } \mathbf{a}$$

Can skip action profiles with probability 0 or utility 0

If Alex's strategy $s_A = \left(\frac{1}{2}, \frac{1}{2}\right)$, Berry's strategy $s_B = (1,0)$

What is the probability of action profile $\mathbf{a}$ =(Concert,Football)?

What is Alex's utility in this action profile?

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Expected Utility

Notation Alert!

Use a, s, u to represent action, strategy, utility of a player

Use $\mathbf{a}, \mathbf{s}, \mathbf{u}$ to represent action, strategy, utility profile (set of users)

Given the strategies of all players,

Expected Utility for player i u_i=

$$\sum_{\mathbf{a}} \text{Prob(action profile } \mathbf{a}\text{)} \times \text{Utility for player } i \text{ in } \mathbf{a}$$

Can skip action profiles with probability 0 or utility 0

If Alex's strategy s_A=$\left(\frac{1}{2}, \frac{1}{2}\right)$, Berry's strategy s_B=$(1,0)$

What is the probability of action profile $\mathbf{a}$ =(Concert,Football)?

$$\frac{1}{2} \times 1 = \frac{1}{2}$$

What is Alex's utility in this action profile?

$$0$$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Poll 1

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $s_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, how many non-zero terms need to be added up when computing the expected utility for player 1?

A. 9

B. 6

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Poll 2

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $s_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, what is the utility of player 1?

A. -1
B. -1/3
C. 0

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Poll 2

In Rock-Paper-Scissors, if $s_1 = (\frac{1}{3}, \frac{2}{3}, 0)$, $s_2 = (0, \frac{1}{2}, \frac{1}{2})$, how many non-zero terms need to be added up when computing the expected utility for player 1?

A. 9

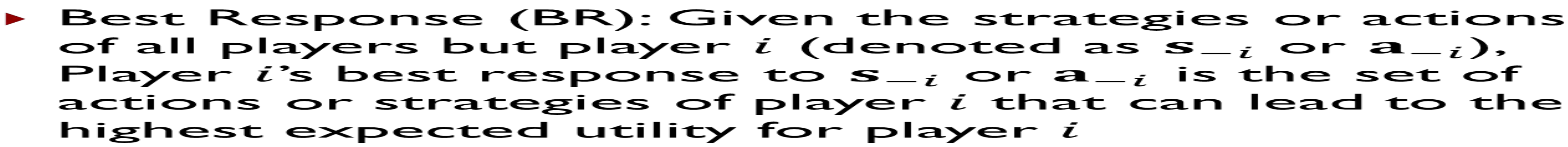
$$\begin{aligned} u_1 = \; & 0 \times \frac{1}{3} \times 0 + (-1) \times \frac{1}{3} \times \frac{1}{2} + 1 \times \frac{1}{3} \times \frac{1}{2} \\ + \; & 1 \times \frac{2}{3} \times 0 + 0 \times \frac{2}{3} \times \frac{1}{2} + (-1) \times \frac{2}{3} \times \frac{1}{2} \\ + & (-1) \times 0 \times 0 + 1 \times 0 \times \frac{1}{2} + 0 \times 0 \times \frac{1}{2} = -\frac{1}{3} \end{aligned}$$

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Best Response



- Best Response (BR): Given the strategies or actions of all players but player i (denoted as s_{-i} or a_{-i}), Player i's best response to s_{-i} or a_{-i} is the set of actions or strategies of player i that can lead to the highest expected utility for player i

In RPS, what is Player 1’s best response to Rock (i.e., assuming Player 2 plays Rock)?

In Prisoner’s Dilemma, what is Player 1’s best response to Cooperate?
What is Player 1’s best response to Defect?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Best Response

- Best Response (BR): Given the strategies or actions of all players but player i (denoted as s_{-i} or a_{-i}), Player i's best response to s_{-i} or a_{-i} is the set of actions or strategies of player i that can lead to the highest expected utility for player i

In RPS, what is Player 1's best response to Rock (i.e., assuming Player 2 plays Rock)?

Paper

In Prisoner's Dilemma, what is Player 1's best response to Cooperate? Defect

What is Player 1's best response to Defect? Defect

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Best Response

- Best Response (BR): Given the strategies or actions of all players but player i (denoted as s_{-i} or a_{-i}), Player i's best response to s_{-i} or a_{-i} is the set of actions or strategies of player i that can lead to the highest expected utility for player i

What is Alex's best response to Berry's mixed strategy $s_B = \left(\frac{1}{2}, \frac{1}{2}\right)$?

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Poll 3

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which actions or strategies are player 2's best responses to s_1?

A. Rock
B. Paper
C. Scissors
D. Lizard
E. $s_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$
F. $s_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Player 2

		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Best Response

Theorem 1 (Nash 1951): A mixed strategy is a BR iff all actions in the support are a BR

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Dominance

s_i and $s_i{}'$ are two strategies for player i

s_i strictly dominates $s_i{}'$ if s_i is always better than $s_i{}'$, no matter what strategies are chosen by other players

s_i strictly dominates s_i' if

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

always better

s_i very weakly dominates s_i' if

$$u_i\ (s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

never worse

s_i weakly dominates s_i' if

$$u_i\ (s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

and $\exists \mathbf{s}_{-i}, u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$

never worse and sometimes better

Dominance

Can you find any dominance relationships between the pure strategies in these games?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Dominance

If s_i strictly dominates $s_i', \forall s_i' \in S_i \setminus \{s_i\}$,

is s_i a best response to $\mathbf{s}_{-i}, \forall \mathbf{s}_{-i}$?

Yes. Remember:

- s_i strictly dominates s_i' if

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

Rewriting the statement at the top:

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i} \quad \forall s_i' \in S_i \setminus \{s_i\}$$

So… for any $\mathbf{s}_{-i}$

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall s_i' \in S_i \setminus \{s_i\}$$

This is the definition of best response ☺

That is, s_i leads to the highest utility compared to all other responses, $s_i{}'$

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- (Minimax strategy)
- (Maximin strategy)
- (Stackelberg Equilibrium)

Dominant Strategy

A strategy could be (always better / never worse / never worse and sometimes better) than any other strategy

s_i is a (strictly/very weakly/weakly) dominant strategy if it (strictly/very weakly/weakly) dominates $s_i', \forall s_i' \in S_i \backslash \{s_i\}$

Focus on single player's strategy

Doesn't always exist

Is there a strictly dominant strategy for player 1 in PD?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Dominant Strategy Equilibrium

Sometimes called dominant strategy solution

Every player plays a dominant strategy

Focus on strategy profile for all players

Doesn’t always exist

What is the dominant strategy equilibrium for PD?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

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- (Stackelberg Equilibrium)

Nash Equilibrium

Nash Equilibrium (NE)

- Every player's strategy is a best response to others' strategy profile
- In other words, one cannot gain by unilateral deviation
- Pure Strategy Nash Equilibrium (PSNE)
  - $a_i \in BR(\mathbf{a}_{-i}), \forall i$
- Mixed Strategy Nash Equilibrium
  - At least one player use a randomized strategy
  - $s_i \in BR(\mathbf{s}_{-i}), \forall i$

Nash Equilibrium

What are the PSNEs in these games?

What is the mixed strategy NE in RPS?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Nash Equilibrium

What are the PSNEs in these games?

RPS: None. Prisoner’s Dilemma: (D,D). Football vs Concert: (F,F),(C,C)

What is the mixed strategy NE in RPS?

(1/3,1/3,1/3) for both players

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Nash Equilibrium

Theorem 2 (Nash 1951): NE always exists in finite games

- Finite number of players, finite number of actions
- NE: can be pure or mixed
- Proof: Through Brouwer's fixed point theorem

Find PSNE

Find pure strategy Nash Equilibrium (PSNE)

- Enumerate all action profile
- For each action profile, check if it is NE
  - For each player, check other available actions to see if he should deviate
- Other approaches?

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4	5, 2
D	0, 10	1, 8	7, 0

Find PSNE

A strictly dominated strategy is one that is always worse than some other strategy

Strictly dominated strategies cannot be part of an NE   Why?

Which are the strictly dominated strategies for player 1?

How about player 2?

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4	5, 2
D	0, 10	1, 8	7, 0

Find PSNE

A strictly dominated strategy is one that is always worse than some other strategy

Strictly dominated strategies cannot be part of an NE   Why?

Which are the strictly dominated strategies for player 1?

How about player 2?

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4 ≥	5, 2
D	0, 10	1, 8	7, 0

Find PSNE through Iterative Removal

Remove strictly dominated actions (pure strategies) and then find PSNE in the remaining game

Can have new strictly dominated actions in the remaining game!

Repeat the process until no actions can be removed

This is the Iterative Removal algorithm (also known as Iterative Elimination of Strictly Dominated Strategies)

Find PSNE in this game using iterative removal

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4	5, 2
D	0, 10	1, 8	7, 0

Find PSNE through Iterative Removal

Remove strictly dominated actions (pure strategies) and then find PSNE in the remaining game

Can have new strictly dominated actions in the remaining game!

Repeat the process until no actions can be removed

This is the Iterative Removal algorithm (also known as Iterative Elimination of Strictly Dominated Strategies)

Find PSNE in this game using iterative removal

	L	C	R
U	10,3	1,5	5,4
M	3,1	2,4	5,2
D	0,10	1,8	7,0

→

	L	C
U	10,3	1,5
M	3,1	2,4
D	0,10	1,8

→

	L	C
U	10,3	1,5
M	3,1	2,4

→

	C
U	1,5
M	2,4

→

	C
M	2,4

Find PSNE through Iterative Removal

When the algorithm terminates, if the remaining game has only one action for each player, then that is the unique NE of the game and the game is called dominance solvable

- It may not be a dominant strategy equilibrium

When the remaining game has more than one action for some players, find PSNE in the remaining game

Order of removal does not matter

Player 2

Player 1

	L	C	R
U	10,3	1,5	5,4
M	3,1	2,4	5,2
D	0,10	1,8	7,0

Find Mixed Strategy Nash Equilibrium

Can we still apply iterative removal?

- Yes! The removed strategies cannot be part of any NE
- You can always apply iterative removal first

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

(The presented technique is for 2x2 games.)

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$$u_A(F, s_B) = u_A(C, s_B) \qquad u_B(s_A, F) = u_B(s_A, C)$$

Why? Remember Theorem 1: A mixed strategy is BR iff all actions in the support are BR.

So…if $s_A \in BR(s_B)$, then $F \in BR(s_B)$and $C \in BR(s_B)$

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

Berry

Alex	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$u_A(F, s_B) = u_A(C, s_B)$ $\quad$ $u_B(s_A, F) = u_B(s_A, C)$

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$$u_A(F, s_B) = u_A(C, s_B) \qquad u_B(s_A, C) = u_B(s_A, C)$$

$$u_A(F, s_B) = 2 * q + 0 * (1-q) = u_A(C, s_B) = 0 * q + 1 * (1-q)$$

So $2q = 1 - q$, we get $q = \frac{1}{3}$

Similarly, $u_B(s_A, F) = 1 * p + 0 * (1-p) = u_B(s_A, C) = 0 * p + 2 * (1-p)$

So $p = 2(1-p)$, we get $p = \frac{2}{3}$

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Minimax strategy
- Maximin strategy
- Stackelberg Equilibrium

Maximin and Minimax Strategy

Both focus on single player's strategy

Maximin Strategy

- Maximize worst case expected utility
- Maximin value (also called safety level)

Minimax Strategy

- Minimize best case expected utility for the other player (just want to harm your opponent)
- Minimax value

Minimax Theorem

Theorem 3 (von Neumann 1928, Nash 1951):

- Minimax=Maximin=NE in 2-player zero-sum games
- All NEs leads to the same utility profile in a two-player zero-sum game

- Formally, every two-player zero-sum game has a unique value v such that
  - Player 1 can guarantee value at least v
  - Player 2 can guarantee loss at most v
  - v is called the value of the game (aka. game value)

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Minimax strategy
- Maximin strategy
- Stackelberg Equilibrium

Power of Commitment

What’s the PSNEs in this game and player 1’s utility?

What action should player 2 choose if player 1 commits to playing b? What is player 1’s utility?

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1’s expected utility?

Player 2

		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Power of Commitment

What’s the PSNEs in this game and player 1’s utility? (a, c), 2

What action should player 2 choose if player 1 commits to playing b? What is player 1’s utility? d, 3

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1’s utility? d, 3.5

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Stackelberg Equilibrium

Stackelberg Game

- Leader commits to a strategy first
- Follower responds after observing the leader's strategy

Stackelberg Equilibrium

- Follower best responds to leader's strategy
- Leader commits to a strategy that maximizes her utility assuming follower best responds

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Stackelberg Equilibrium

If the leader can only commit to a pure strategy, or you know that the leader's strategy in equilibrium is a pure strategy, the equilibrium can be found by enumerating leader's pure strategy

If ties for the follower are broken by the follower such that the leader benefits, the leader can exploit this. This is the strong Stackelberg equilibrium (SSE)

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

		Player 2	
		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2