Plan

Last Time

- Linear programming (LP) formulation
  - Problem description
  - Optimization representation
  - Graphical representation

Today

- LP: Continue graphical representation
- Solving linear programs
- Higher dimensions than just 2
- Integer programs

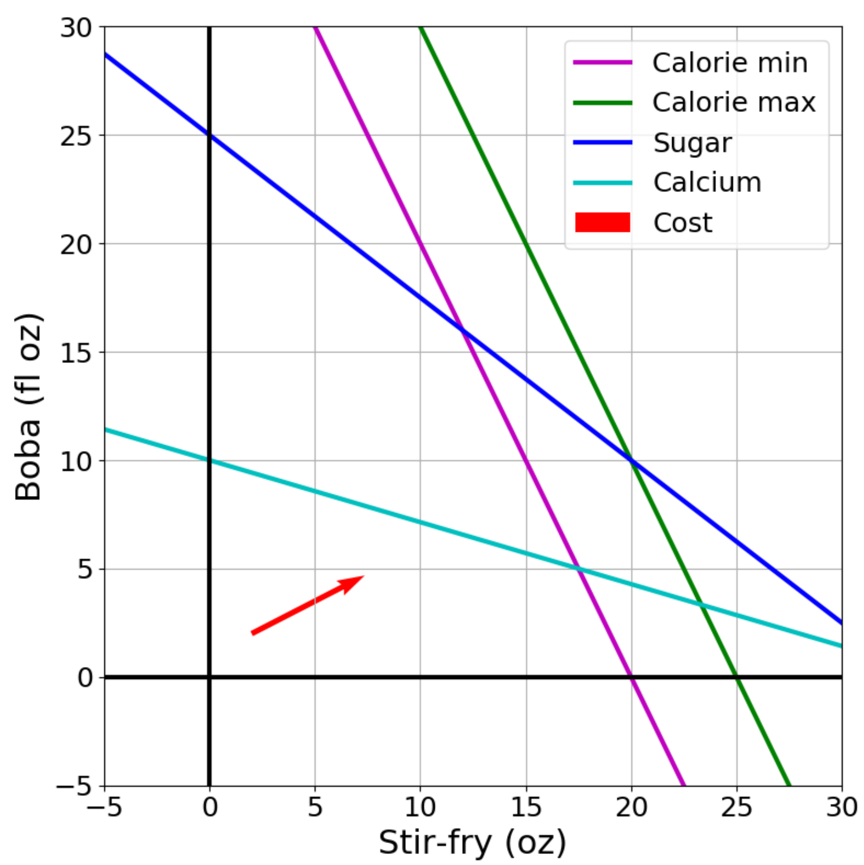
Optimization

Problem Description

Optimization Representation

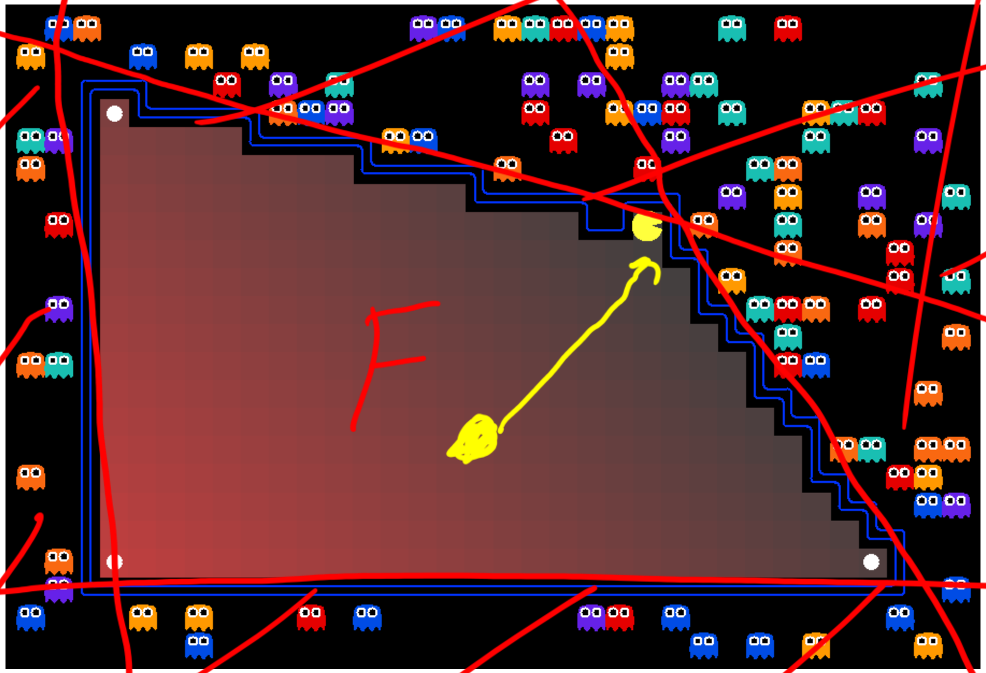
$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Graphical Representation



AI: Representation and Problem Solving

Linear and Integer Programming



Instructor: Pat Virtue

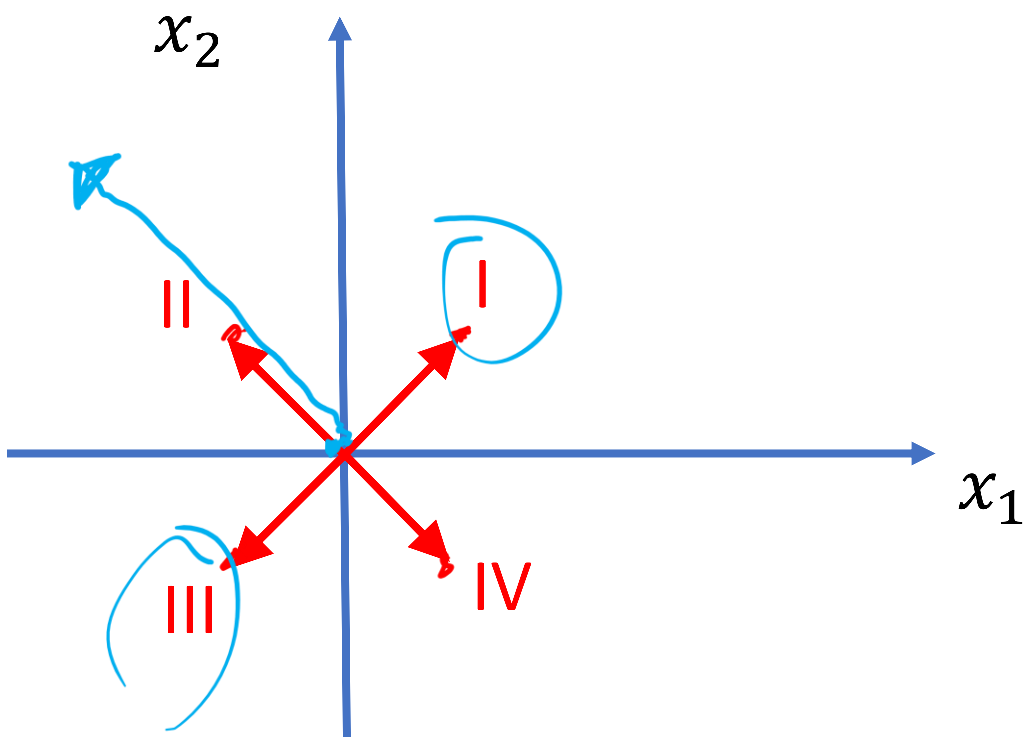
Slide credits: CMU AI with drawings from http://ai.berkeley.edu

Poll 1

Which of these points have cost $\mathbf{c}^\top \mathbf{x} = 0$?

for cost vector: $\mathbf{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow c_1 x_1 \ldots$$



$$c_1 x_1 + c_2 x_2$$
$$-2 \;\; ? \qquad 2 \;\; ?$$

Question

Given the cost vector $[c_1, c_2]^\top$ where will

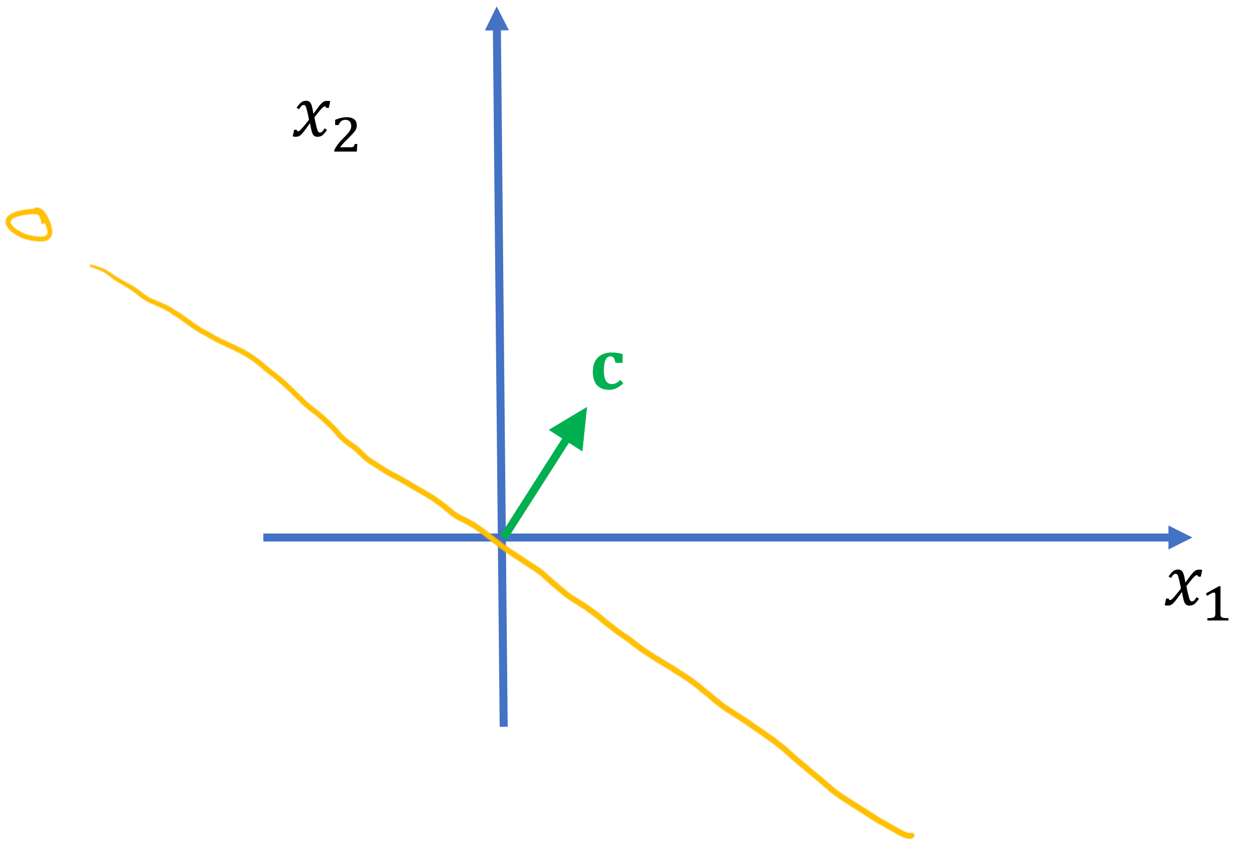
$\mathbf{c}^\top\mathbf{x} = 0$?

$\mathbf{c}^\top\mathbf{x} = 1$?

$\mathbf{c}^\top\mathbf{x} = 2$?

$\mathbf{c}^\top\mathbf{x} = -1$?

$\mathbf{c}^\top\mathbf{x} = -2$?



Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

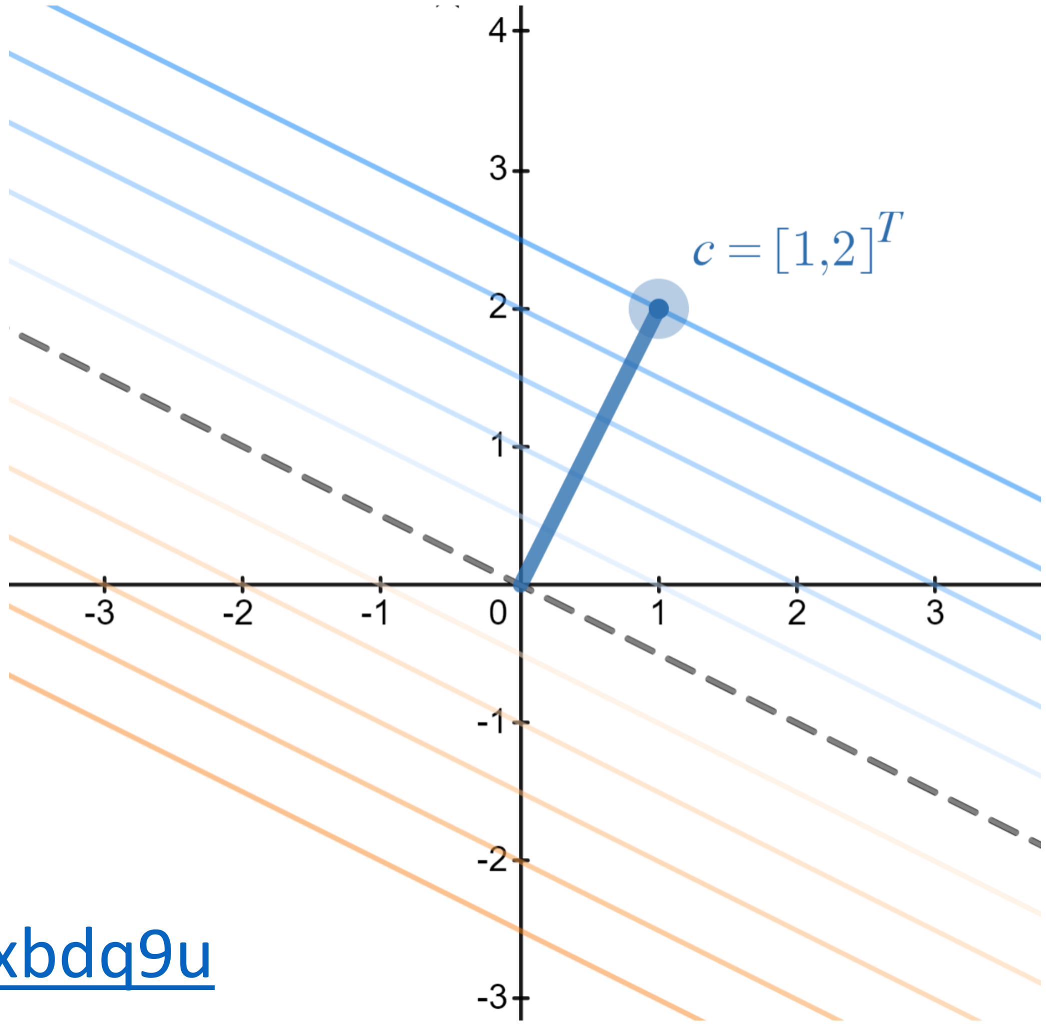
$\mathbf{c}^\top \mathbf{x} = 0$?

$\mathbf{c}^\top \mathbf{x} = 1$?

$\mathbf{c}^\top \mathbf{x} = 2$?

$\mathbf{c}^\top \mathbf{x} = -1$?

$\mathbf{c}^\top \mathbf{x} = -2$?



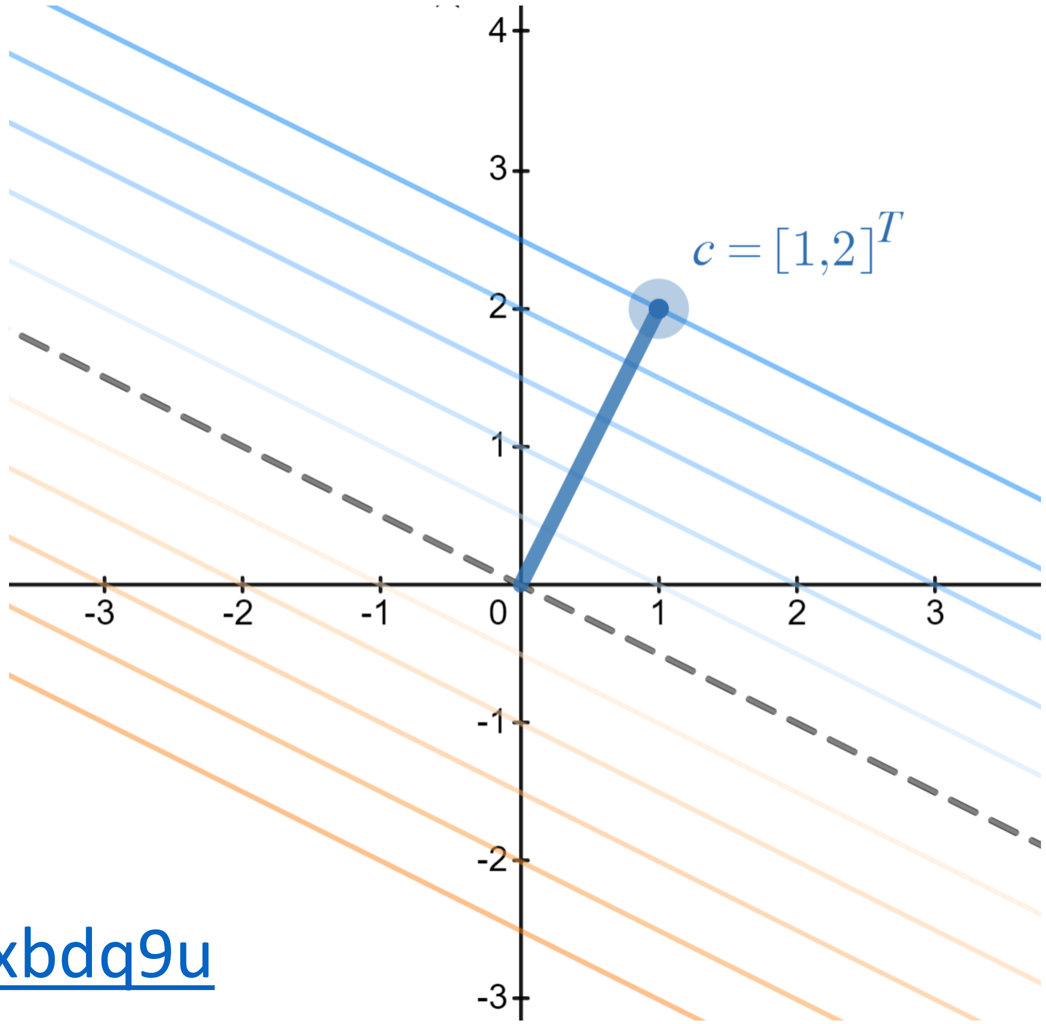
https://www.desmos.com/calculator/8d9kxbdq9u

Question

As the magnitude of $\mathbf{c}$ increases, the distance between the contours lines of the objective $\mathbf{c}^\top \mathbf{x}$:

A) Increases

B) Decreases



https://www.desmos.com/calculator/8d9kxbdq9u

Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

Graphics Representation

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

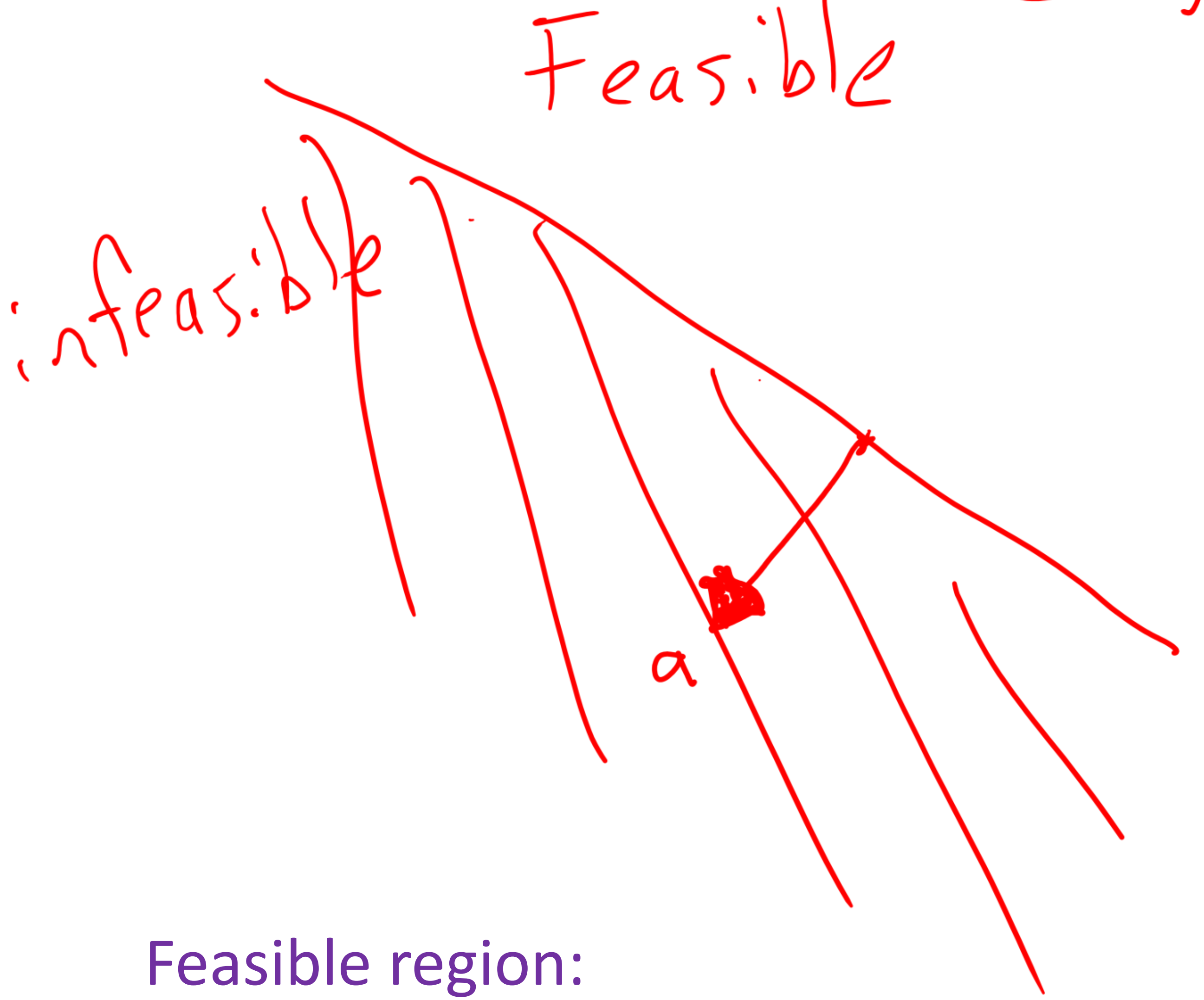
Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\, x_1 + a_2\, x_2 = \underline{b_1}$

2. $\underline{a_1}\, x_1 + \underline{a_2}\, x_2 \leq b_1$

3. $$\begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\leq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\leq b_4 \end{aligned}$$



Feasible region:
All points x that satisfy the constraints

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\, x_1 + a_2\, x_2 = b_1$

2. $a_1\, x_1 + a_2\, x_2 \leq b_1$

3. $$\begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\leq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\leq b_4 \end{aligned}$$

Feasible region:
All points x that satisfy the constraints

Graphics Representation

Geometry / Algebra I Quiz

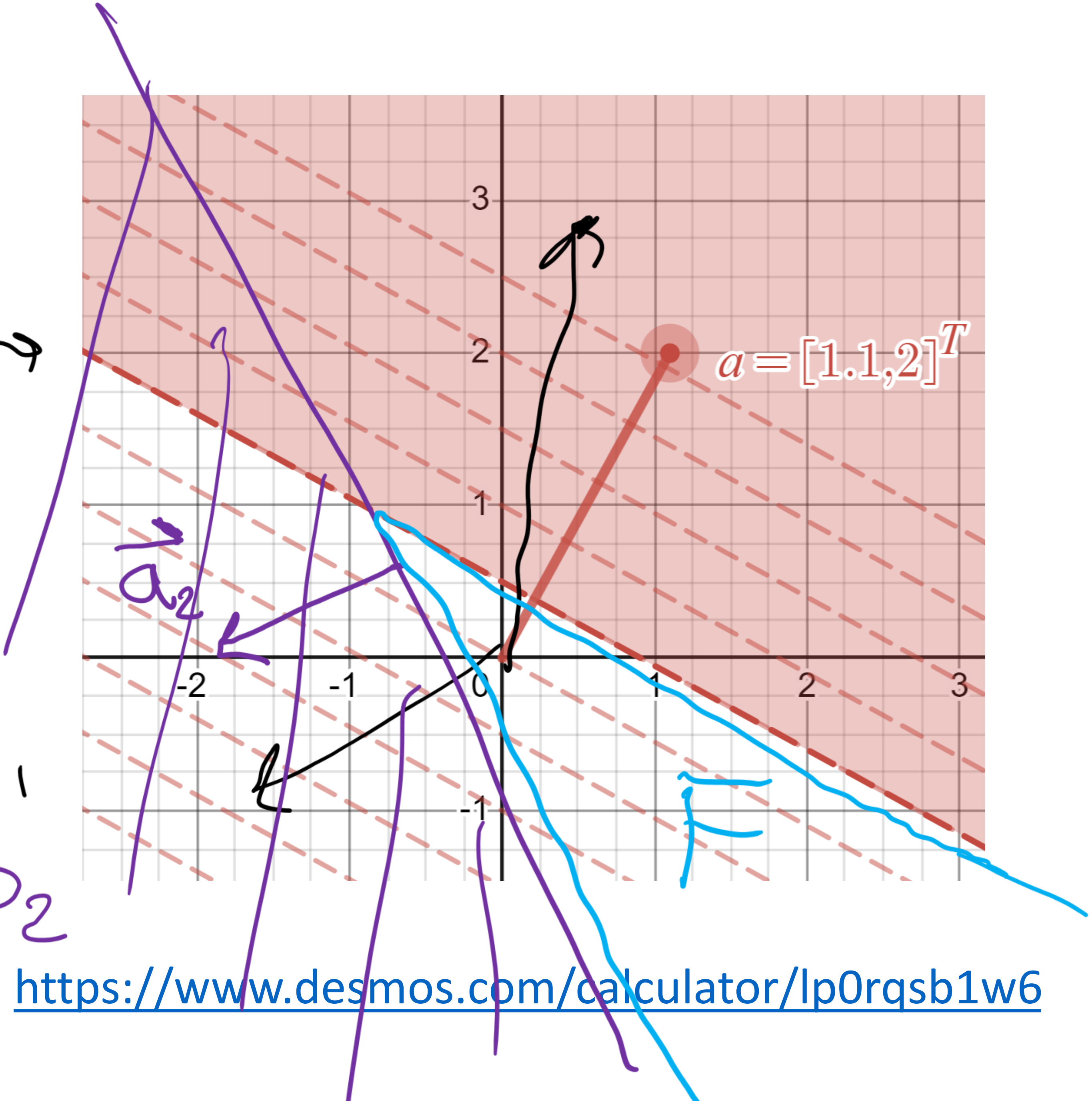
What shape do these represent?

1. $a_1 x_1 + a_2 x_2 = b_1$

2. $a_1 x_1 + a_2 x_2 \leq b_1$

3. $a_{11} x_1 + a_{12} x_2 \leq b_1$

$a_{21} x_1 + a_{21} x_2 \leq b_2$



https://www.desmos.com/calculator/lp0rqsb1w6

Graphics Representation

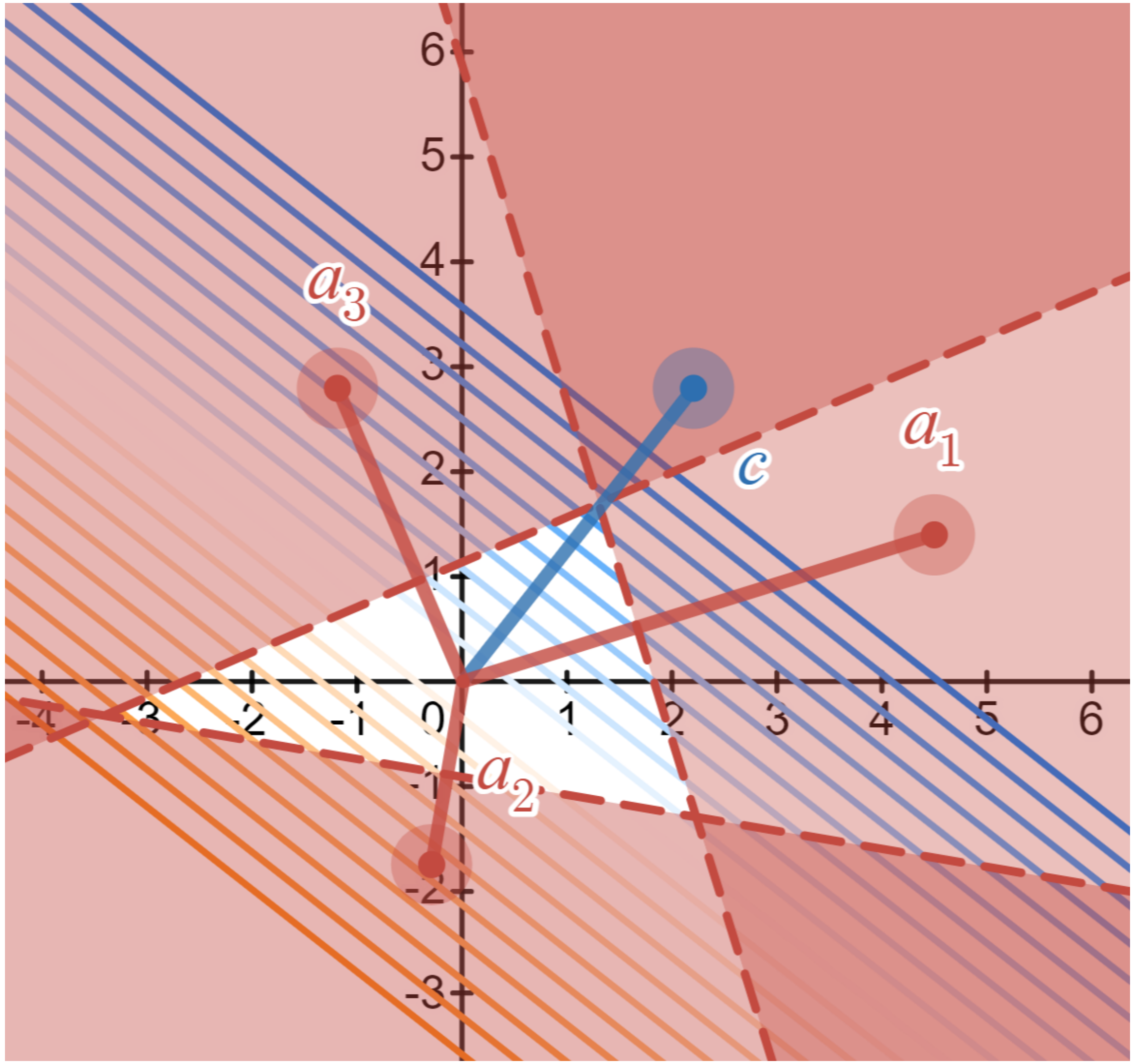
Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\,x_1 + a_2\,x_2 = b_1$

2. $a_1\,x_1 + a_2\,x_2 \leq b_1$

3. $$\begin{aligned} a_{1,1}\,x_1 + a_{1,2}\,x_2 &\leq b_1 \\ a_{2,1}\,x_1 + a_{2,2}\,x_2 &\leq b_2 \\ a_{3,1}\,x_1 + a_{3,2}\,x_2 &\leq b_3 \\ \cancel{a_{4,1}\,x_1 + a_{4,2}\,x_2} &\cancel{\leq b_4} \end{aligned}$$



https://www.desmos.com/calculator/plp1thgsbh

Reminder: Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

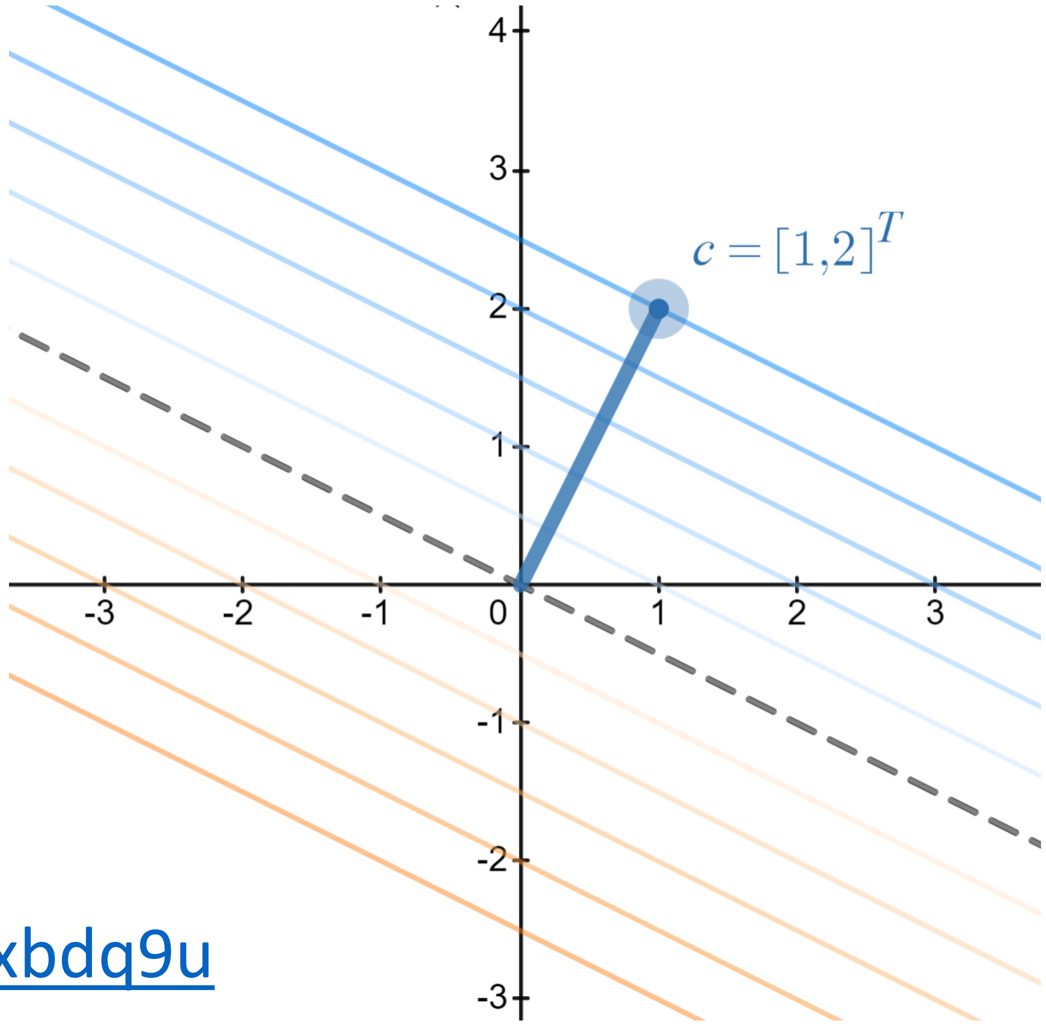
$\mathbf{c}^\top\mathbf{x} = 0$?

$\mathbf{c}^\top\mathbf{x} = 1$?

$\mathbf{c}^\top\mathbf{x} = 2$?

$\mathbf{c}^\top\mathbf{x} = -1$?

$\mathbf{c}^\top\mathbf{x} = -2$?



https://www.desmos.com/calculator/8d9kxbdq9u

Poll 2

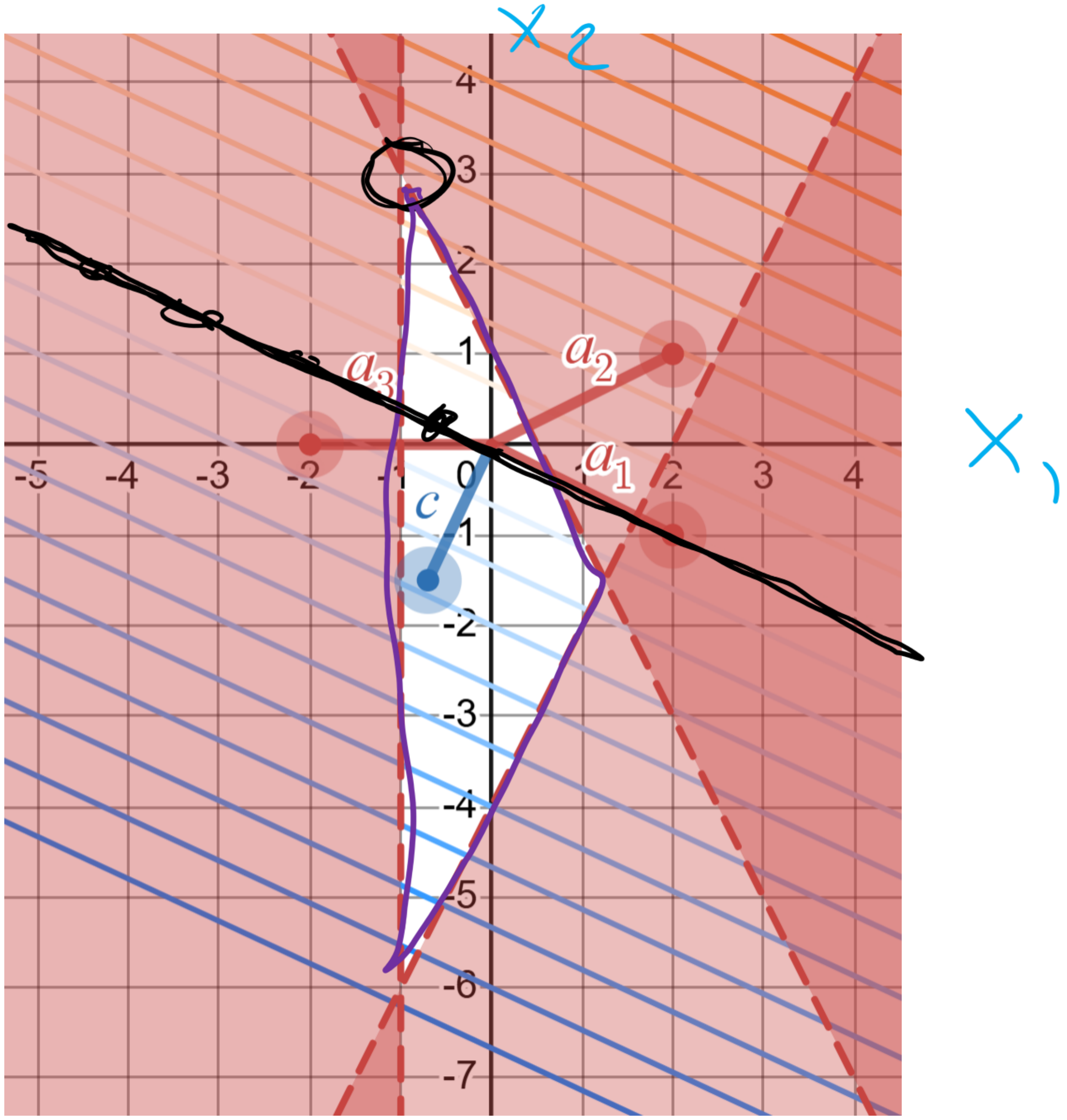
What is the solution to this LP?

x_1 $\boxed{}$

x_2 $\boxed{}$

$$\min_x \; c^T x$$

$$\text{s.t. } Ax \le b$$

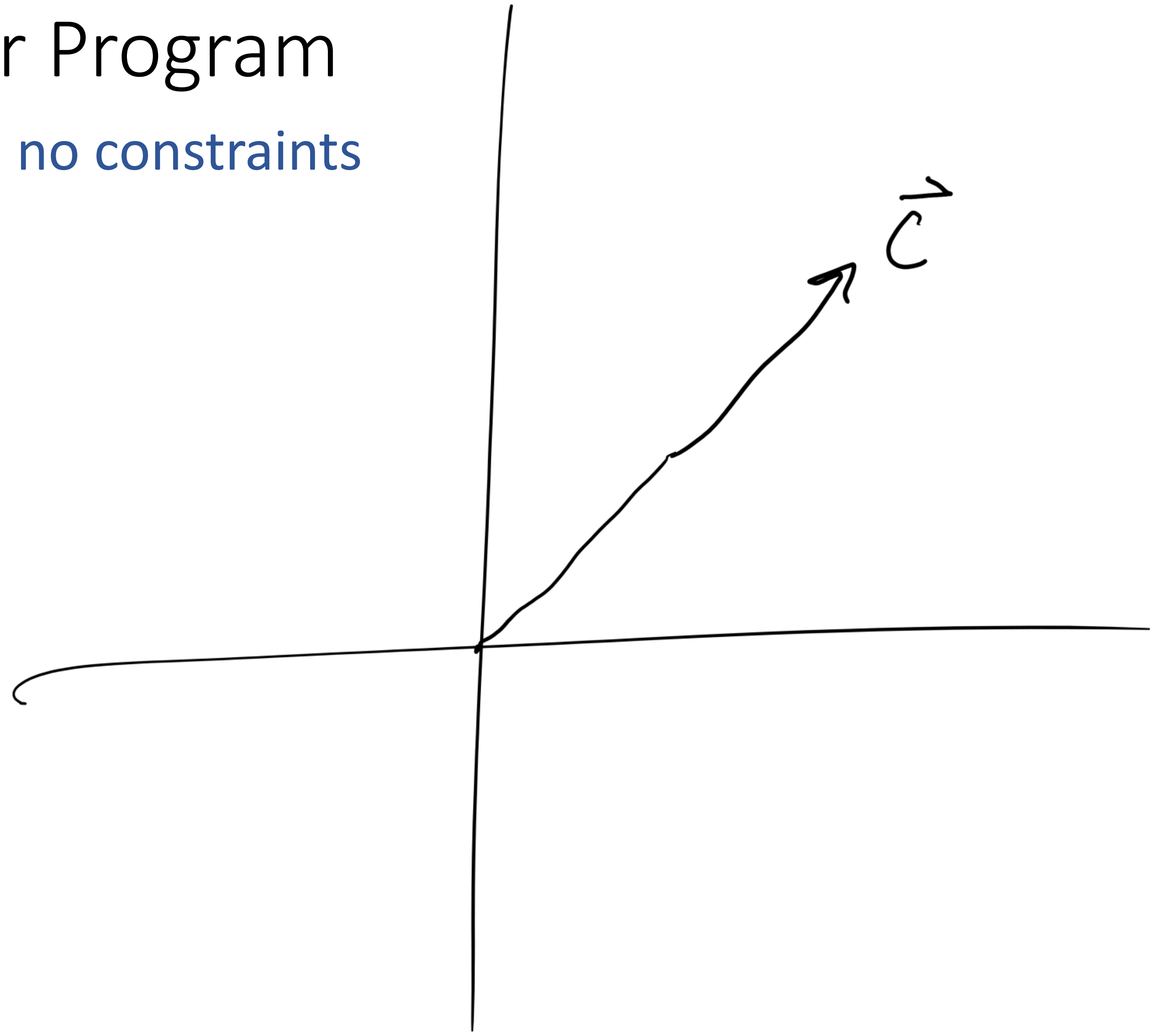


https://www.desmos.com/calculator/tnlo7p5plp

Solving a Linear Program

Inequality form, with no constraints

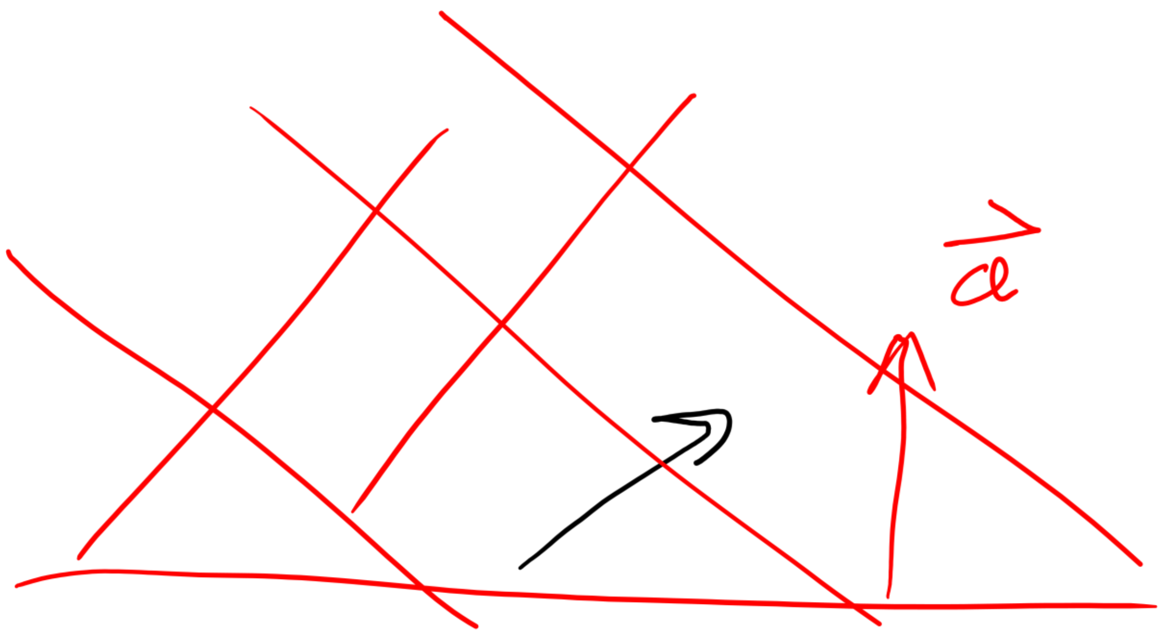
$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$



Solving a Linear Program

Inequality form, with one constraint

$$
\begin{aligned}
\min_{\mathbf{x}}.\quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.}\quad & a_1 x_1 + a_2 x_2 \leq b
\end{aligned}
$$

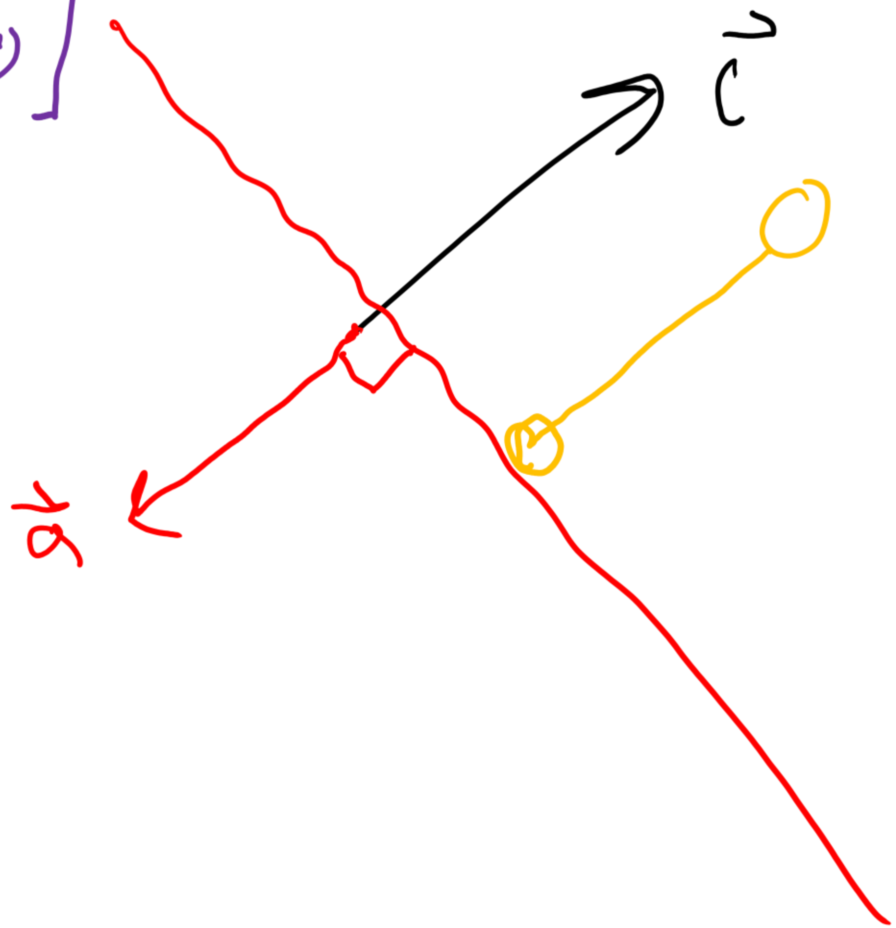


Poll 3

True or False: A minimizing LP with exactly on constraint, will always have a minimum objective at $-\infty$.

F if $\vec{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$



Solving an LP

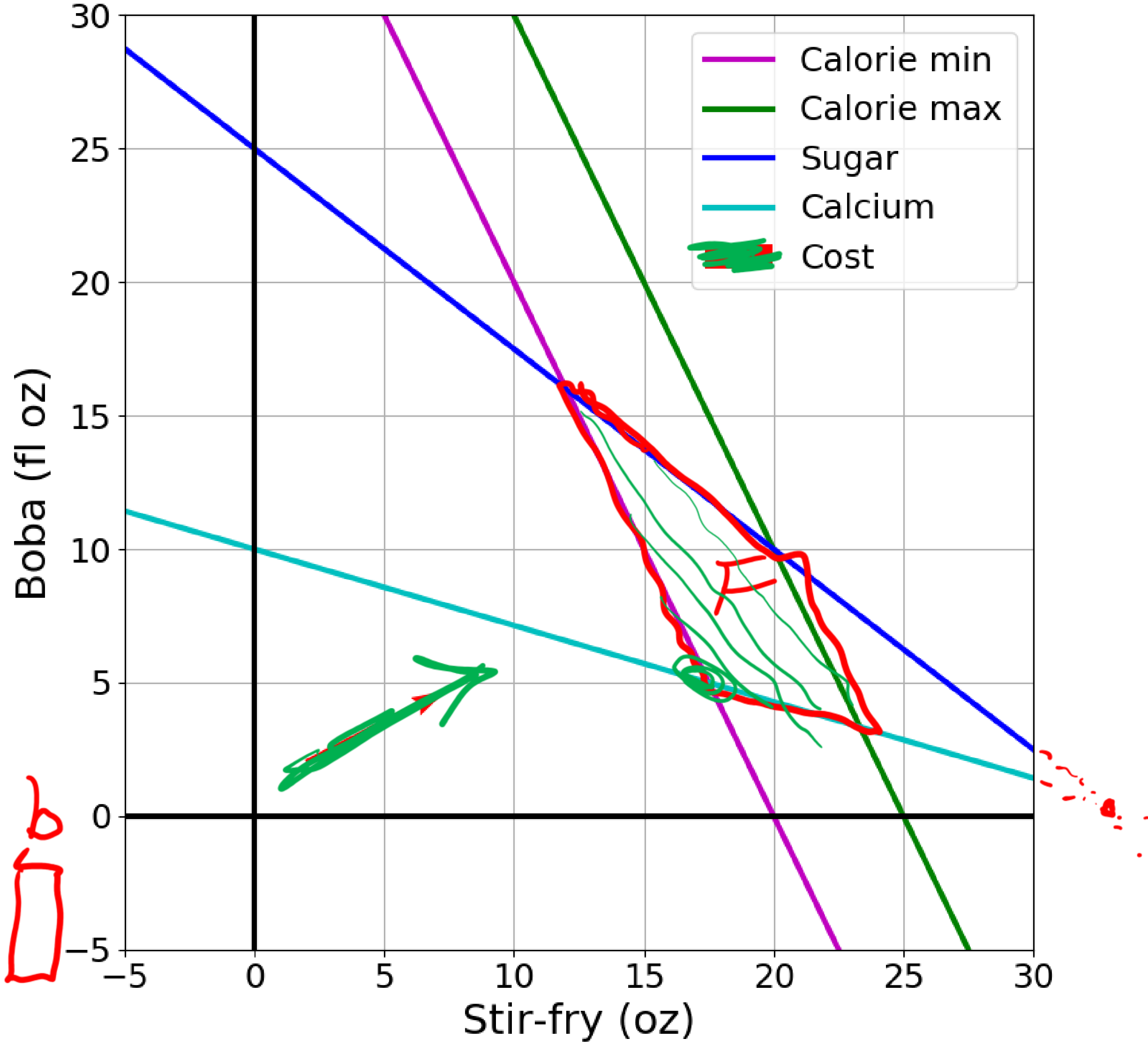
Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections

In more detail:

1. Enumerate all intersections
2. Keep only those that are feasible (satisfy *all* inequalities) $Ax \leq b$
3. Return feasible intersection with the lowest objective value $c^T x$



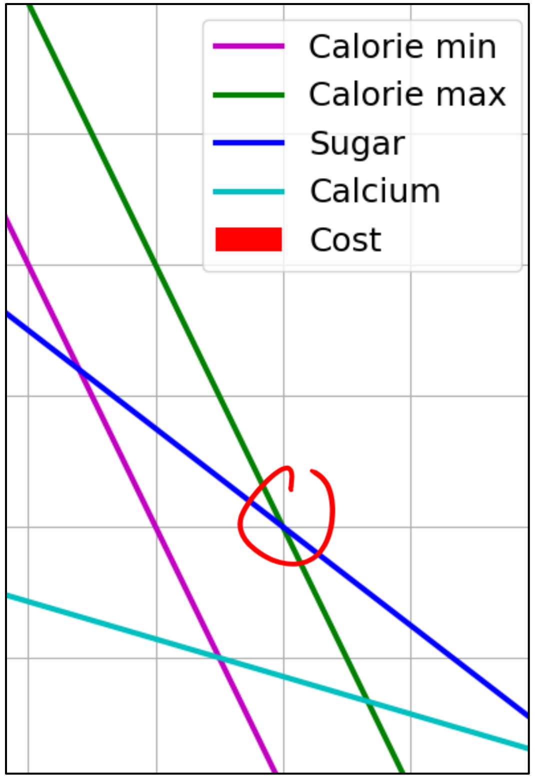
Solving an LP

But, how do we find the intersection between boundaries?

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



$$A' \begin{bmatrix} 100 & 50 \\ 3 & 4 \end{bmatrix} \quad b' = \begin{bmatrix} 2500 \\ 100 \end{bmatrix}$$

$$x' = (A')^{-1} b'$$

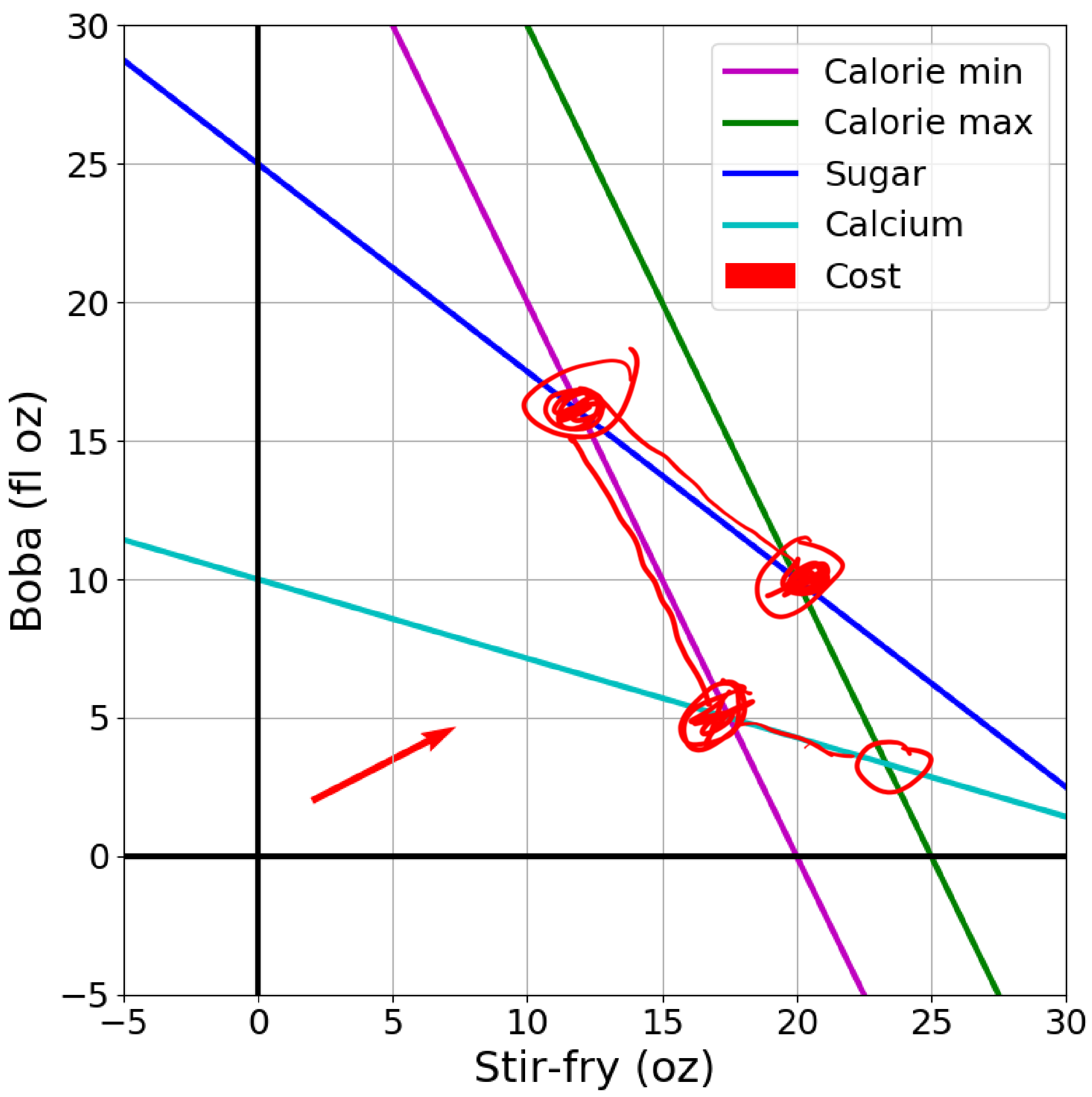
If invertible (full rank)

Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

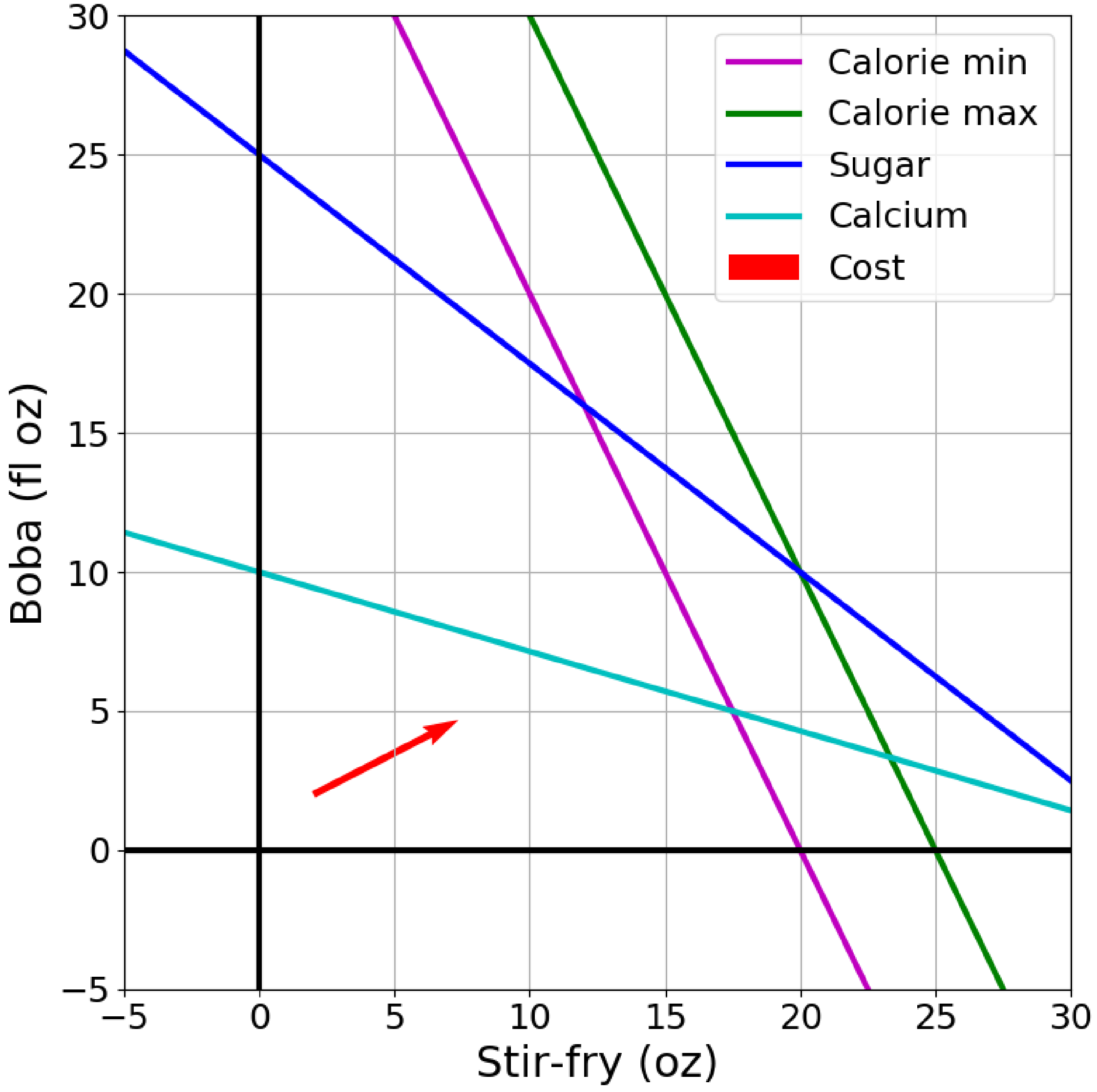
- Check objective at all feasible intersections
- Simplex



Solving an LP

Simplex algorithm

- Start at a feasible intersection (if not trivial, can solve another LP to find one)
- Define successors as “neighbors” of current intersection
  - i.e., remove one row from our square subset of A, and add another row not in the subset; then check feasibility
- Move to any successor with lower objective than current intersection
  - If no such successors, we are done



Greedy local hill-climbing search! … but always finds *optimal* solution

Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior Point

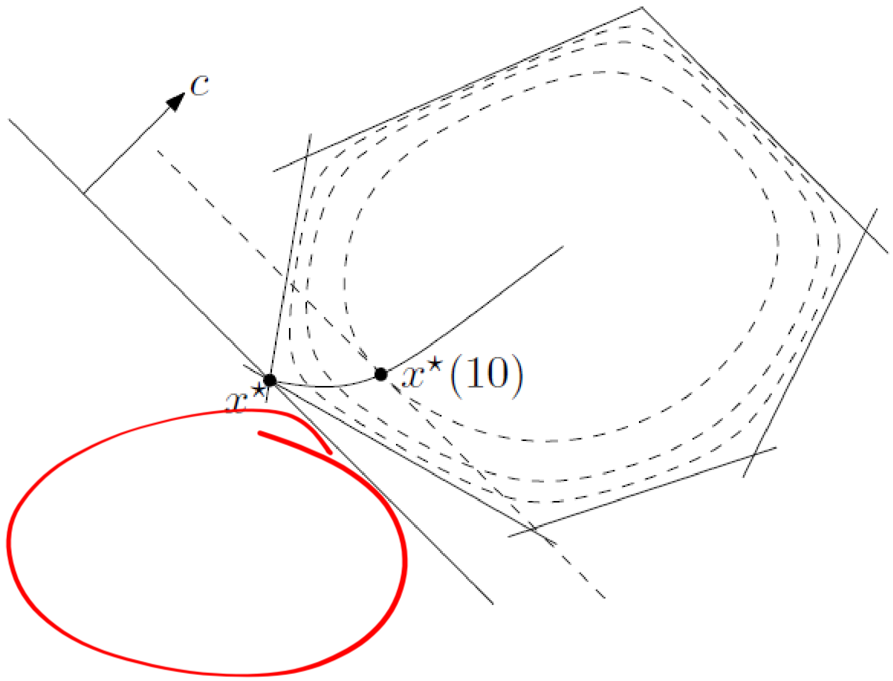


Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

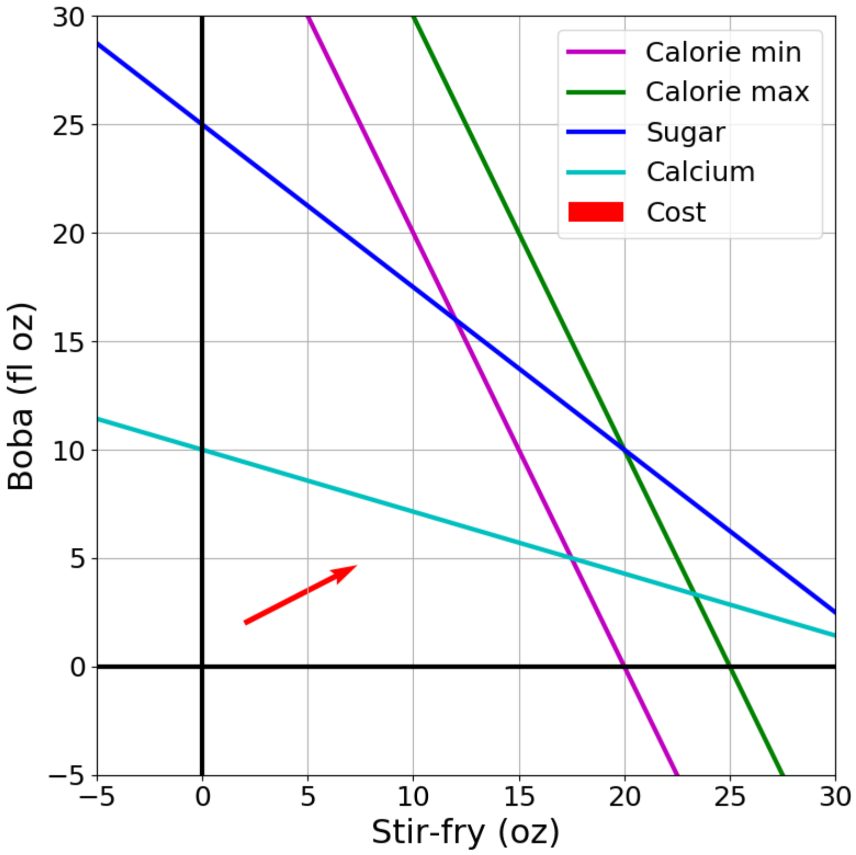
What about higher dimensions?

Problem Description

Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Graphical Representation



"Marty, you're not thinking fourth-dimensionally"

https://www.youtube.com/watch?v=CUcNM7OsdsY

Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

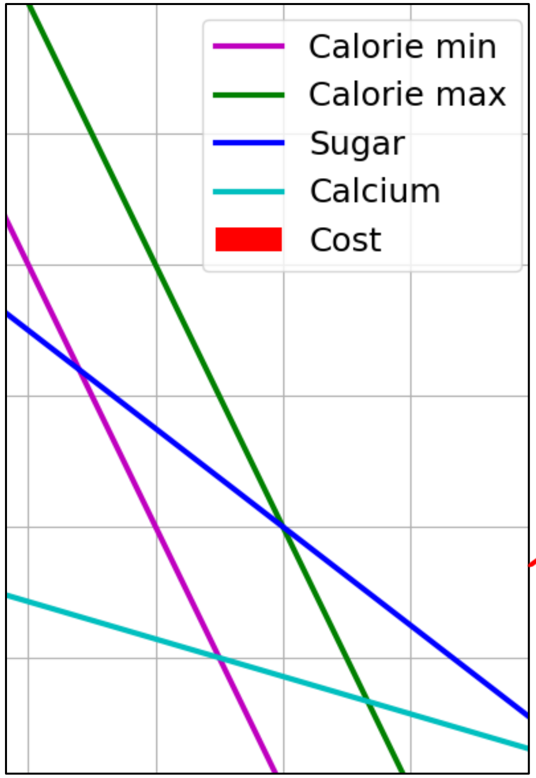
	2-D	3-D	N-D
$a_1\, x_1 + a_2\, x_2 = b_1$	line	plane	hyperplane
$a_1 x_1 + a_2 x_2 + a_3 x_3 = b_1$			
$a_1\, x_1 + a_2\, x_2 \leq b_1$	half plane	half space	half space
$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$ $a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$ $a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$ $a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$	polygon	polyhedron	polytope

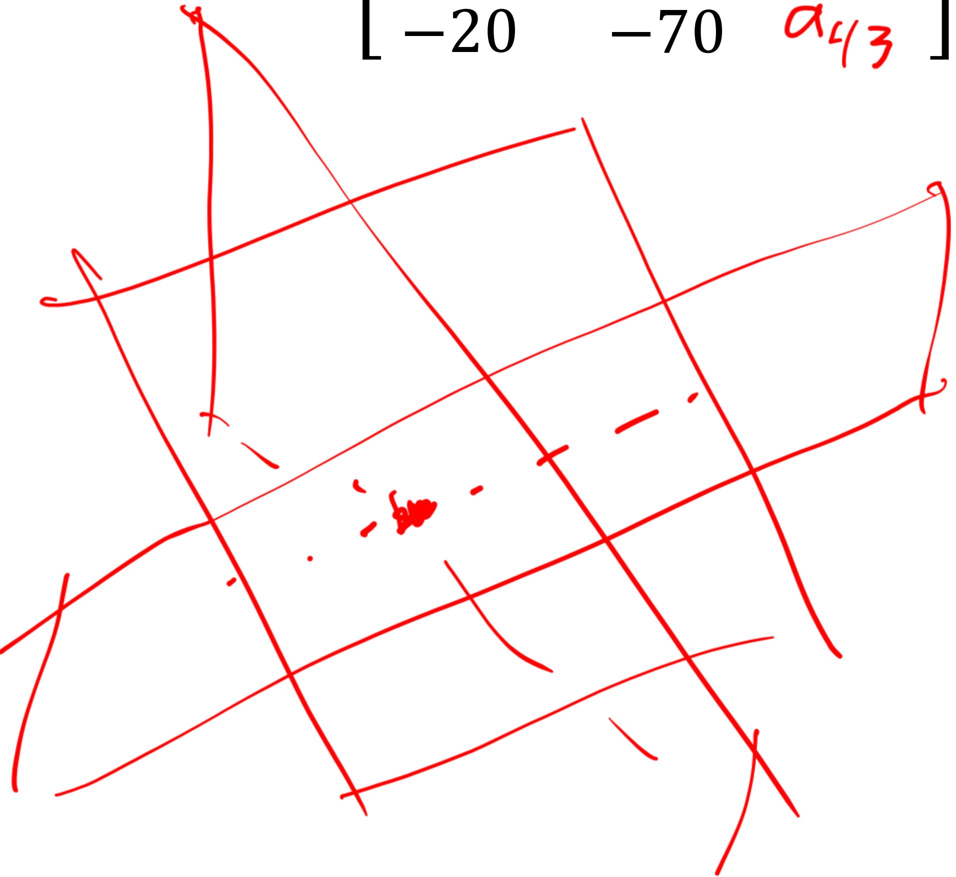
What are intersections in higher dimensions?

How do these linear shapes extend to 3-D, N-D?

$$\min_{\mathbf{x}} \quad \mathbf{c}^{\mathsf{T}}\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 & a_{13} \\ 100 & 50 & a_{23} \\ 3 & 4 & a_{33} \\ -20 & -70 & a_{43} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \begin{matrix} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{matrix}$$



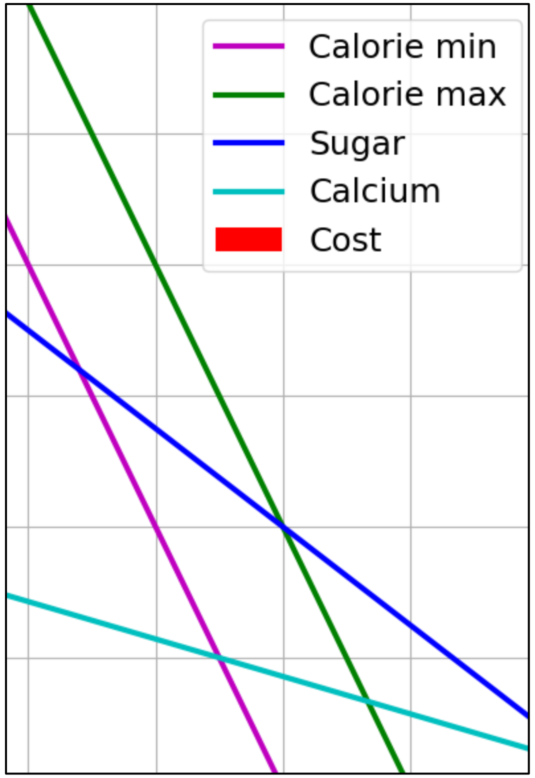


How do we find intersections in higher dimensions?

Still looking at subsets of A matrix

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 & \\ 100 & 50 & \\ 3 & 4 & \\ -20 & -70 & \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$



Integer Programming

Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	~~0.5~~ 5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per bowl)	1	100	3	20
Boba (per glass)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

LP

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

$\vec{x} \in \mathbb{R}^N$

IP

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \underline{\mathbf{x} \in \mathbb{Z}^N} \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

Notation Alert!

Integer Programming: Graphical Representation

Just add a grid of integer points onto our LP representation

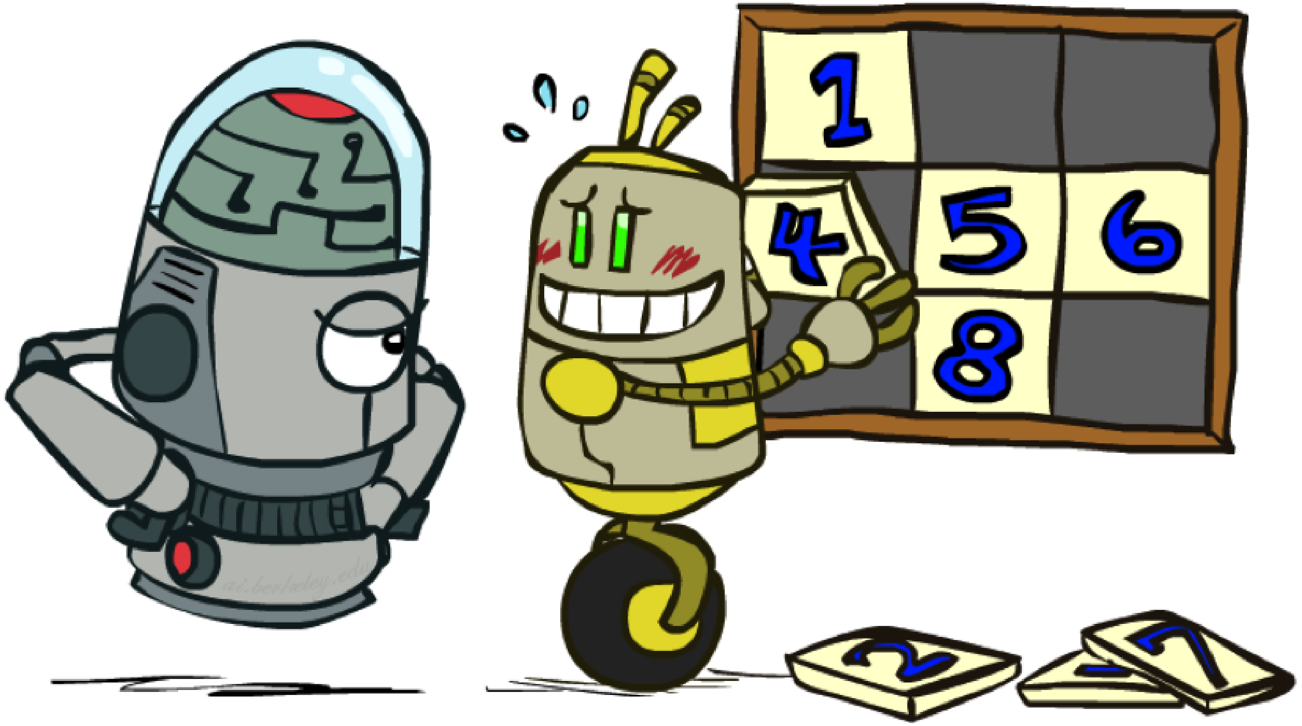
$$
\begin{aligned}
\min_{\mathbf{x}}. \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \in \mathbb{Z}^N
\end{aligned}
$$

Relaxation

Relax IP to LP by dropping integer constraints

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \cancel{\mathbf{x} \in \mathbb{Z}^N} \end{aligned}$$

Remember heuristics?



Notation Alert

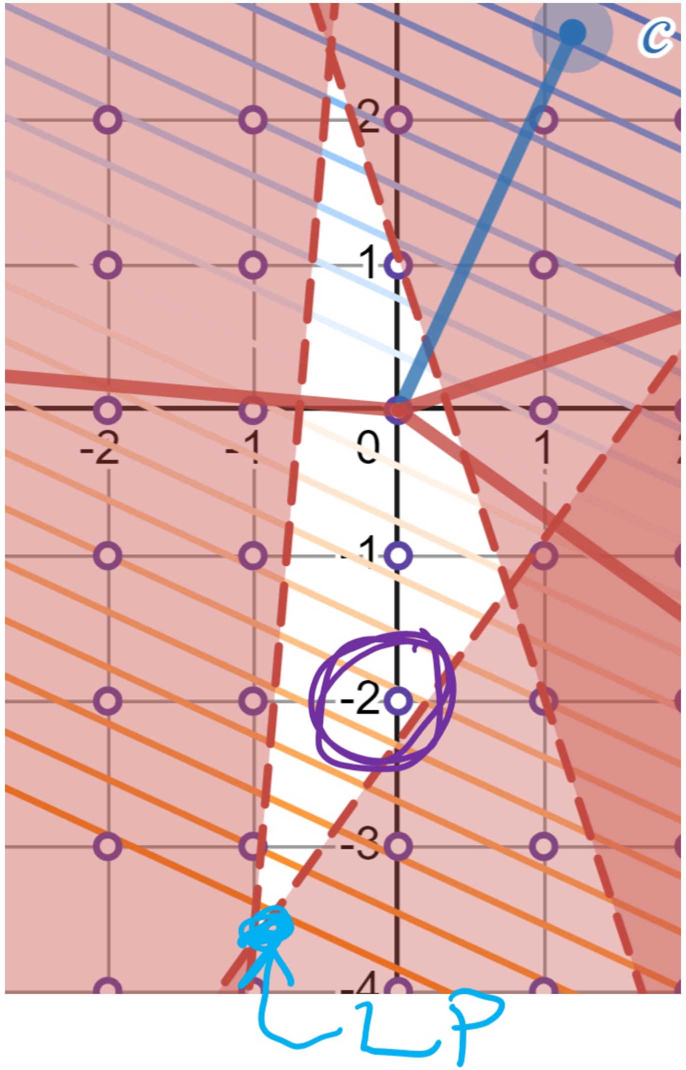
Let y represent the objective value (total cost), $y = \mathbf{c}^\top \mathbf{x}$

Pay attention to argmin. vs just min.

$$x^*_{IP} = \underset{\mathbf{x}}{\text{argmin.}} \quad \mathbf{c}^\top \mathbf{x} \qquad = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \in \mathbb{Z}^N$$

$$y^*_{IP} = \underset{\mathbf{x}}{\text{min.}} \quad \mathbf{c}^\top \mathbf{x} \qquad = -5.1$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \in \mathbb{Z}^N$$

$$y^*_{LP} = c^T x^*_{IP}$$



Poll 4:

Let y_{IP}^* be the optimal objective of an integer program P.
Let $\mathbf{x}_{IP}^*$ be an optimal point of an integer program P.
Let y_{LP}^* be the optimal objective of the LP-relaxed version of P.
Let $\mathbf{x}_{LP}^*$ be an optimal point of the LP-relaxed version of P.
Assume that P is a minimization problem.

$$\boxed{\begin{aligned} y_{IP}^* = \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}}$$

$$\boxed{\begin{aligned} y_{LP}^* = \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \end{aligned}}$$

Which of the following are true?

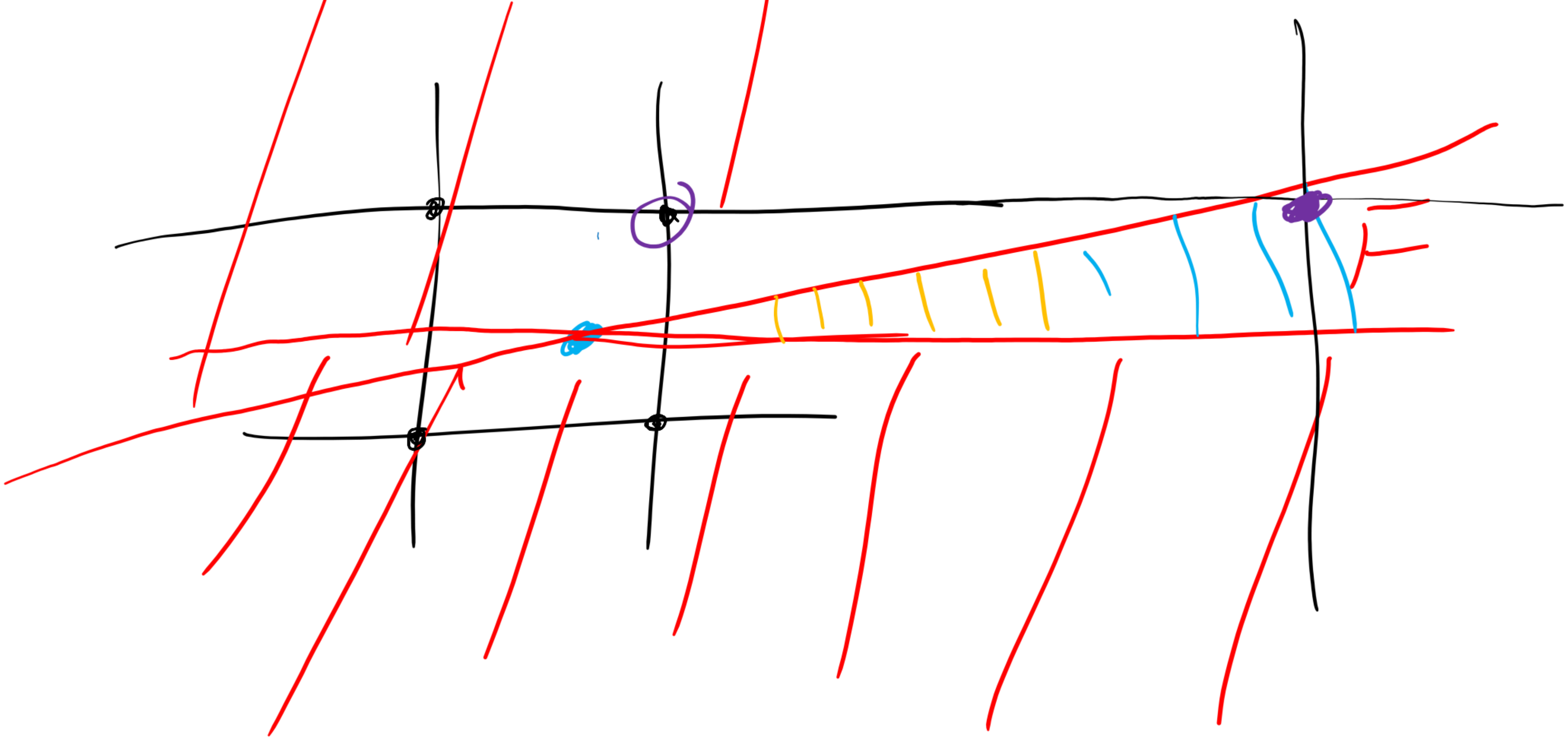
A) $\mathbf{x}_{IP}^* = \mathbf{x}_{\mathrm{LP}}^*$

B) $y_{IP}^* \leq y_{LP}^*$

C) $y_{IP}^* \geq y_{LP}^*$

Poll 5:

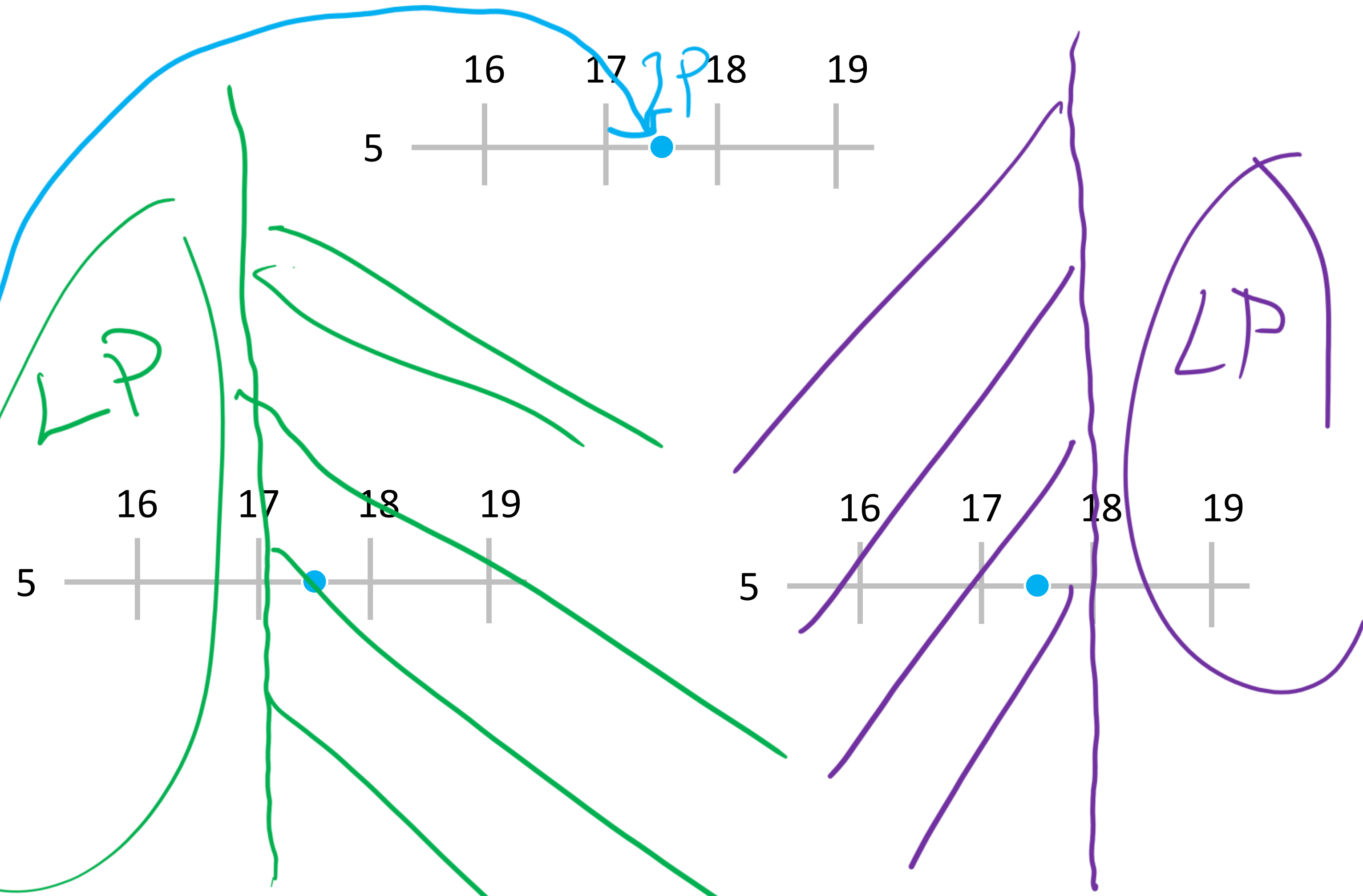
True/False: It is sufficient to consider the integer points around the corresponding LP solution?



Solving an IP

Branch and Bound algorithm

Core steps:



Use LP solver to find $\mathbf{x}^\star_{LP}$

If $\mathbf{x}^\star_{LP}$ is all integer valued,

return solution

Otherwise:

Create two branches

→ *Left branch*: Added constraint $x_i \leq floor(x_i)$ 17.5 → 17

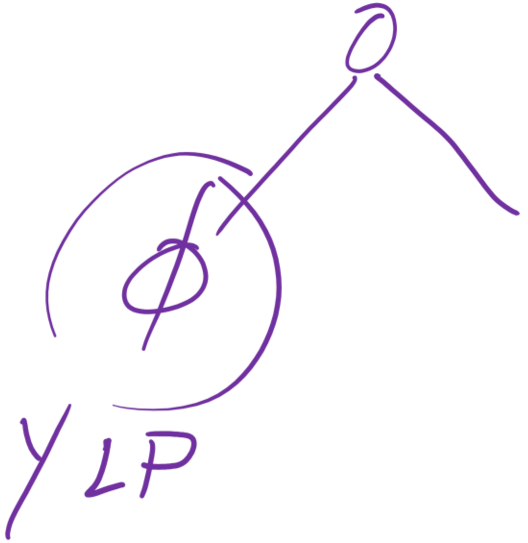
→ *Right branch*: Added constraint $x_i \geq ceil(x_i)$ 17.5 → 18

Solving an IP

Branch and Bound algorithm

1. Push LP solution of problem into priority queue,

   ordered by objective value of LP solution



2. Repeat:
   - If queue is empty, return IP is infeasible
   - Pop candidate solution $\mathbf{x}_{LP}^{\star}$ from priority queue ()
   - If $\mathbf{x}_{LP}^{\star}$ is all integer valued, we are done; return solution
   - Otherwise, select a coordinate x_i that is not integer valued, and add two additional LPs to the priority queue:

     Left branch: Added constraint $x_i \leq floor(x_i)$

     Right branch: Added constraint $x_i \geq ceil(x_i)$

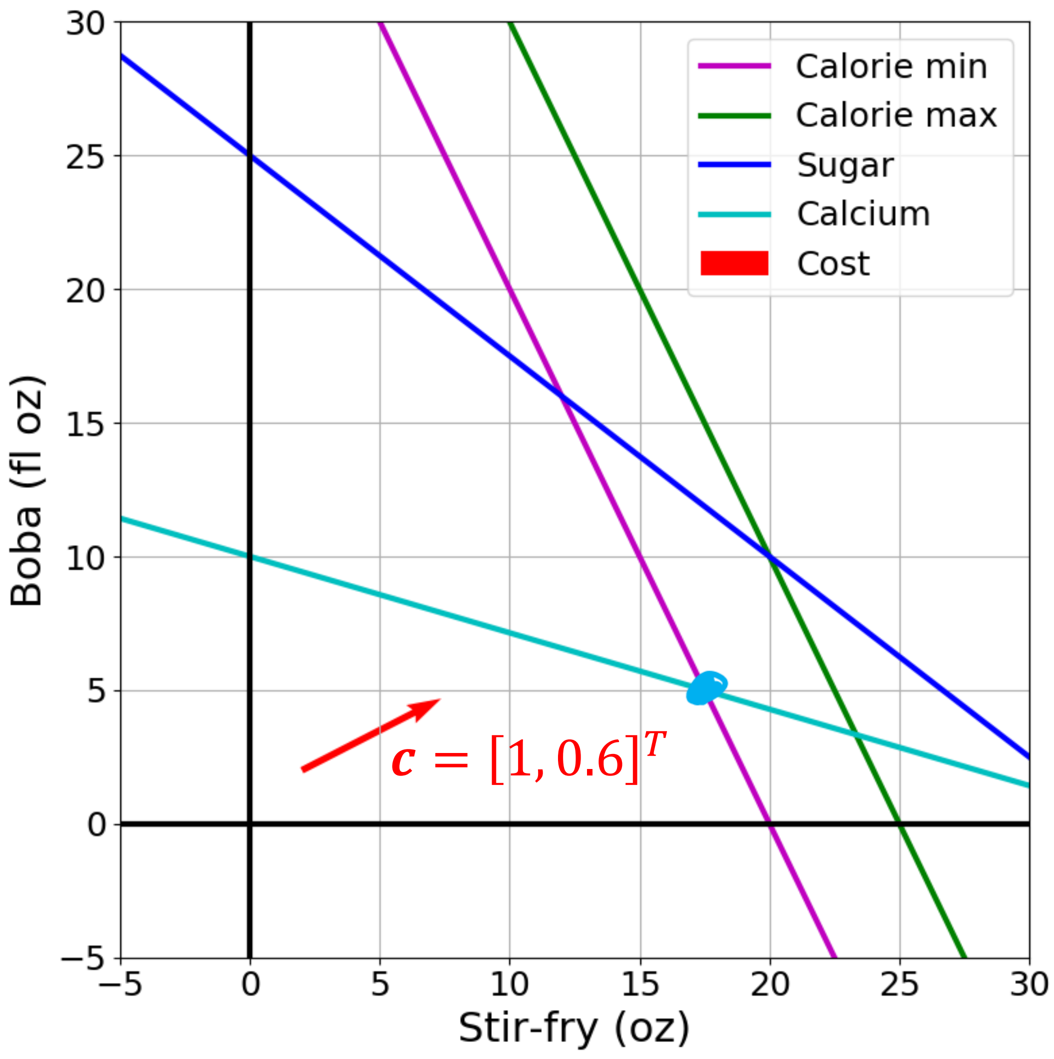
Note: Only add LPs to the queue if they are feasible

Branch and Bound Example

$$\mathbf{x}^\star = (17.5,5)$$
$$y^\star = 20.5$$

$x_1 \leq 17$

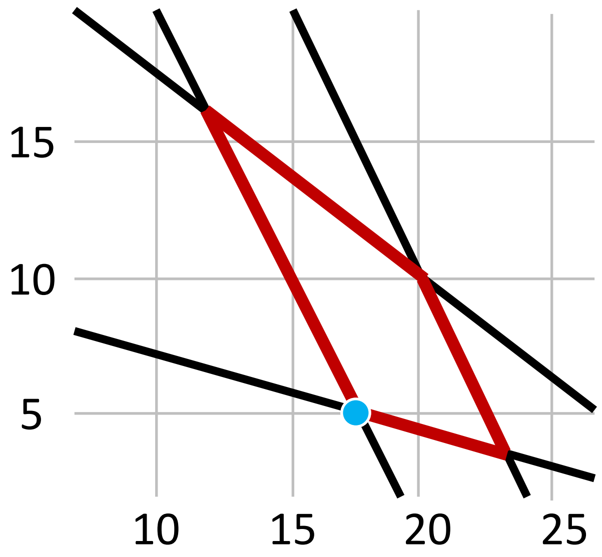
$x_1 \geq 18$



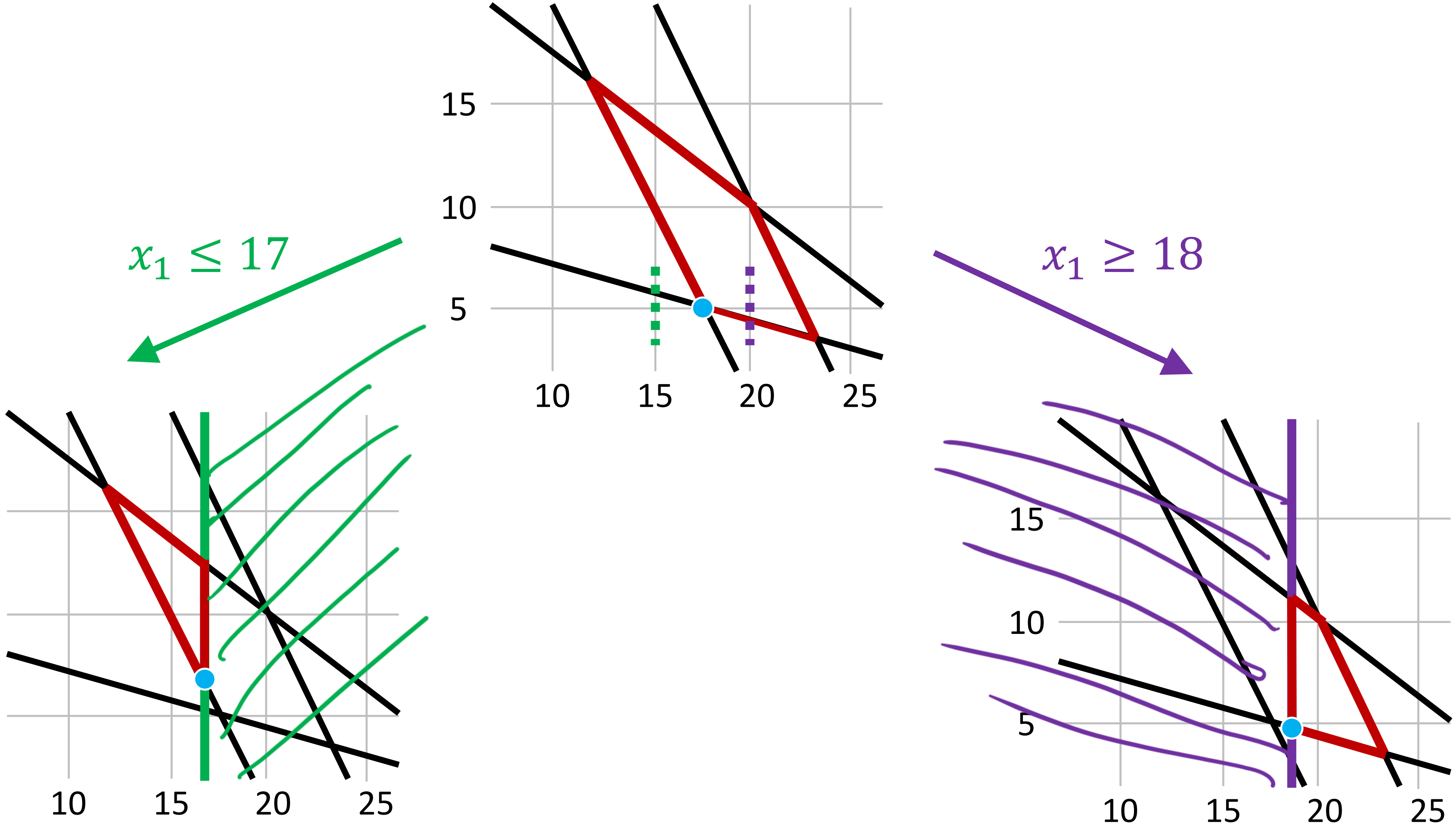
Priority Queue:

1. ~~$\mathbf{x}^\star = (17.5,5), \; y^\star = 20.5$~~

Branch and Bound Example



Branch and Bound Example



Branch and Bound Example

$$\mathbf{x}^\star = (17.5,5)$$
$$y^\star = 20.5$$

$x_1 \leq 17$ | $x_1 \geq 18$

$$\mathbf{x}^\star = (17,6)$$
$$y^\star = 20.6$$

$$\mathbf{x}^\star = (18,4.85)$$
$$y^\star = 20.91$$

Priority Queue:

1. $\mathbf{x}^\star = (17,6), \quad y^\star = 20.6$
2. $\mathbf{x}^\star = (18,4.85), \quad y^\star = 20.91$