Plan

Last Time

- Adversarial search
  - Minimax
  - Evaluation functions
  - Pruning

Today

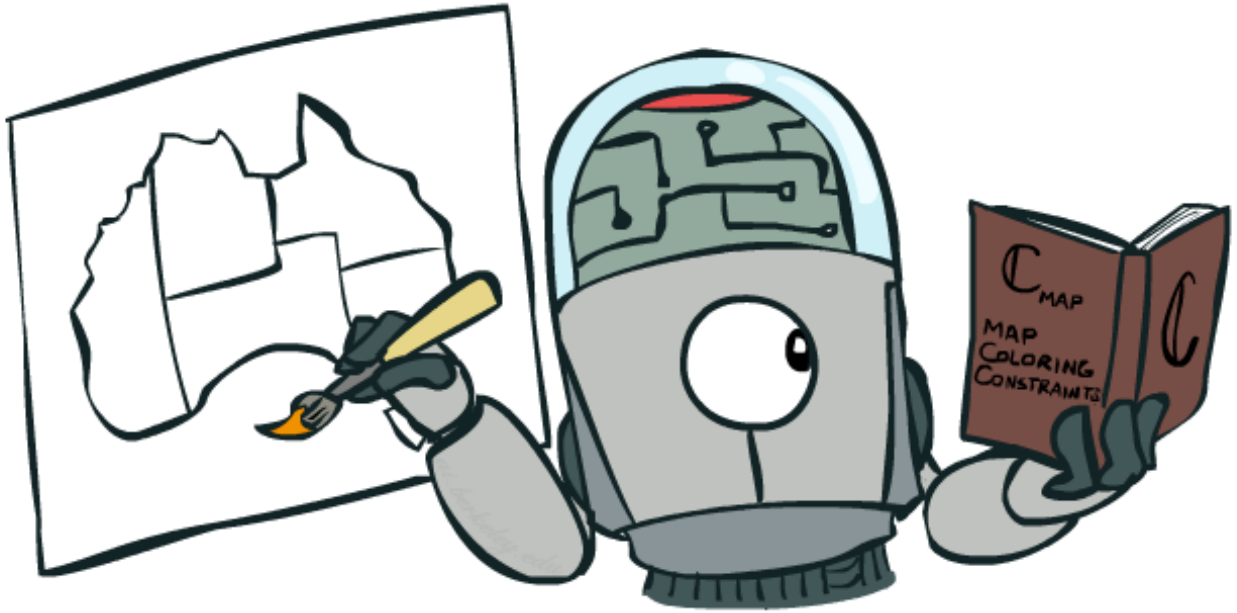
- Adversarial search: Expectimax
- Constraint Satisfaction Problems

Expectimax

Adversarial search slides

AI: Representation and Problem Solving

Constraint Satisfaction Problems (CSPs)

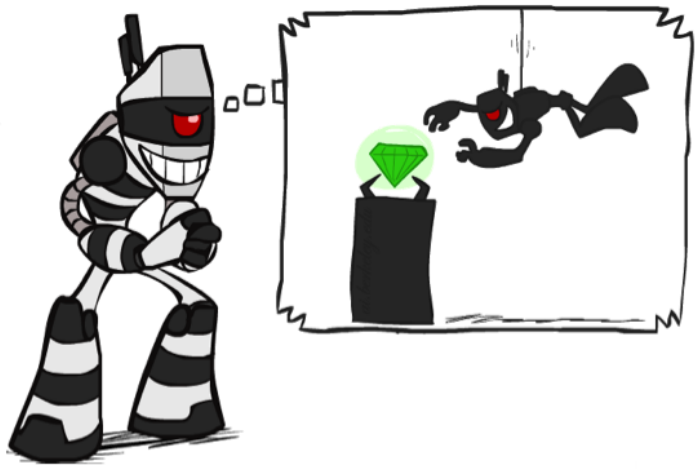


Instructor: Pat Virtue

Slide credits: CMU AI, http://ai.berkeley.edu

What is Search For?

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

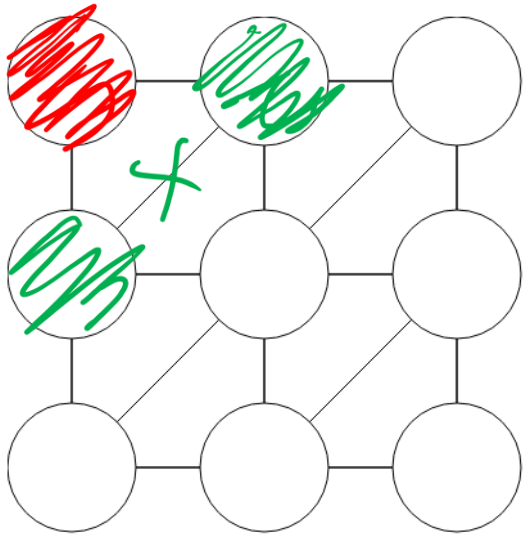


- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)

Are the warm-up assignments
planning or identification problems?

Warm-up as You Walk In

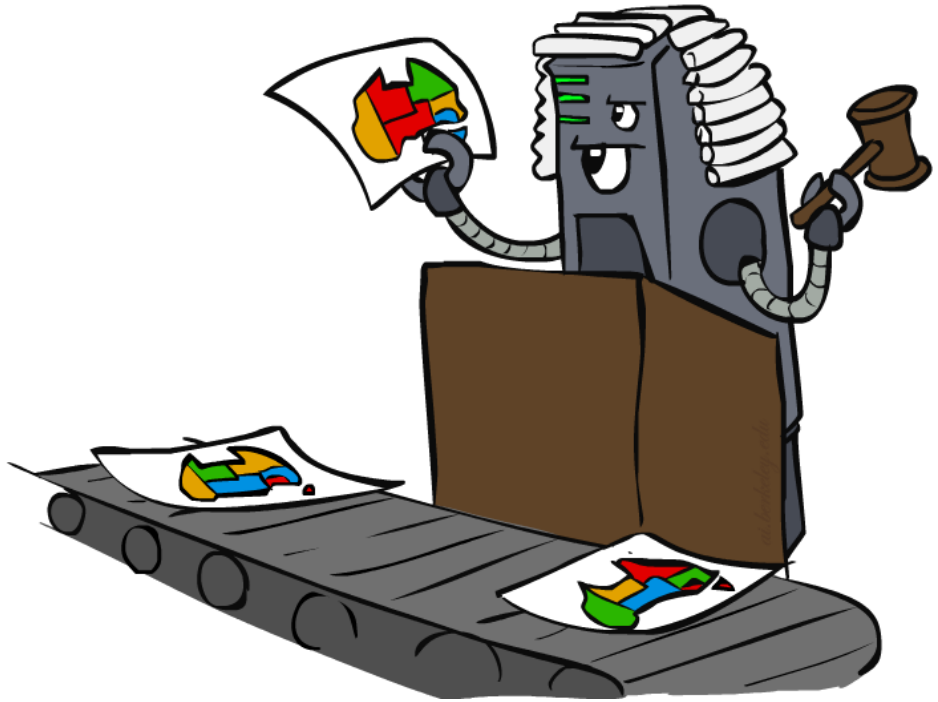
Assign Red, Green, or Blue
Neighbors must be different



Sudoku

1			
	2	1	
		3	
			4

Constraint Satisfaction Problems



CSP is a special class of search problems

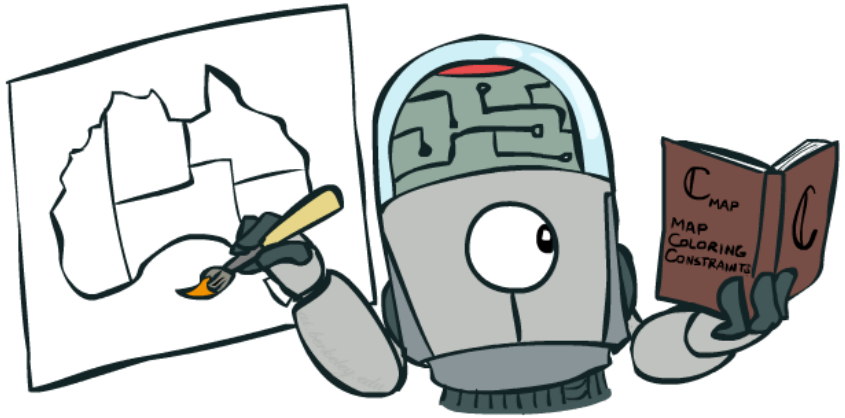
- Mostly identification problems
- Have specialized algorithms for them

Standard search problems:

- State is an arbitrary data structure
- Goal test can be any function over states

Constraint satisfaction problems (CSPs):

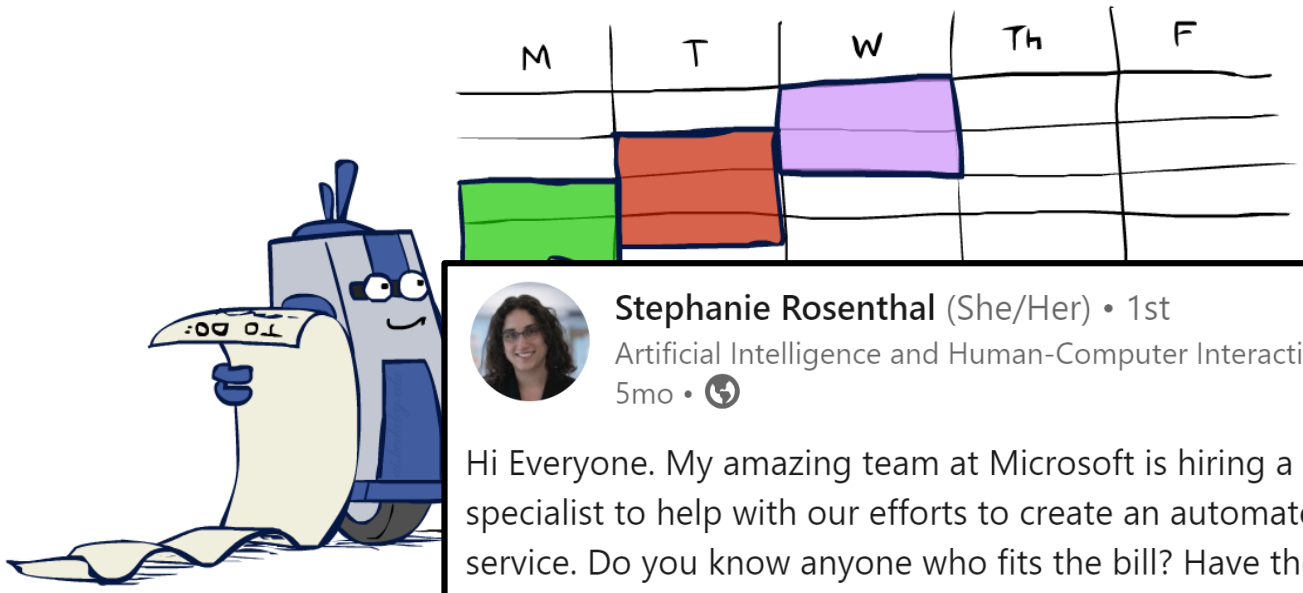
- State is defined by variables X_i with values from a domain D (sometimes D depends on i)
- Goal test is a set of constraints specifying allowable combinations of values for subsets of variables



Why study CSPs?

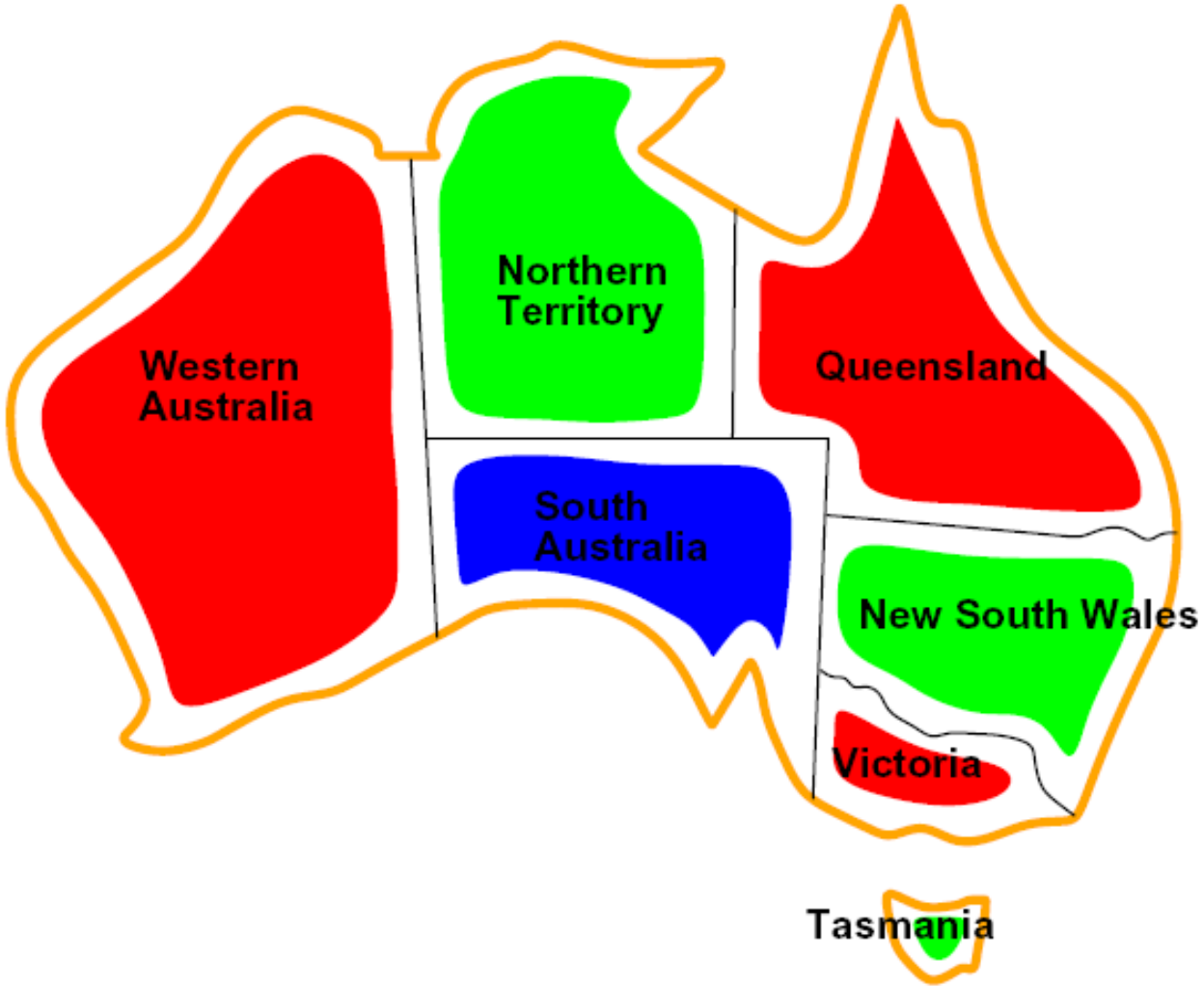
Many real-world problems can be formulated as CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



- Sometimes involve real-valued variables…

CSP Examples

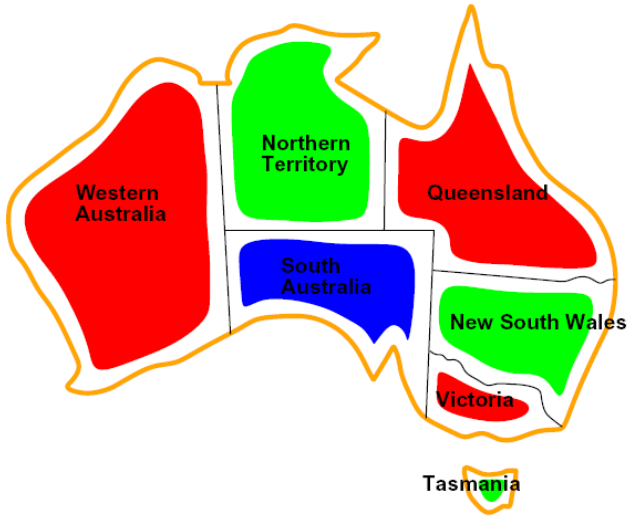


Example: Map Coloring

- Variables: $\mathrm{WA},\ \mathrm{NT},\ \mathrm{Q},\ \mathrm{NSW},\ \mathrm{V},\ \mathrm{SA},\ \mathrm{T}$
- Domains: $\mathrm{D} = \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$
- Constraints: adjacent regions must have different colors

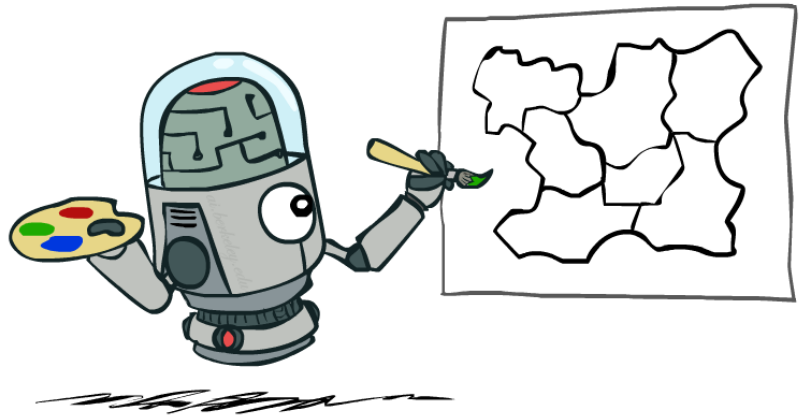
  Implicit: $\mathrm{WA} \neq \mathrm{NT}$

  Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\mathrm{red}, \mathrm{green}), (\mathrm{red}, \mathrm{blue}), \ldots\}$

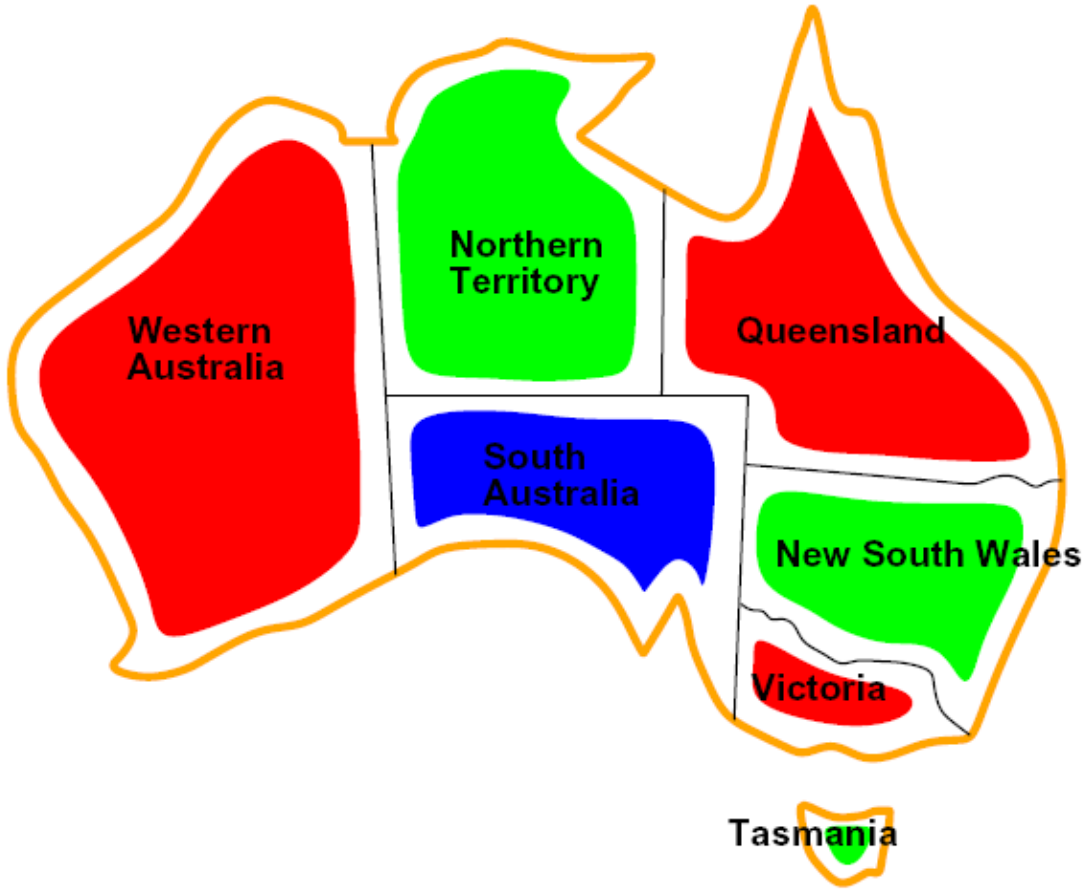


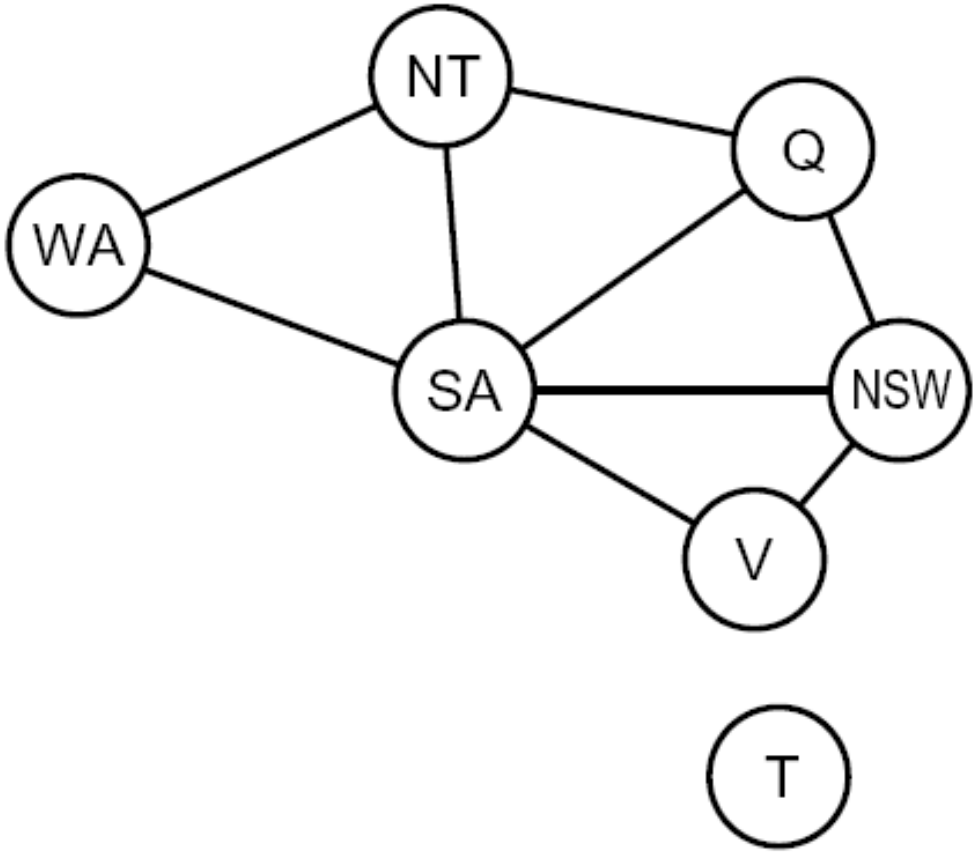
- Solutions are assignments satisfying all constraints, e.g.:

  $\{\mathrm{WA}{=}\mathrm{red},\ \mathrm{NT}{=}\mathrm{green},\ \mathrm{Q}{=}\mathrm{red},\ \mathrm{NSW}{=}\mathrm{green},$
  $\mathrm{V}{=}\mathrm{red},\ \mathrm{SA}{=}\mathrm{blue},\ \mathrm{T}{=}\mathrm{green}\}$



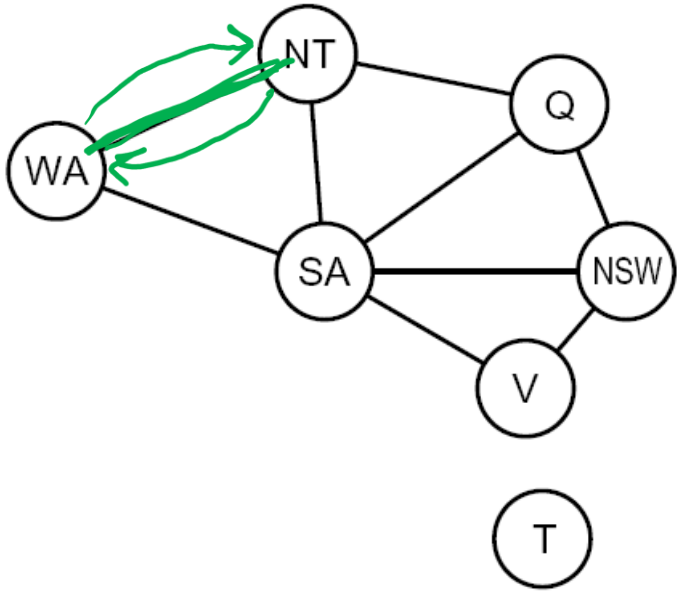
Constraint Graphs





Constraint Graphs

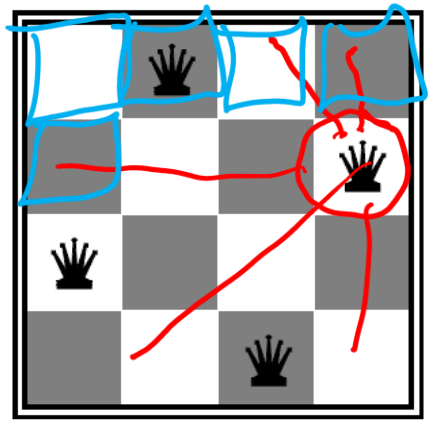
- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

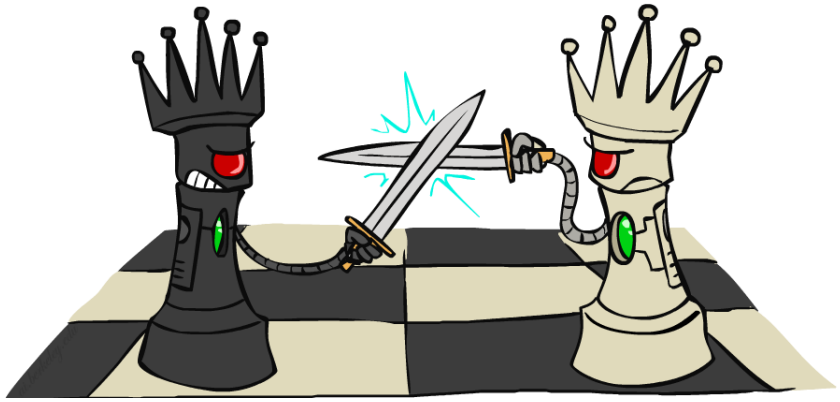


Varieties of CSPs and Constraints

Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



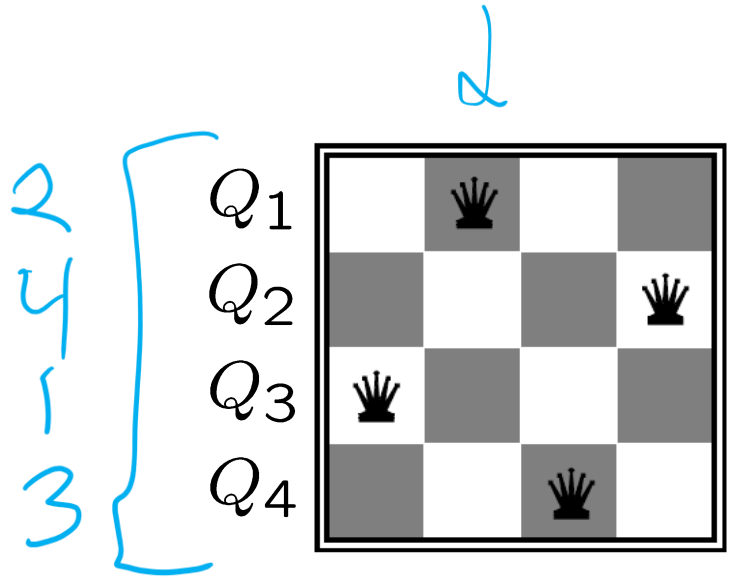


$$\forall i, j, k \quad (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

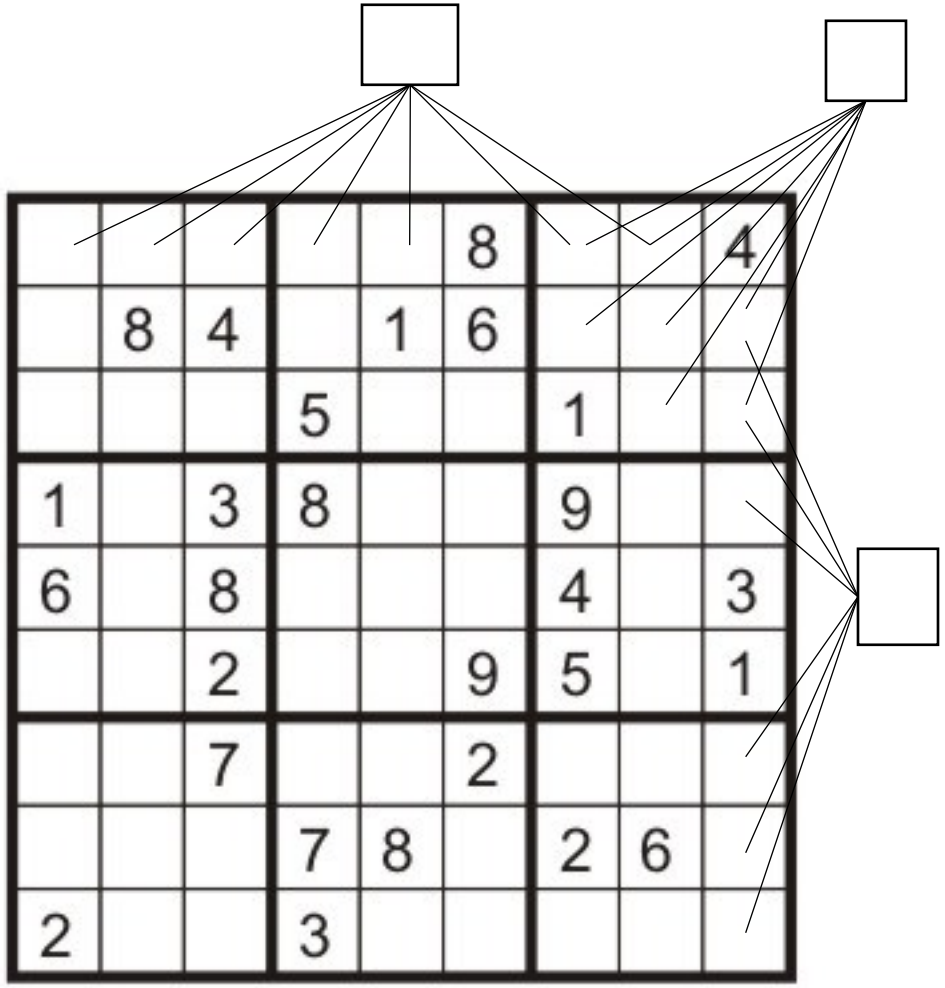


Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\ldots$

Example: Sudoku



- Variables: Each (open) square
- Domains: {1,2,…,9}
- Constraints:
  - 9-way alldiff for each column
  - 9-way alldiff for each row
  - 9-way alldiff for each region
  - (or can have a bunch of pairwise inequality constraints)

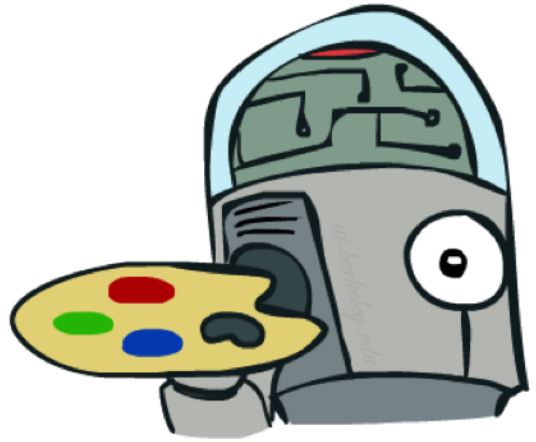
Varieties of CSPs

n #variables
d #values

- Discrete Variables

We will cover today

  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable



We will cover in a later lecture (linear programming)

- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

    $SA \neq green$

    Focus of today
  - Binary constraints involve pairs of variables, e.g.:

    $SA \neq WA$
  - Higher-order constraints involve 3 or more variables:

    e.g., sudoku constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems

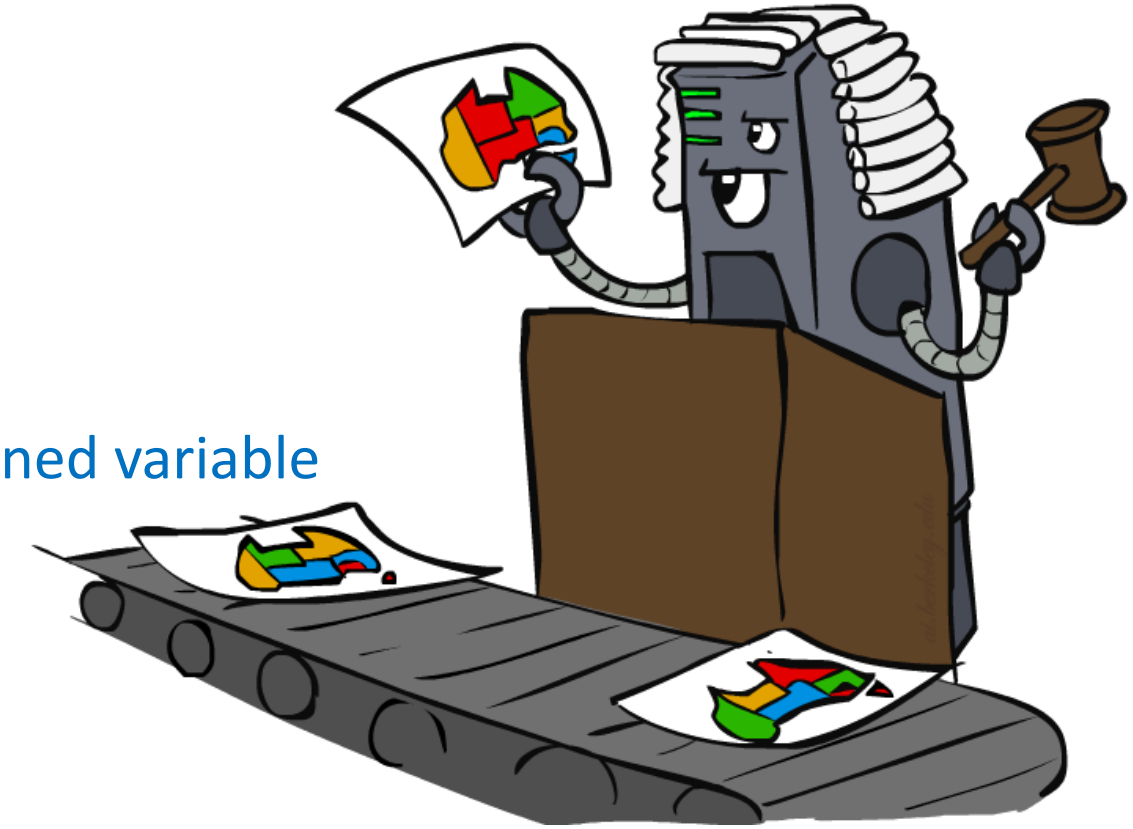
Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

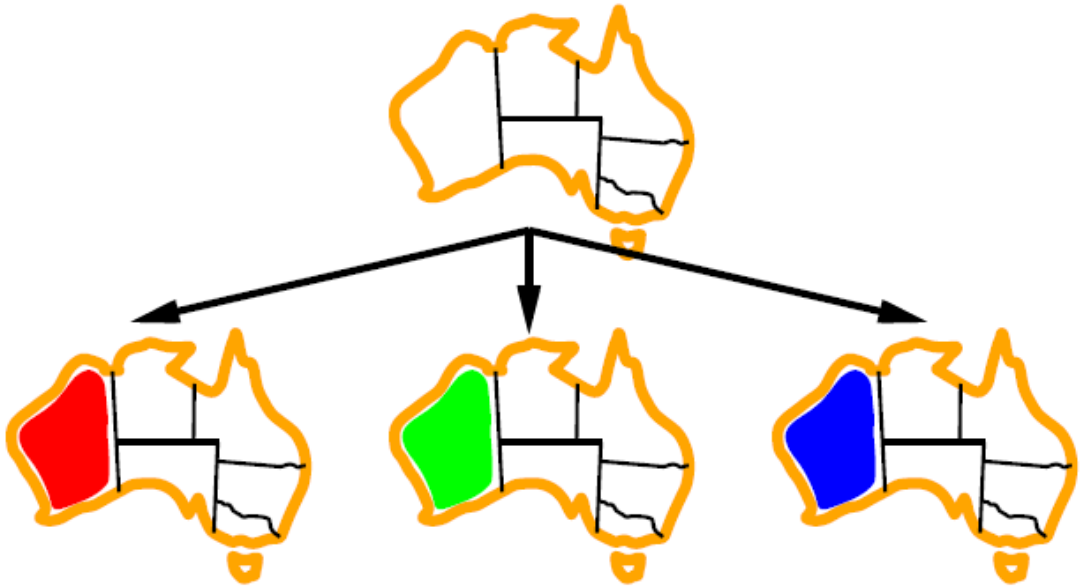
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable →Can be any unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it



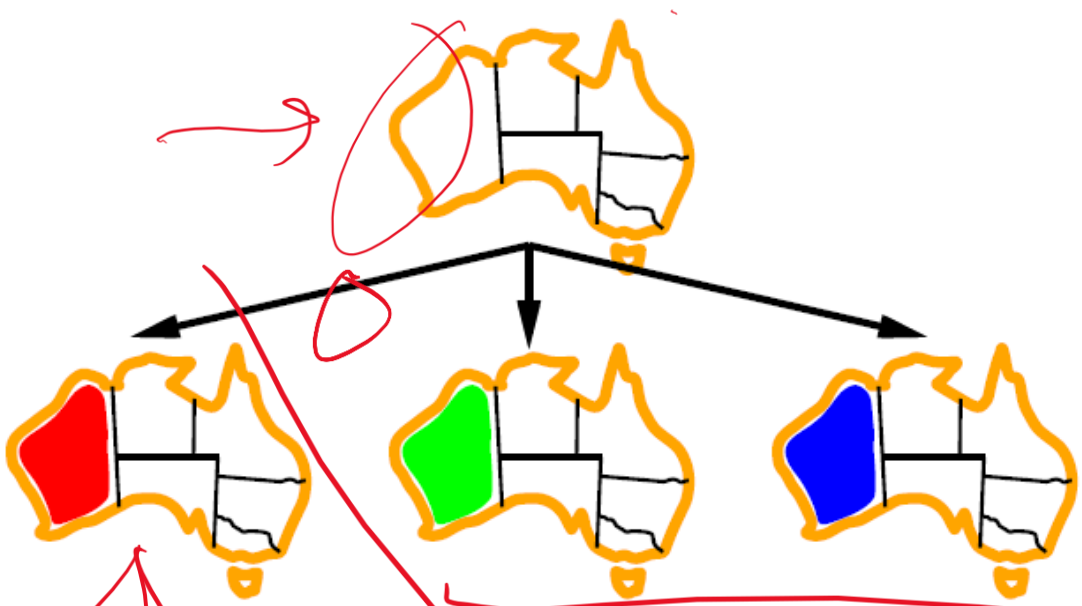
Poll 1: Search for CSPs

Should we use BFS or DFS?

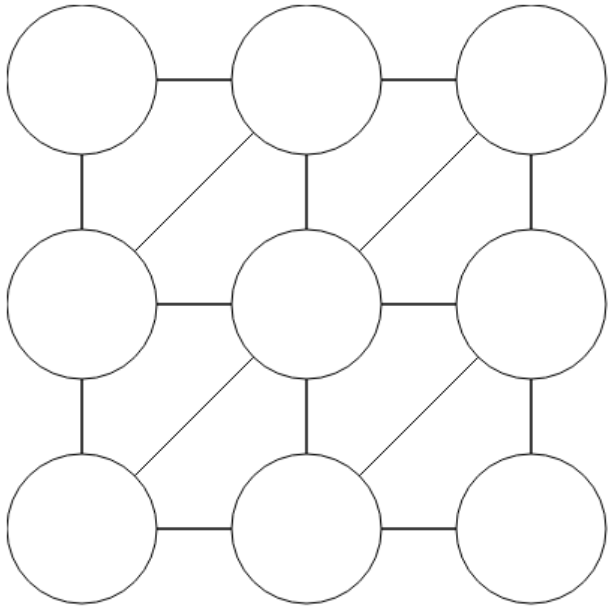


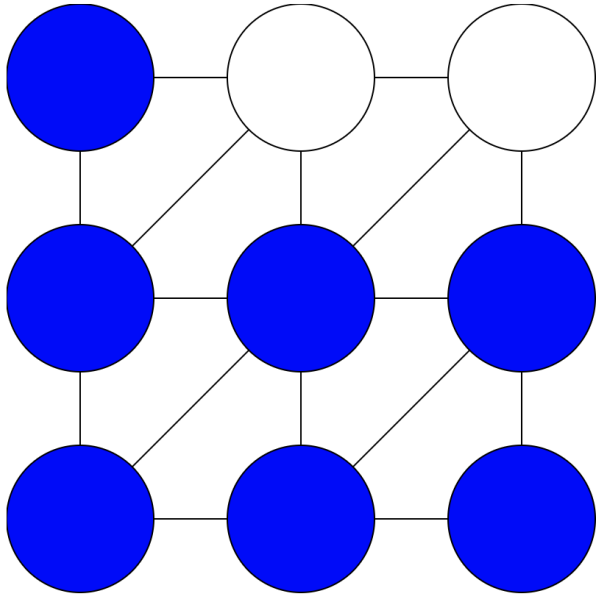
Depth First Search

- At each node, assign a value from the domain to the variable
- Check feasibility (constraints) when the assignment is complete



Demo – Naïve Search

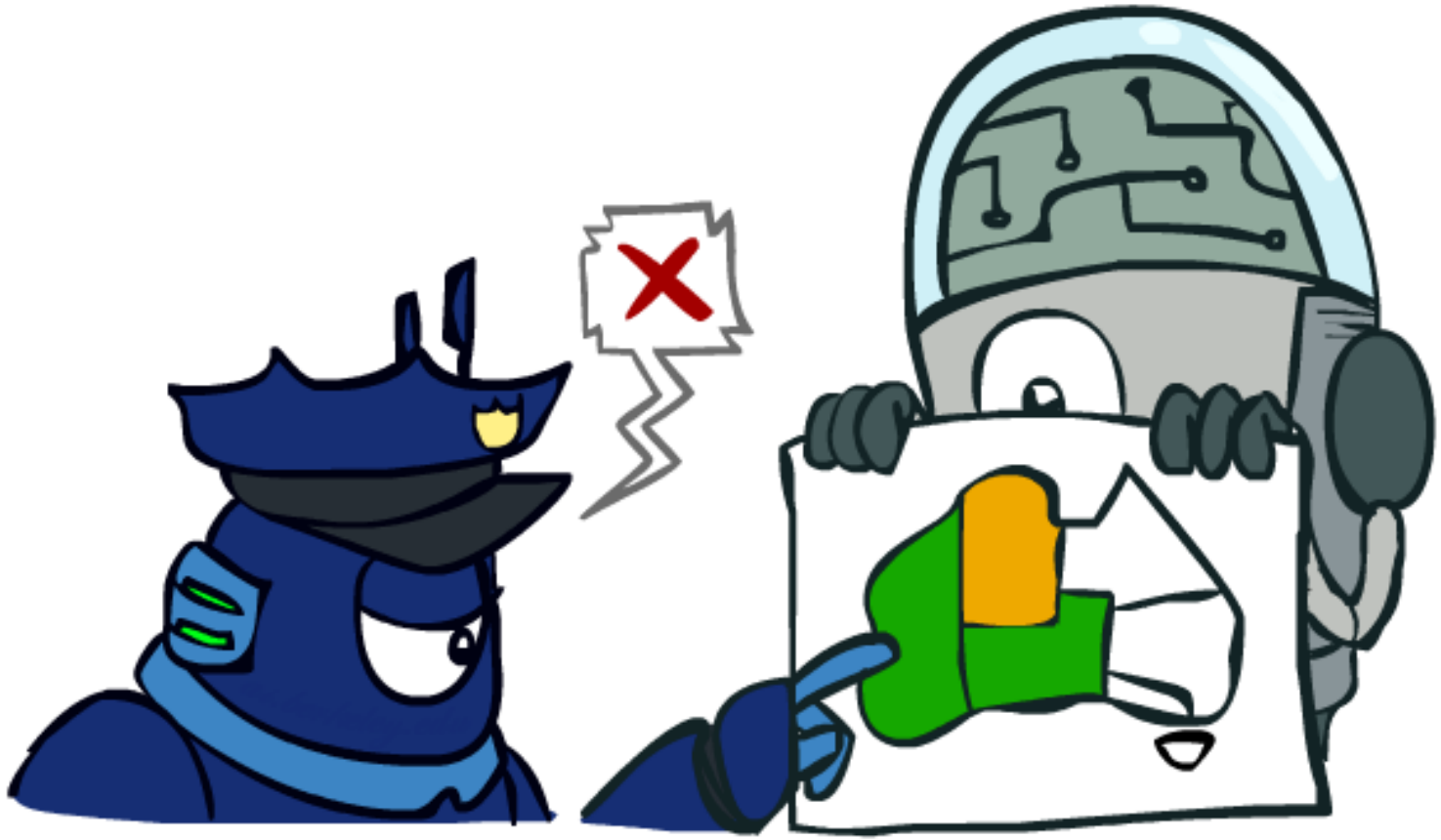




15-281: Artificial Intelligence | OH | Schedule | Recitations | Exams | Assignments | Policies | Course Notes

9/5 Thu	Adversarial Search	AIMA Ch. 5.1-2, 5.5	pptx (inked) pdf (inked)
9/10 Tue	Contraint Satisfaction Problems	AIMA Ch. 6.1-3, 6.5 CSP Demo	

Backtracking Search



Backtracking Search

Backtracking search is the basic uninformed algorithm for solving CSPs

Backtracking search = DFS + two improvements

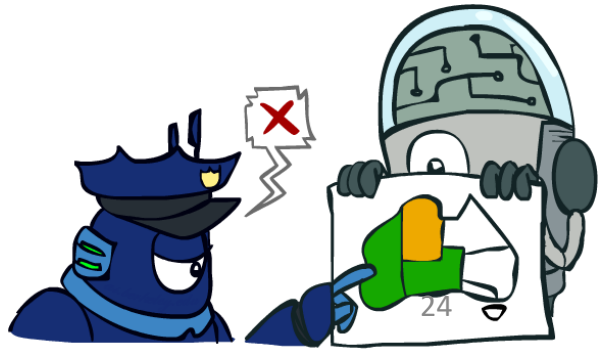
Idea 1: One variable at a time

- Variable assignments are commutative
  - [WA = red then NT = green] same as [NT = green then WA = red]
- Only need to consider assign value to a single variable at each step

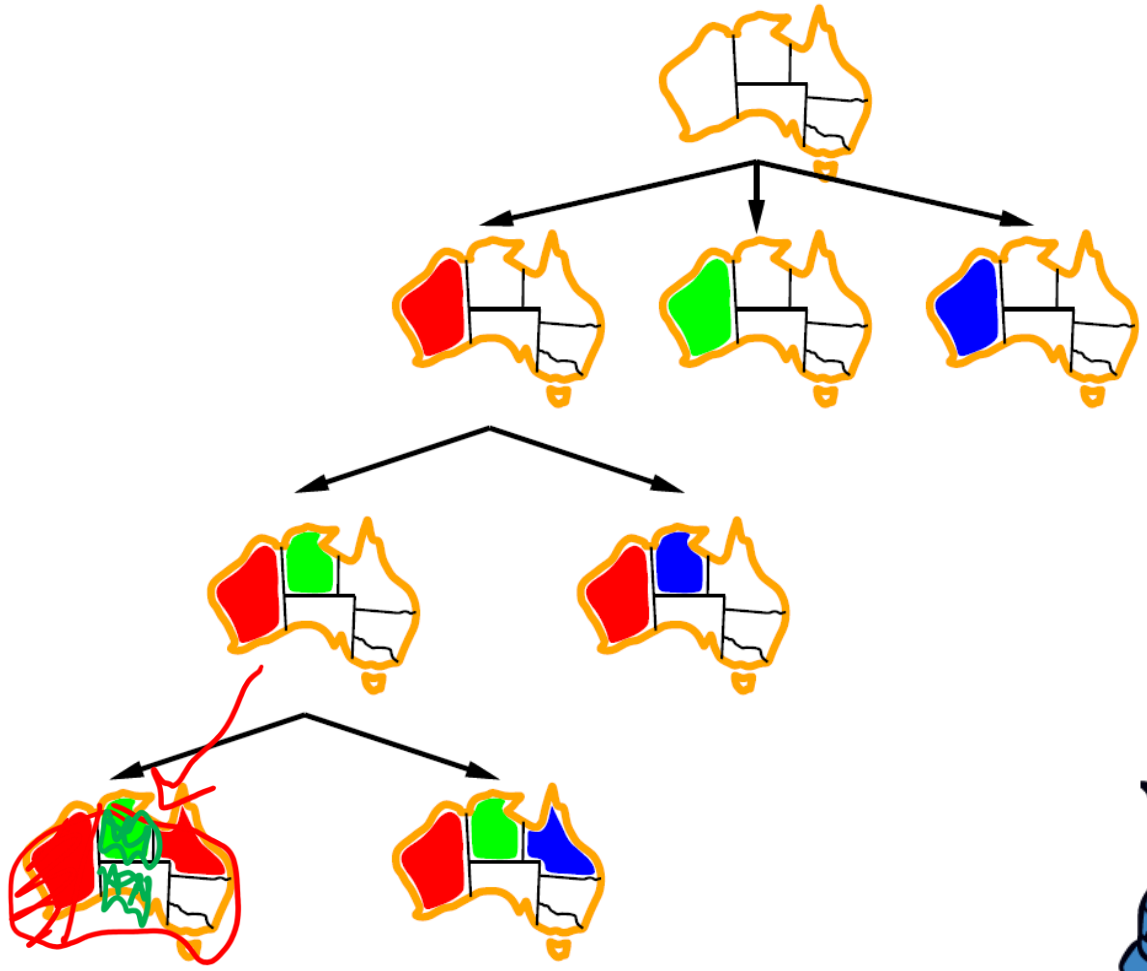
Idea 2: Check constraints as you go

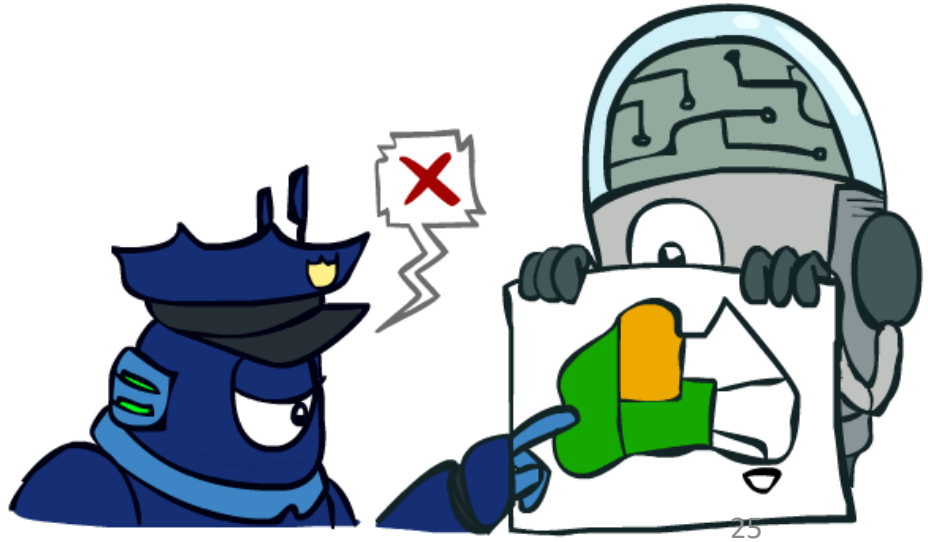
- Consider only values which do not conflict previous assignments
- May need some computation to check the constraints
- “Incremental goal test”

Can solve n-queens for n ≈ 25



Backtracking Example





Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
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No need to check constraints for a complete assignment

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Checks consistency at each assignment

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```
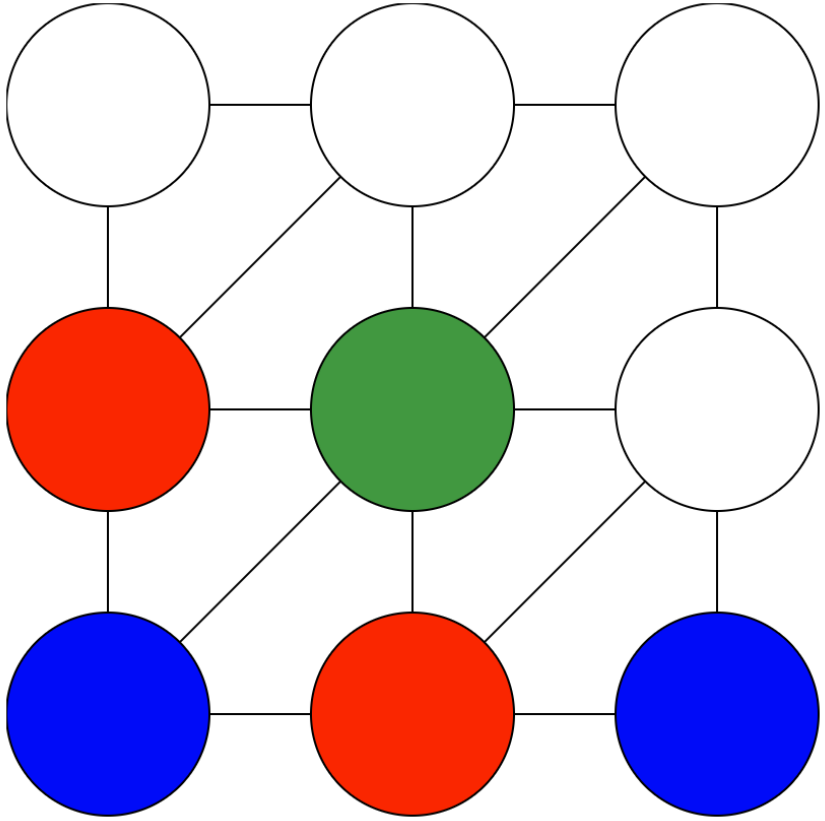

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the decision points?

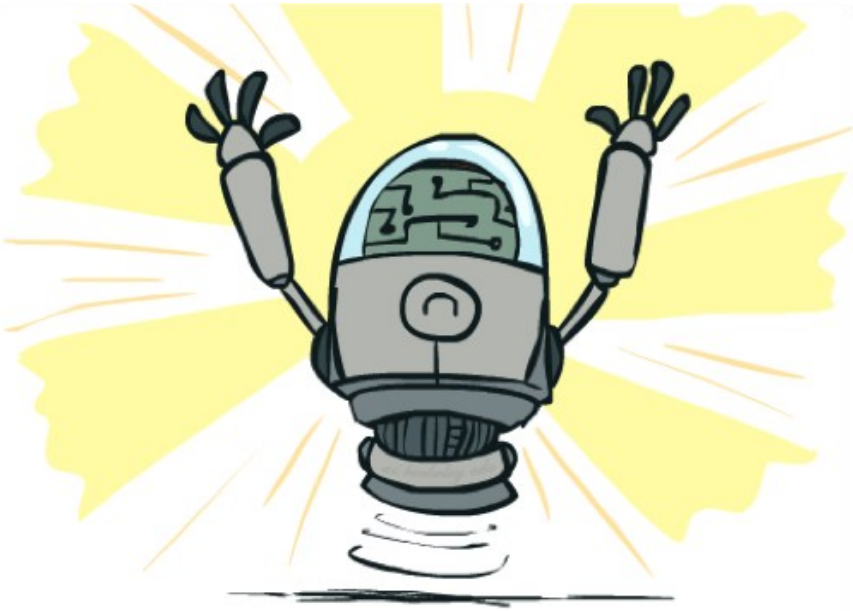
Demo – Backtracking

https://www.cs.cmu.edu/~15281/demos/csp_backtracking



Improving Backtracking

- General-purpose ideas give huge gains in speed
- Filtering: Can we detect inevitable failure early?
  Pruning
- Ordering: ← Heuristics
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Structure: Can we exploit the problem structure?



Filtering

Filtering: Forward Checking

Filtering: Keep track of domains for unassigned variables and cross off bad options

Forward checking: A simple way for filtering

- After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
- Failure detected if some variables have no values remaining

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
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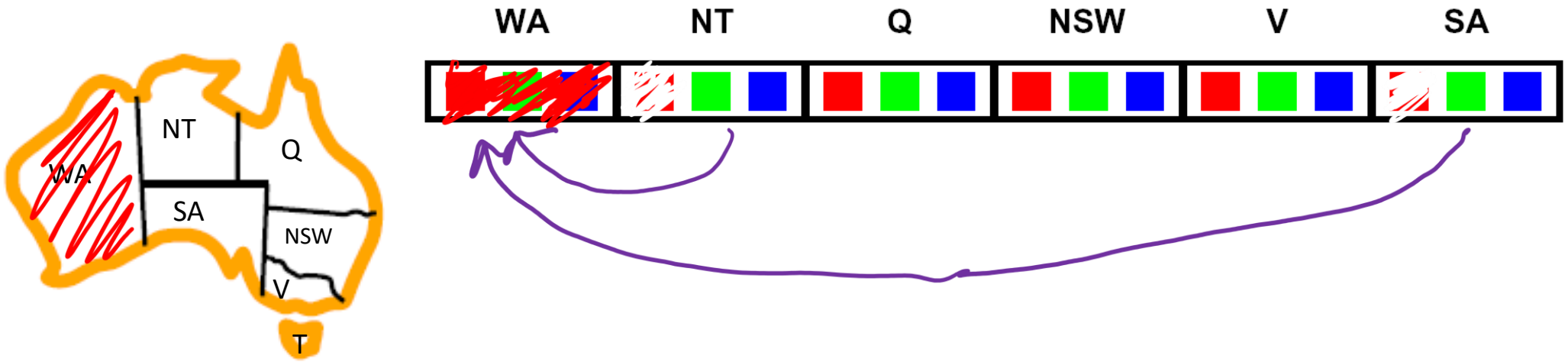

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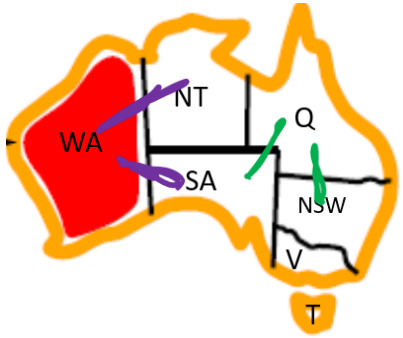

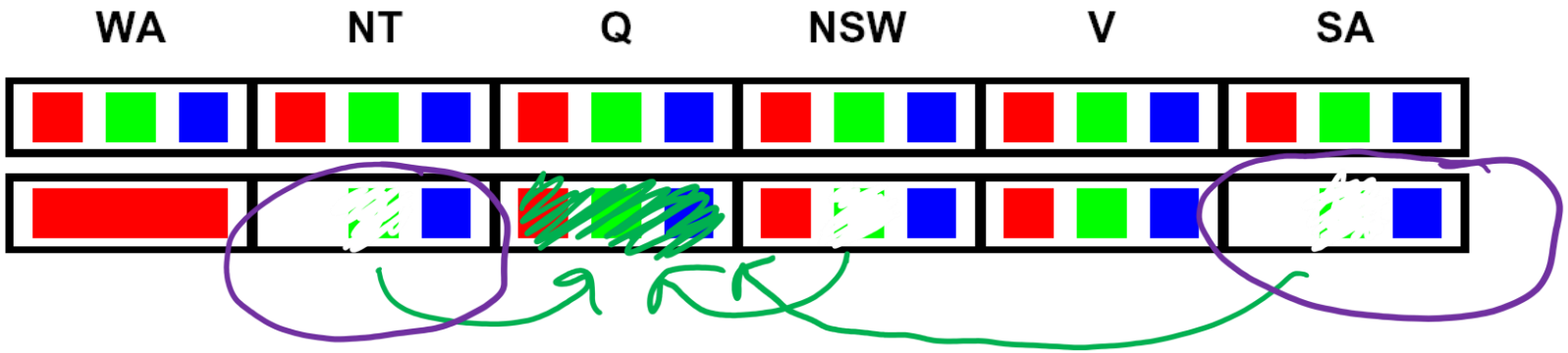

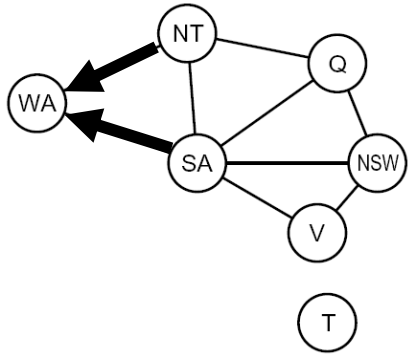

Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
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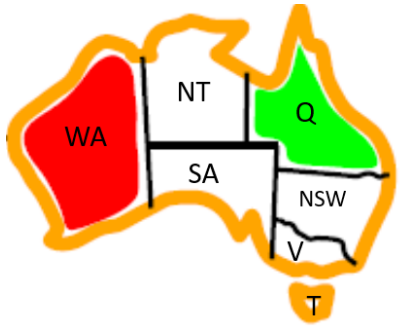
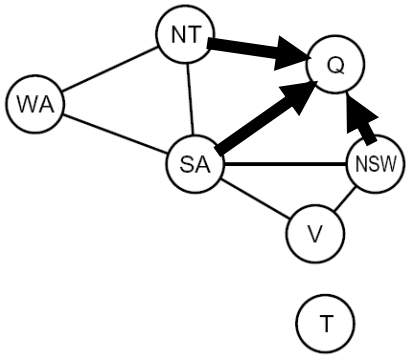
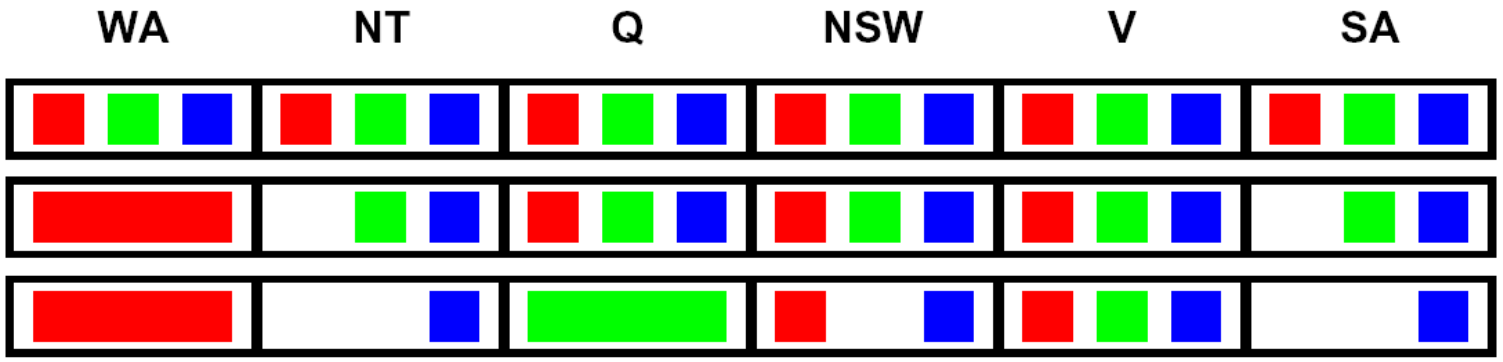
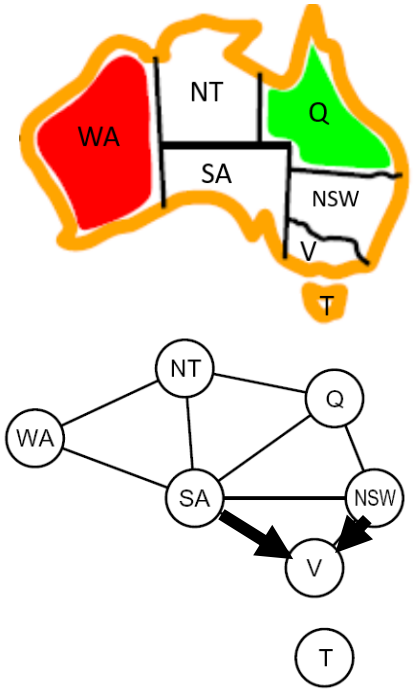


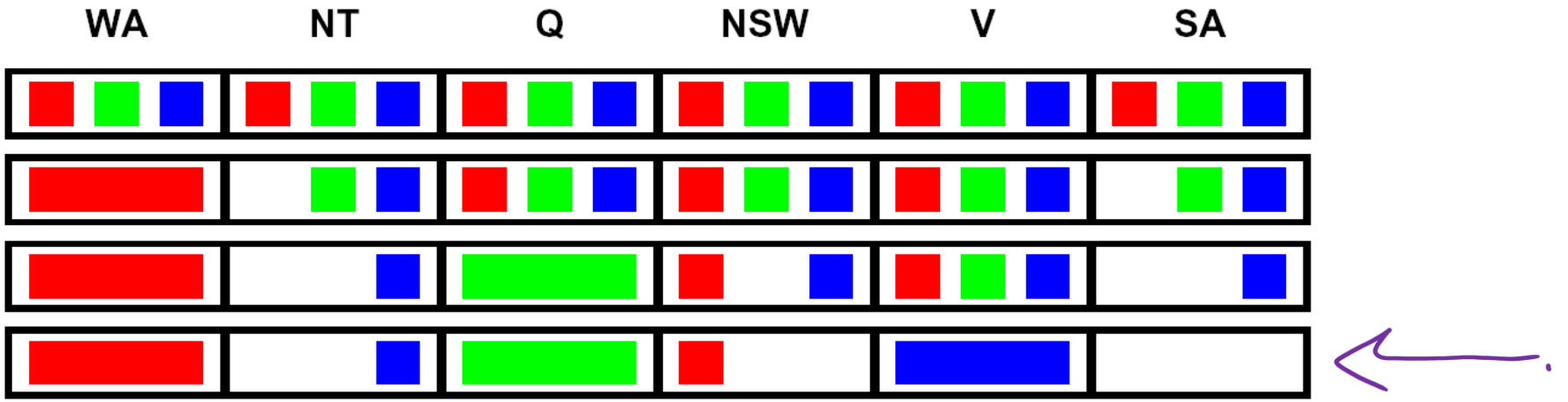




Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
  - After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
  - Failure detected if some variables have no values remaining

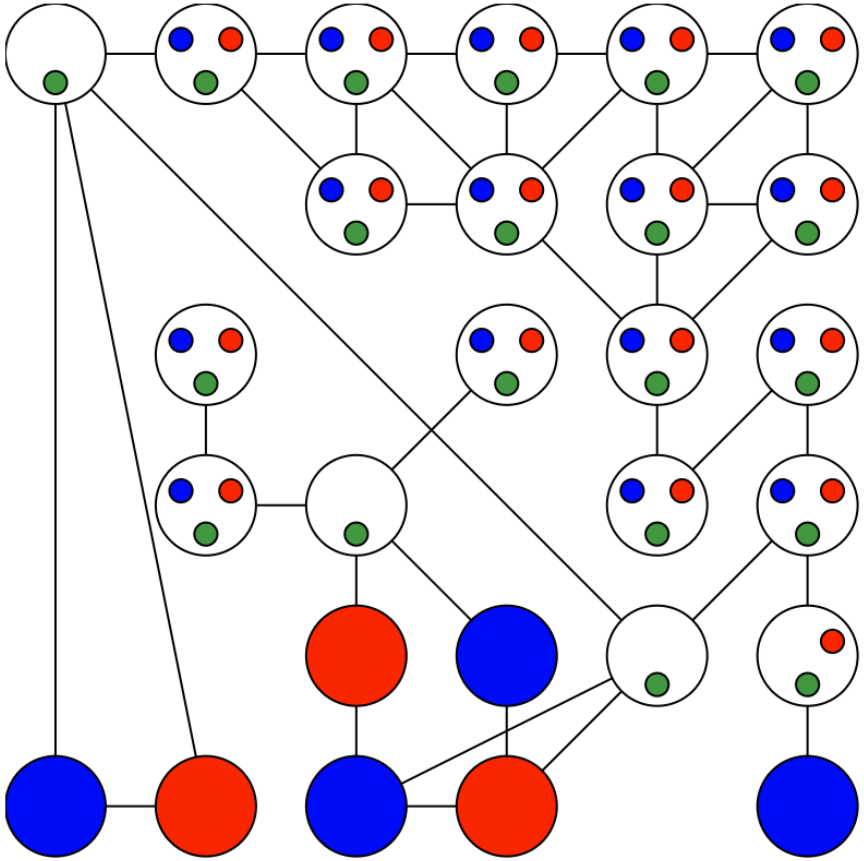


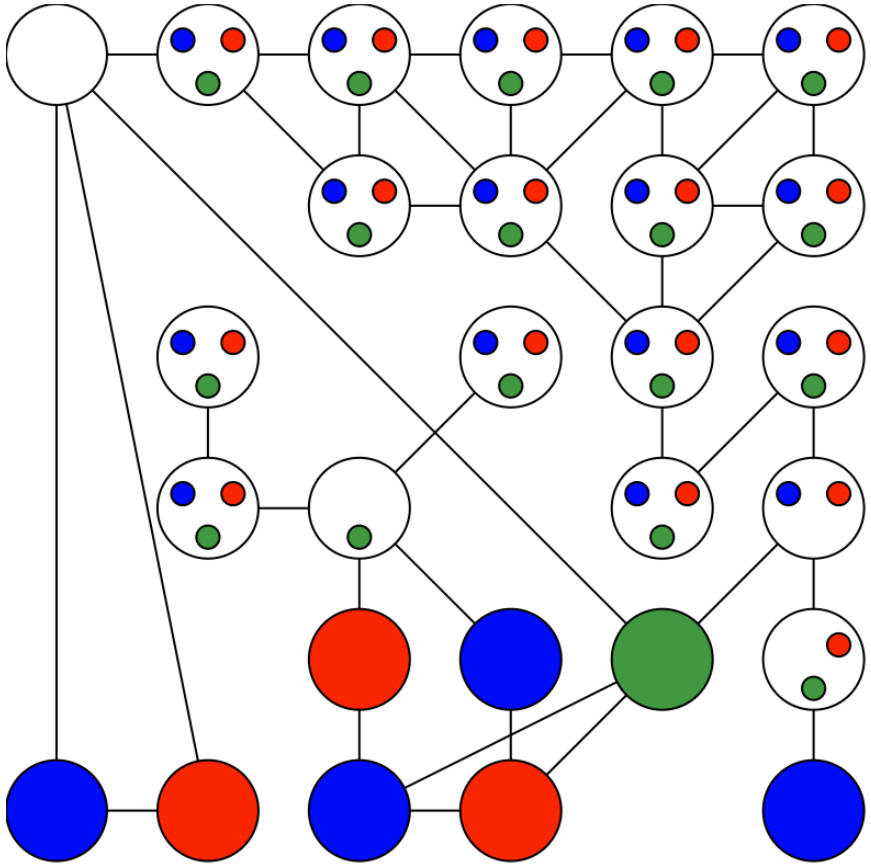


FAIL – variable with no possible values

Demo – Backtracking with Forward Checking

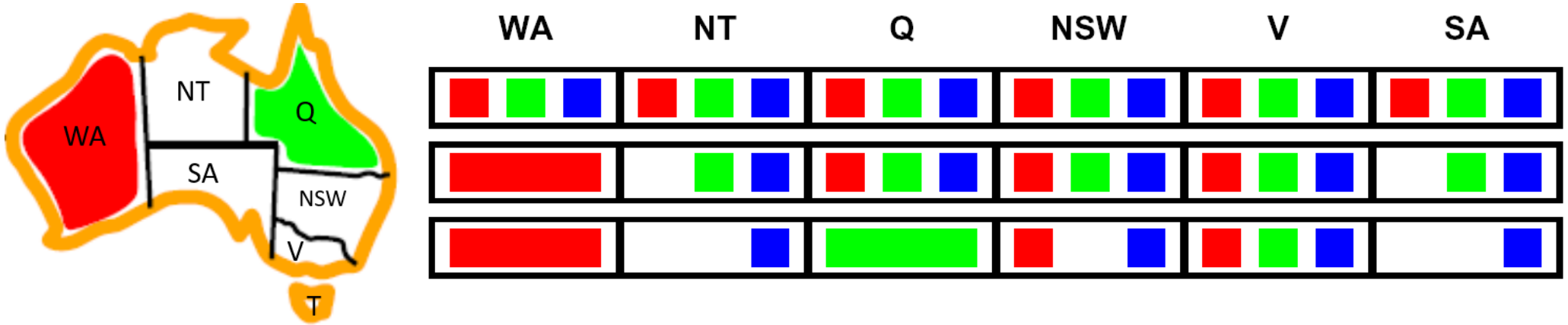
https://www.cs.cmu.edu/~15281/demos/csp_backtracking





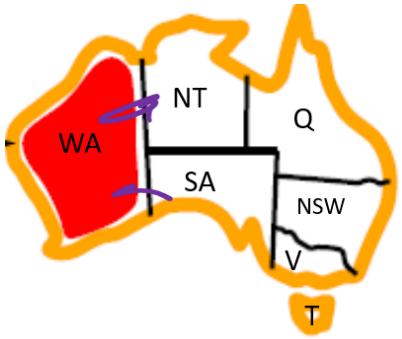
Filtering: Constraint Propagation

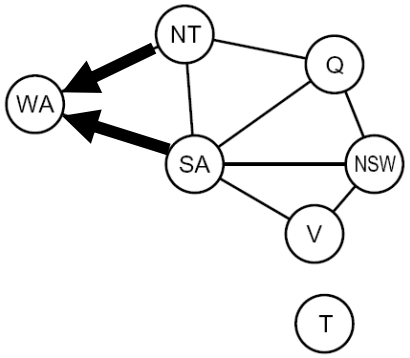
- Limitations of simple forward checking: propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures
  - NT and SA cannot both be blue! Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint



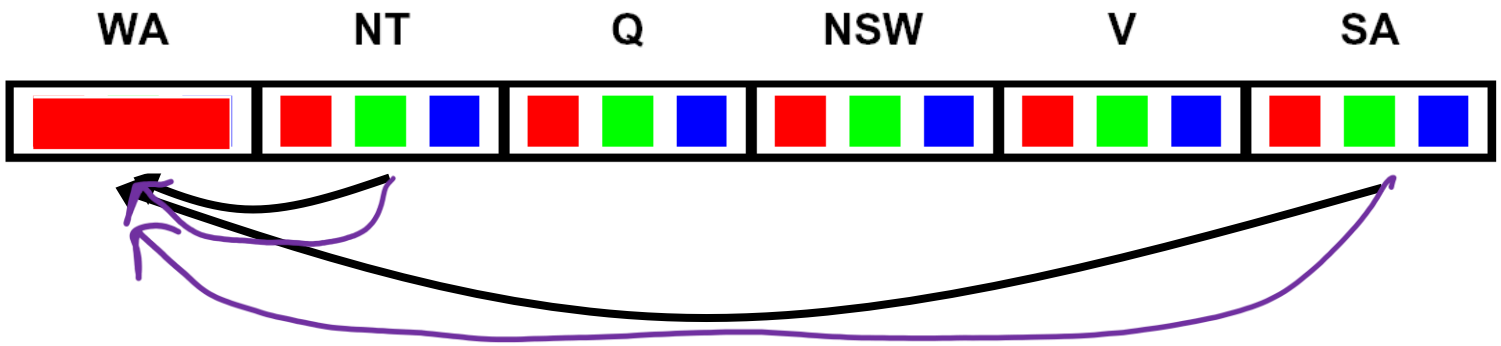
Consistency of A Single Arc

- An arc X $\rightarrow$ Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall (X, Y) \in E$ and Y newly assigned





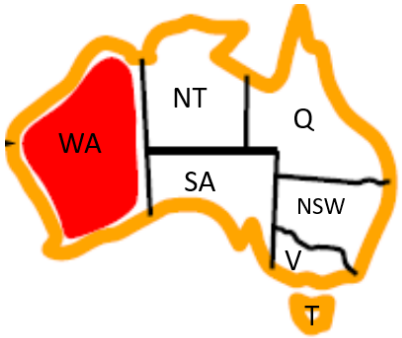
(Remove values from the tail!)

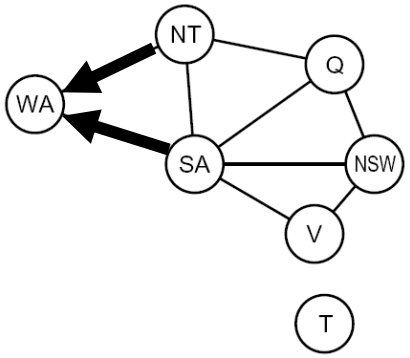


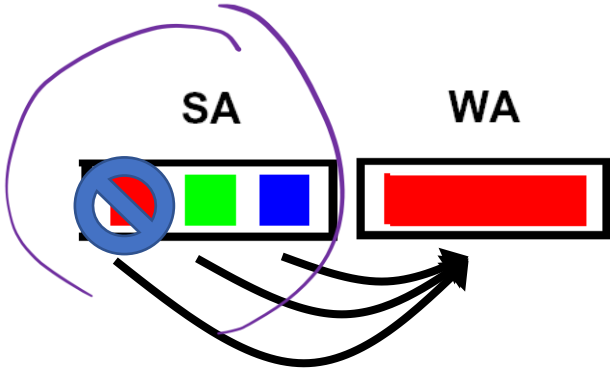
Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

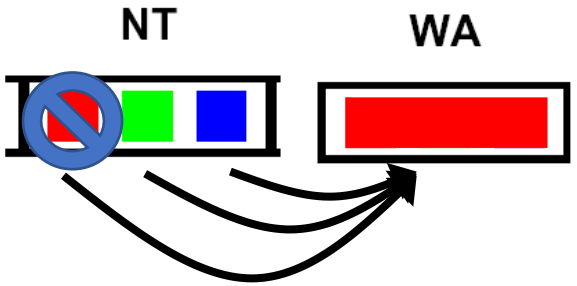
Consistency of A Single Arc

- An arc X $\rightarrow$ Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall(X, Y) \in E$ and Y newly assigned



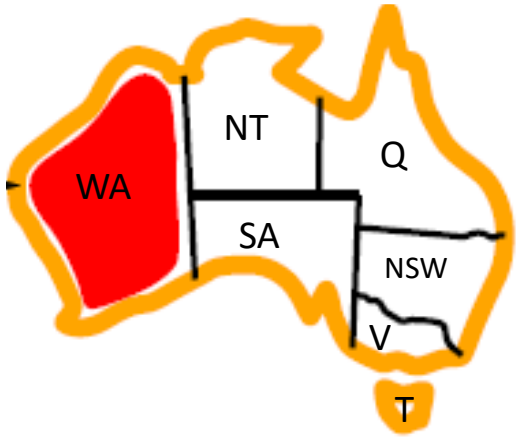


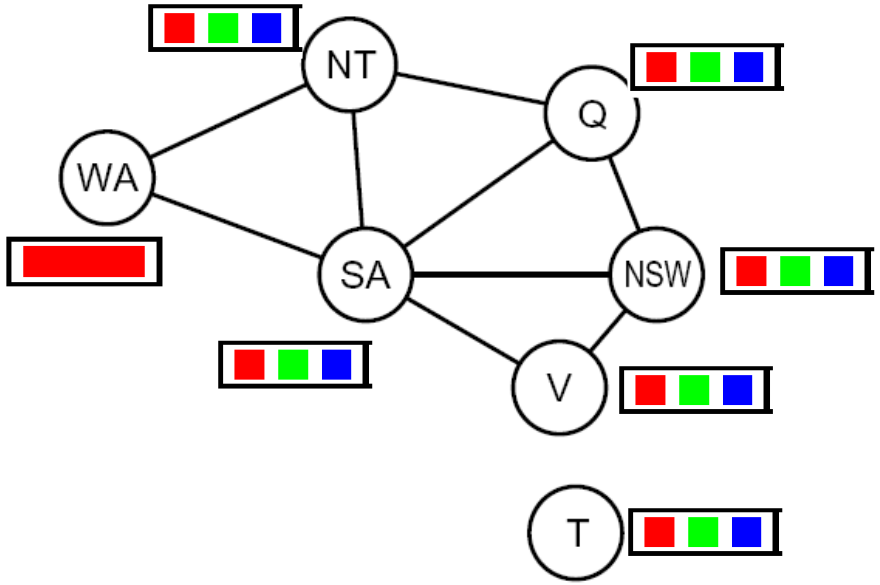




How to Enforce Arc Consistency of Entire CSP

- A simplistic algorithm: Cycle over the pairs of variables, enforcing arc-consistency, repeating the cycle until no domains change for a whole cycle
- AC-3 (short for Arc Consistency Algorithm #3): A more efficient algorithm ignoring constraints that have not been modified since they were last analyzed





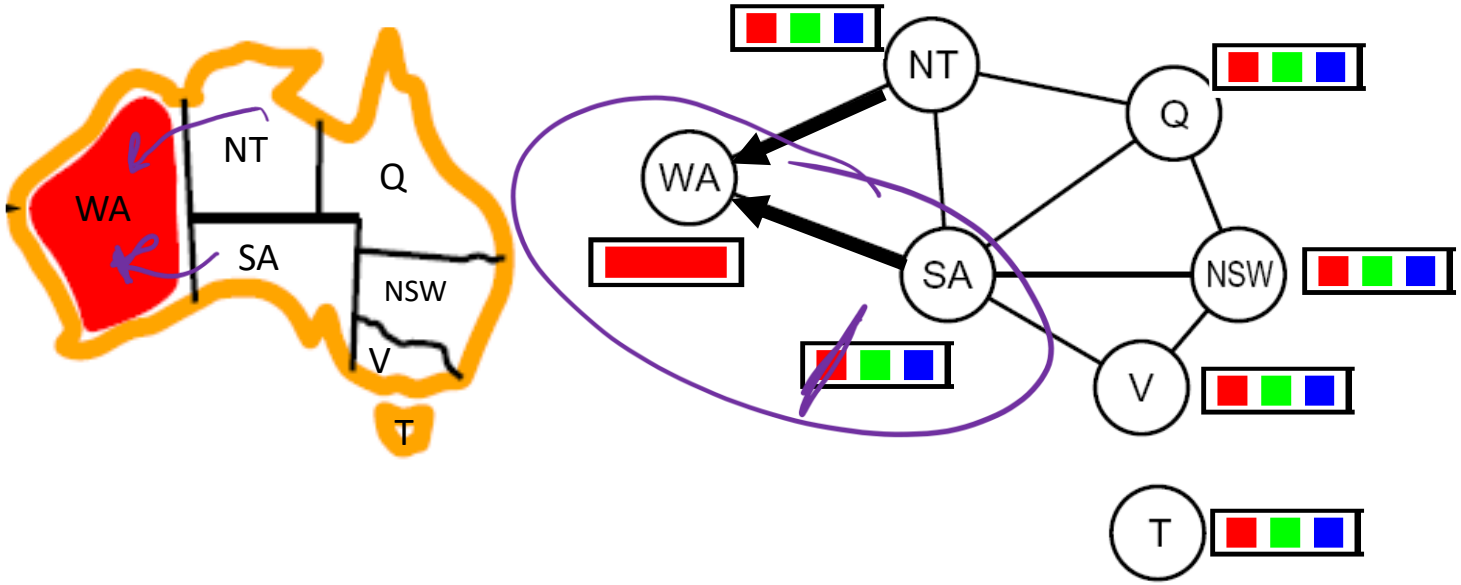
AC-3: Enforce Arc Consistency of Entire CSP

```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp
  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue

function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

Constraint Propagation!

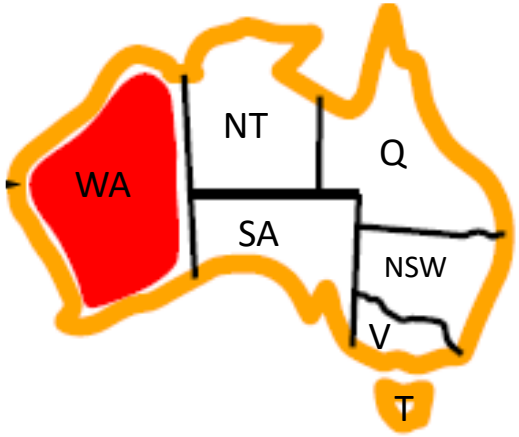
AC-3: Enforce Arc Consistency of Entire CSP

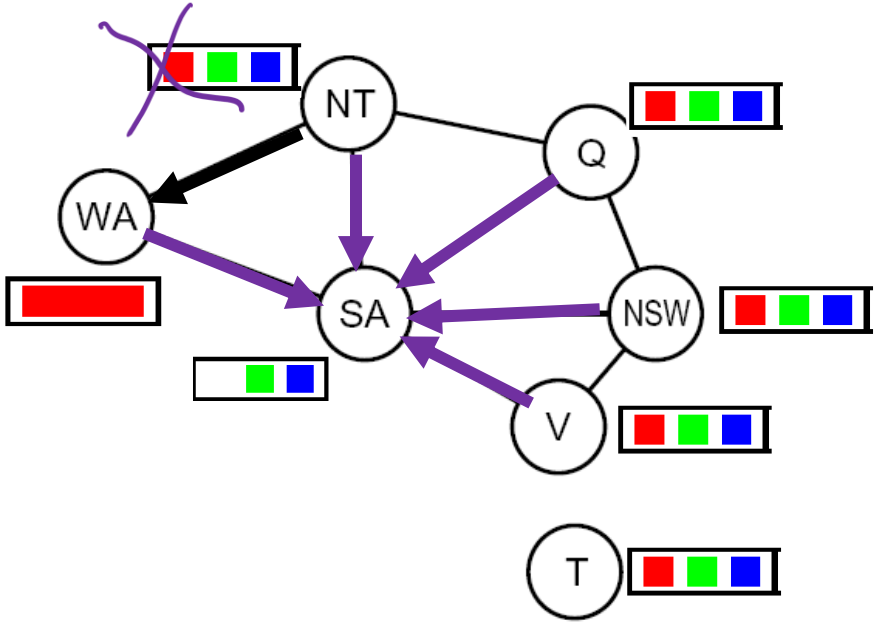


Queue:
SA->WA
NT->WA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

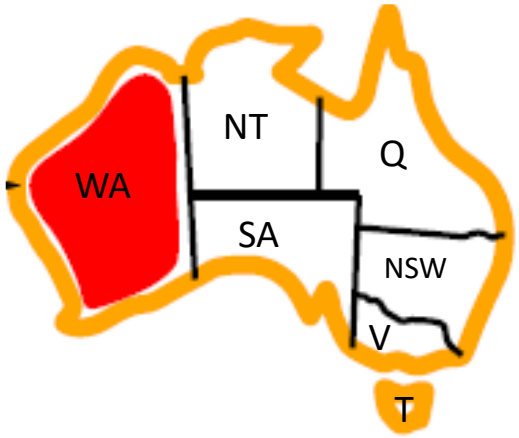


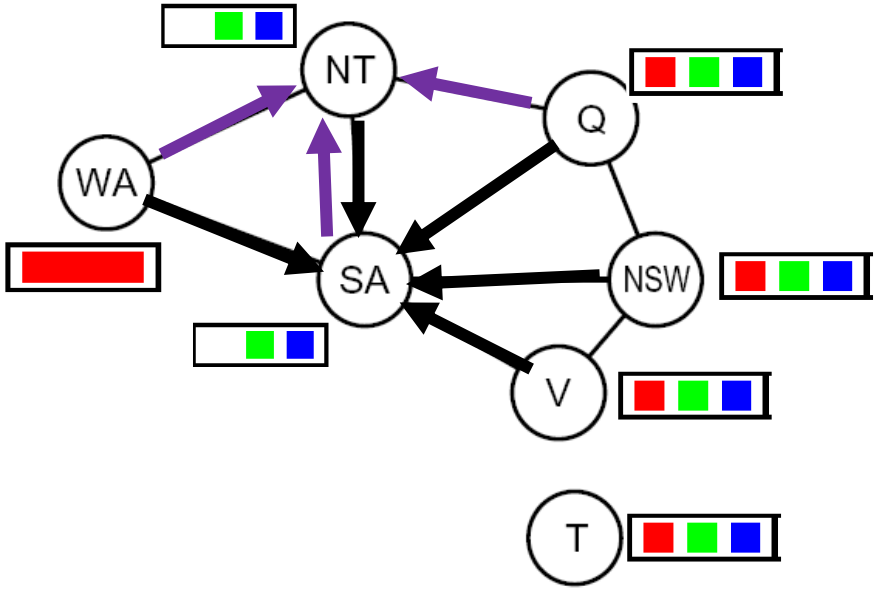


Queue:
NT->WA
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

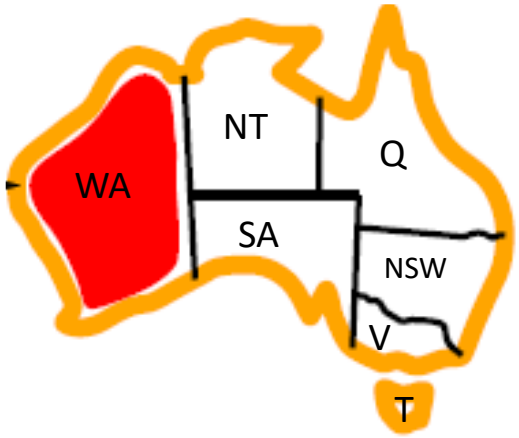


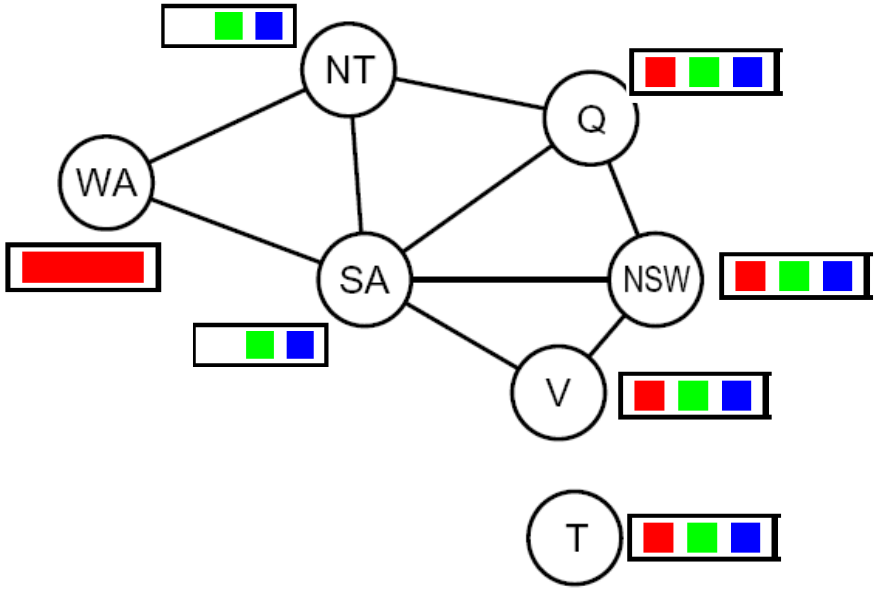


Queue:
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

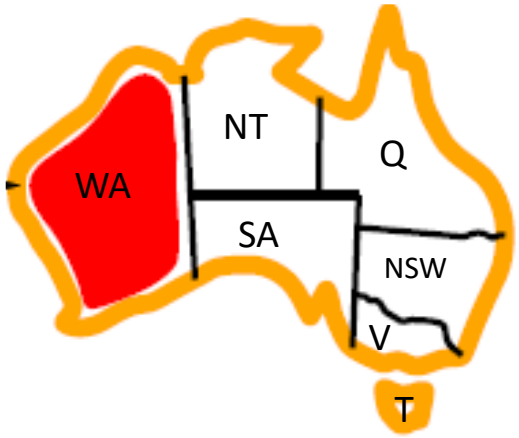
WA->NT

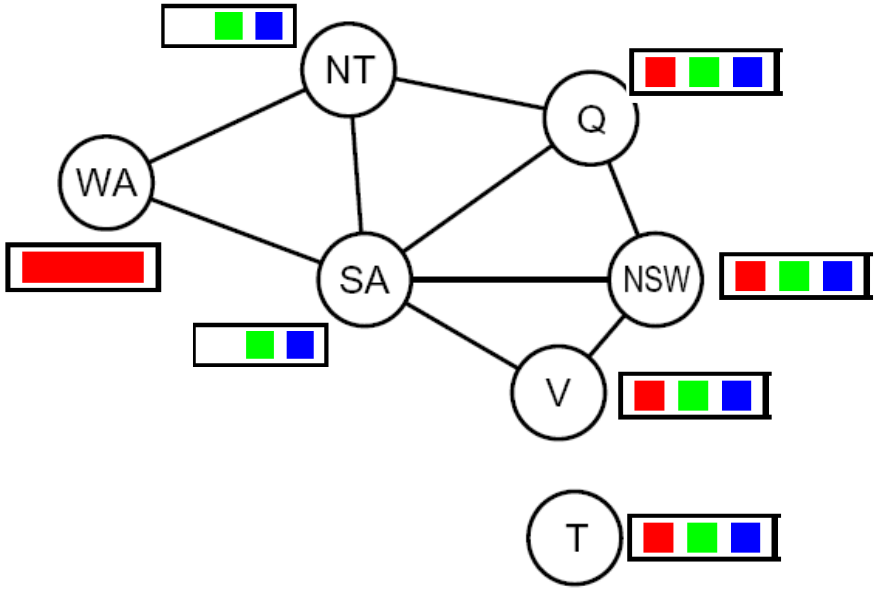
SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

NT->SA

Q->SA

NSW->SA

V->SA

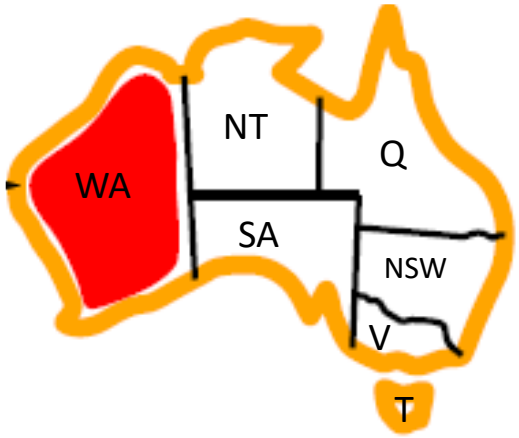
WA->NT

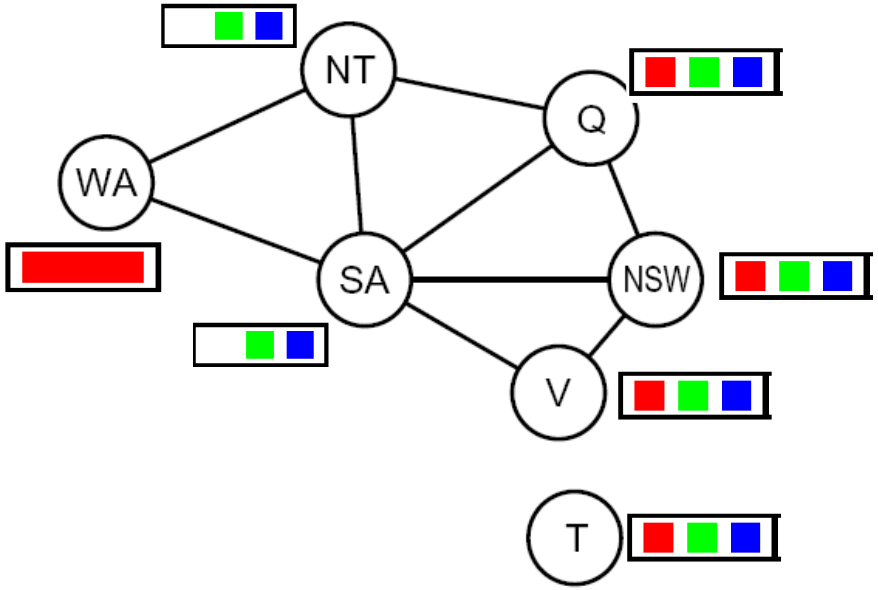
SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

Q->SA

NSW->SA

V->SA

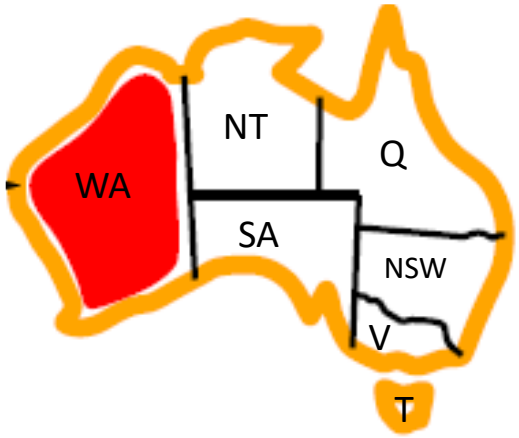
WA->NT

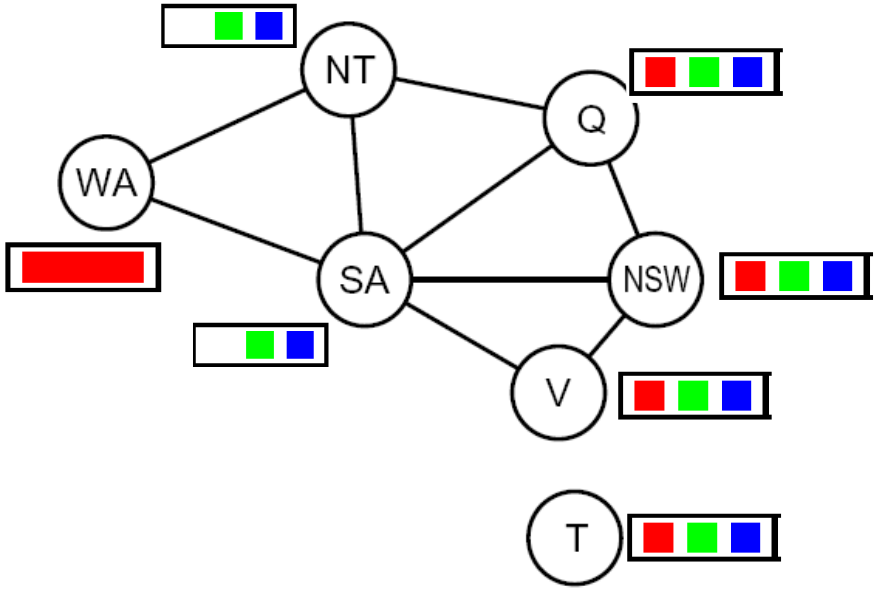
SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

NSW->SA

V->SA

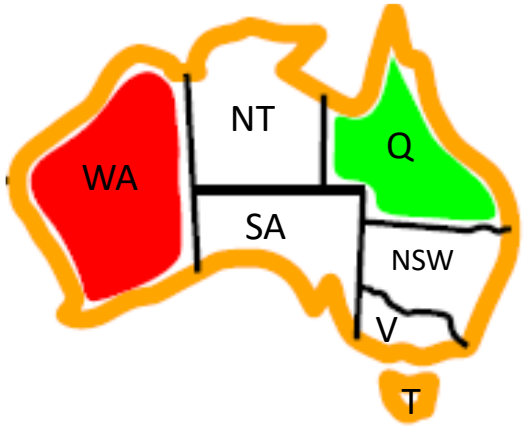
WA->NT

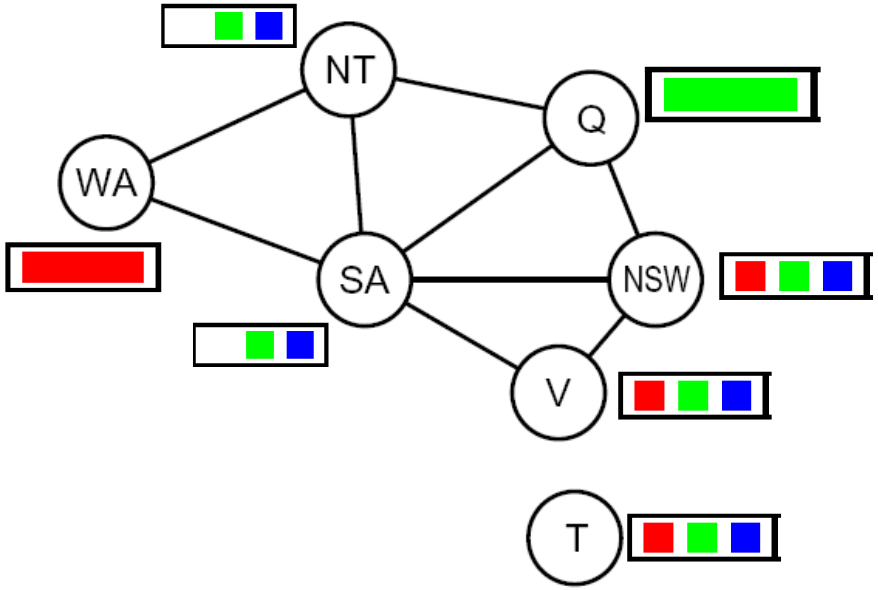
SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

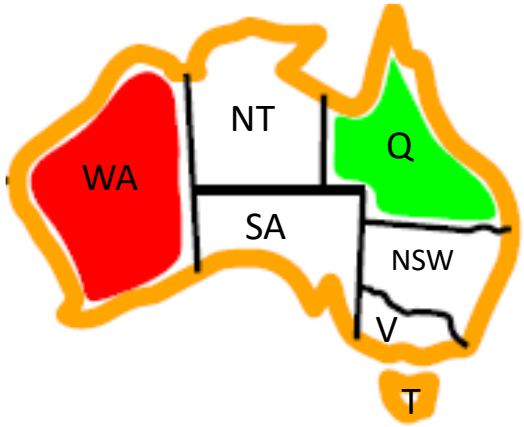


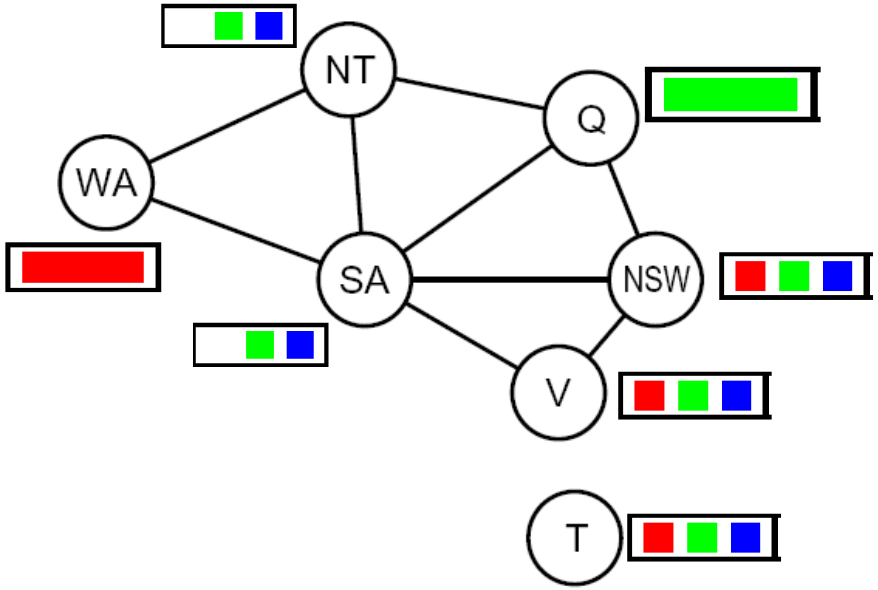


Queue:

Remember: Delete from the tail!

Poll 2: After assigning Q to Green, what gets added to the Queue?



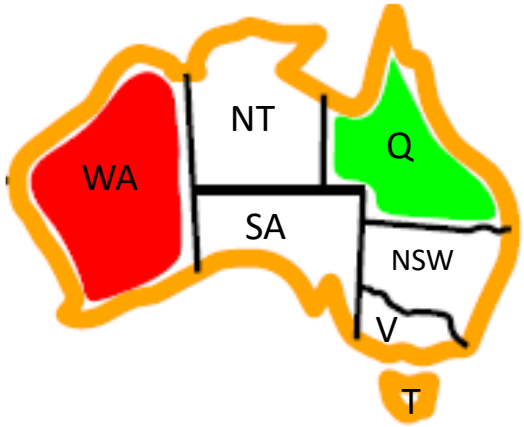


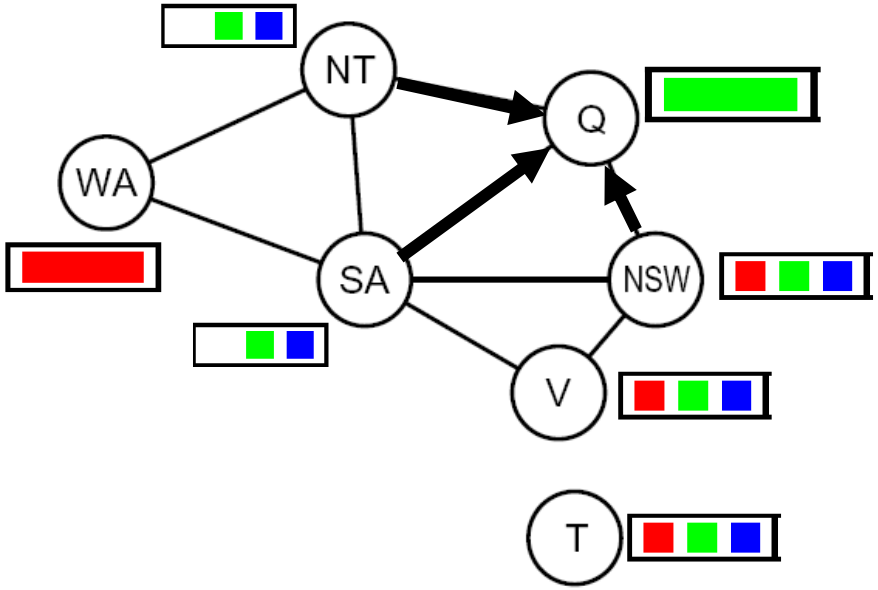
Queue:

A: NSW->Q, SA->Q, NT->Q

B: Q->NSW, Q->SA, Q->NT

AC-3: Enforce Arc Consistency of Entire CSP

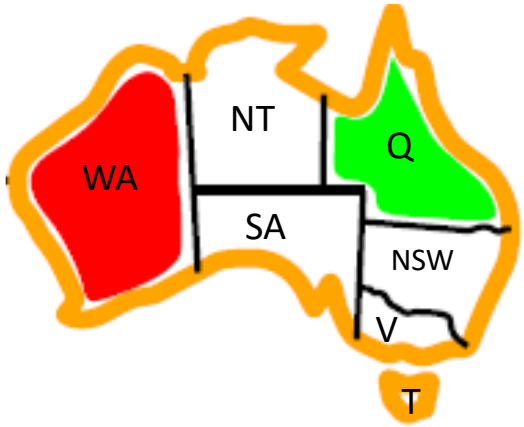


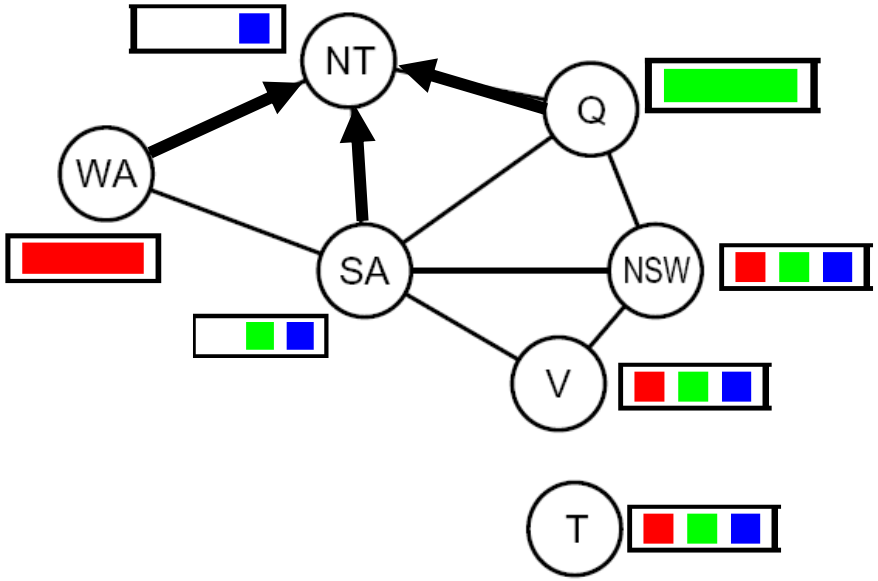


Queue:
NT->Q
SA->Q
NSW->Q

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

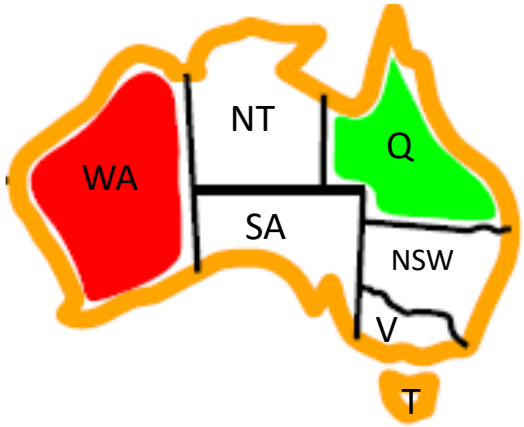


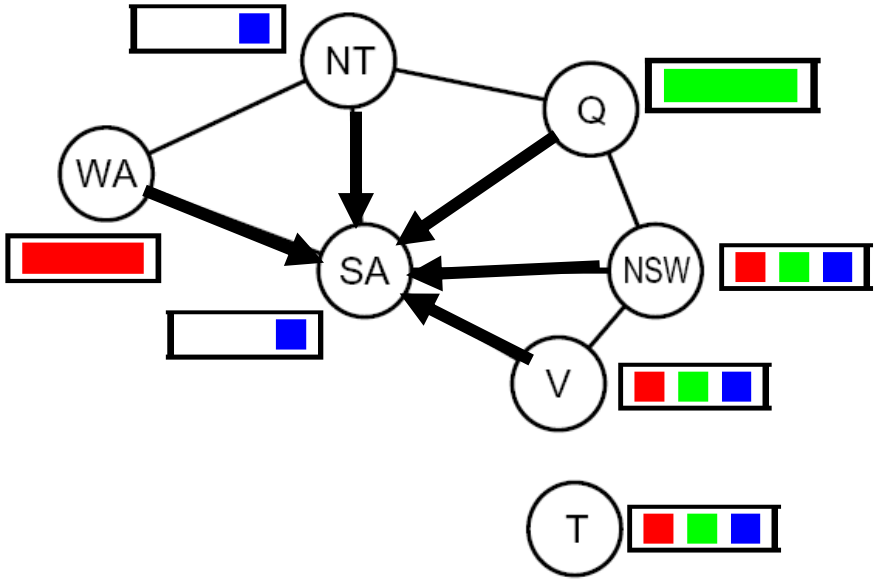


Queue:
SA->Q
NSW->Q
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

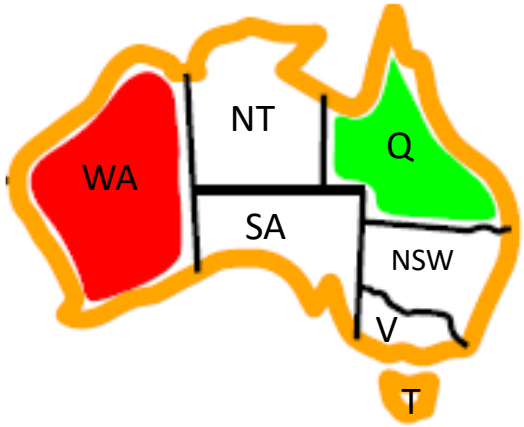


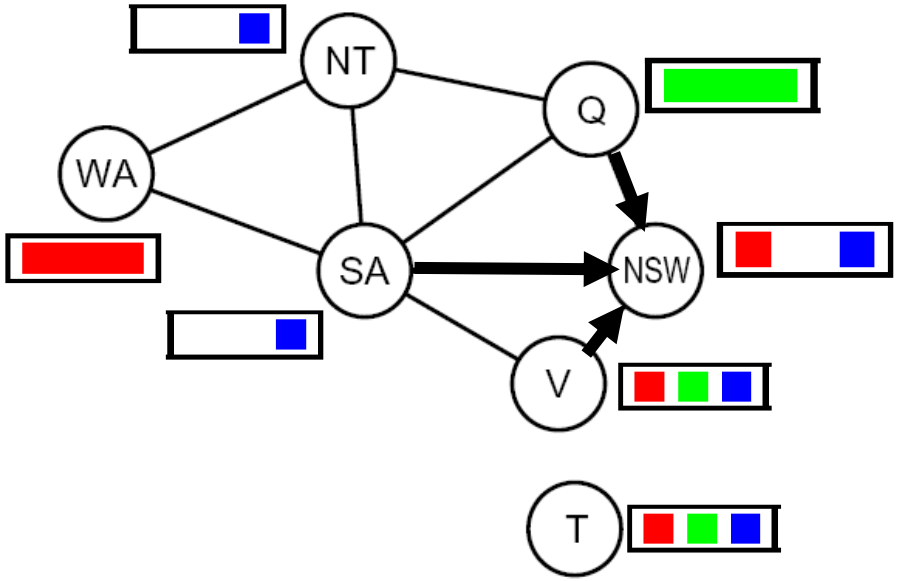


Queue:
NSW->Q
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP

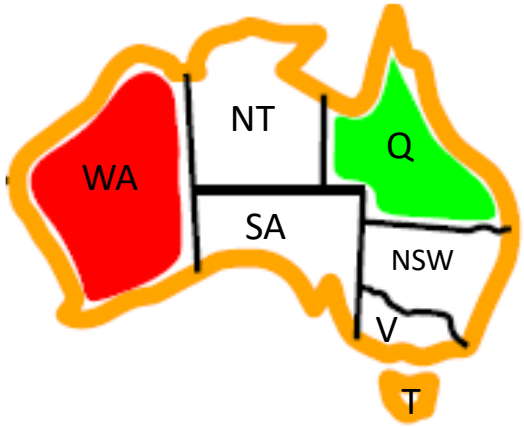


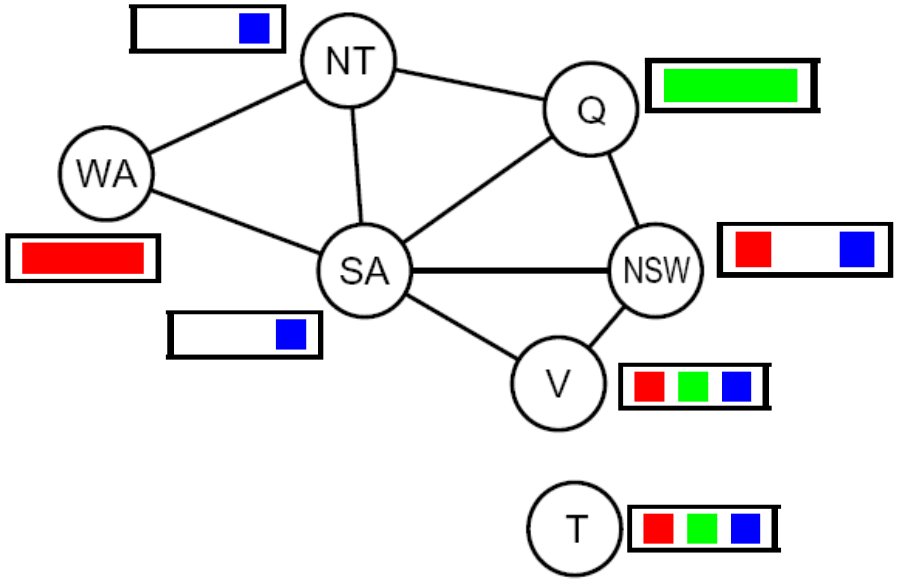


Queue:
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

WA->NT

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

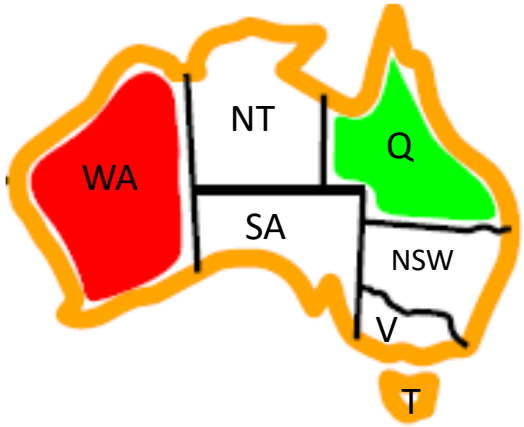
V->NSW

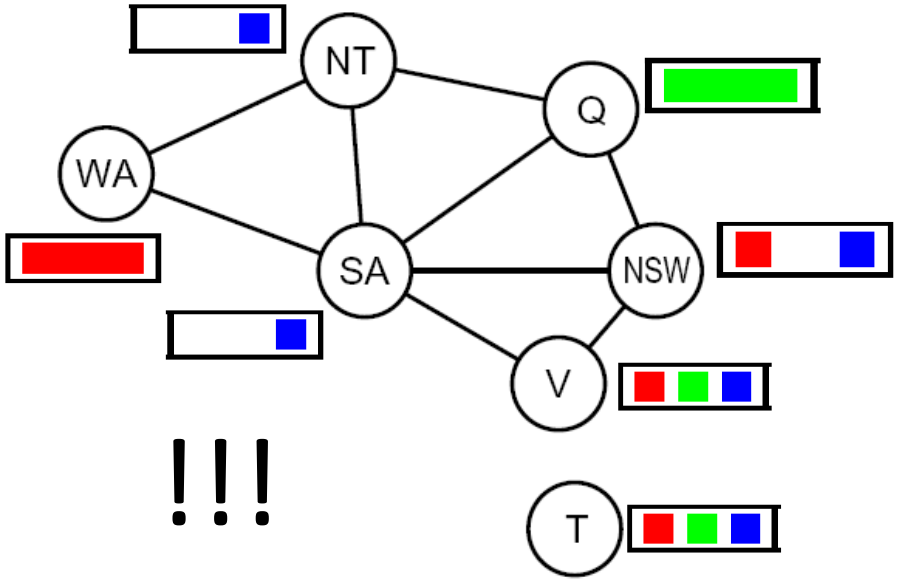
Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





Queue:

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

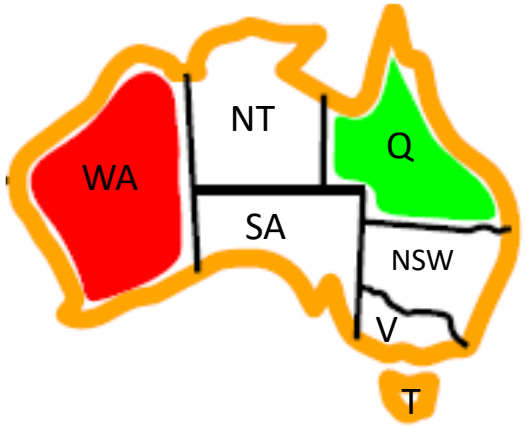
V->NSW

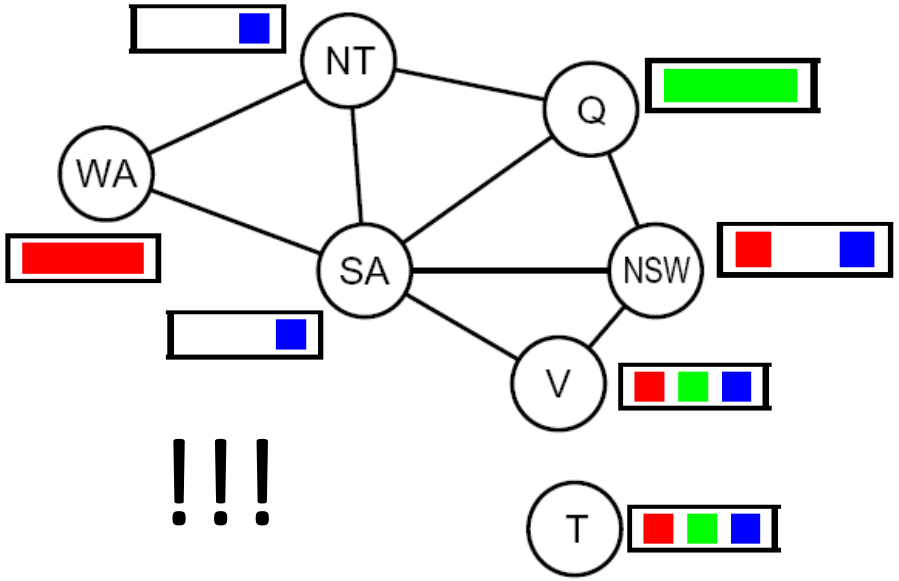
Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP





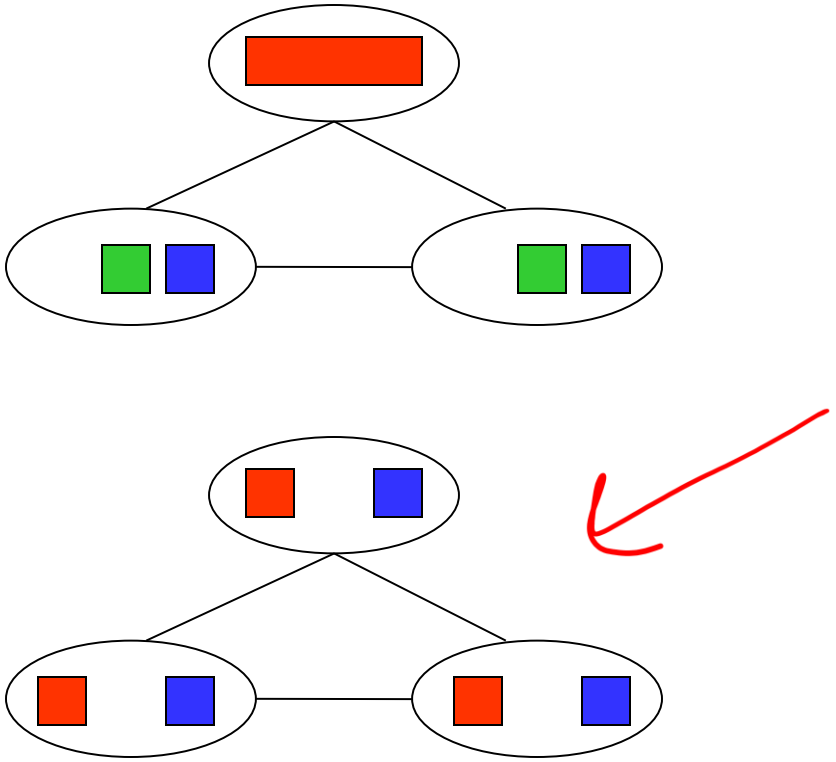
Queue:
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA
V->NSW
Q->NSW
SA->NSW

- Backtrack on the assignment of Q
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)
- Arc consistency only checks local consistency conditions
- Arc consistency still runs inside a backtracking search!



What went wrong here?

Backtracking Search with AC-3

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Var

Val

AC-3

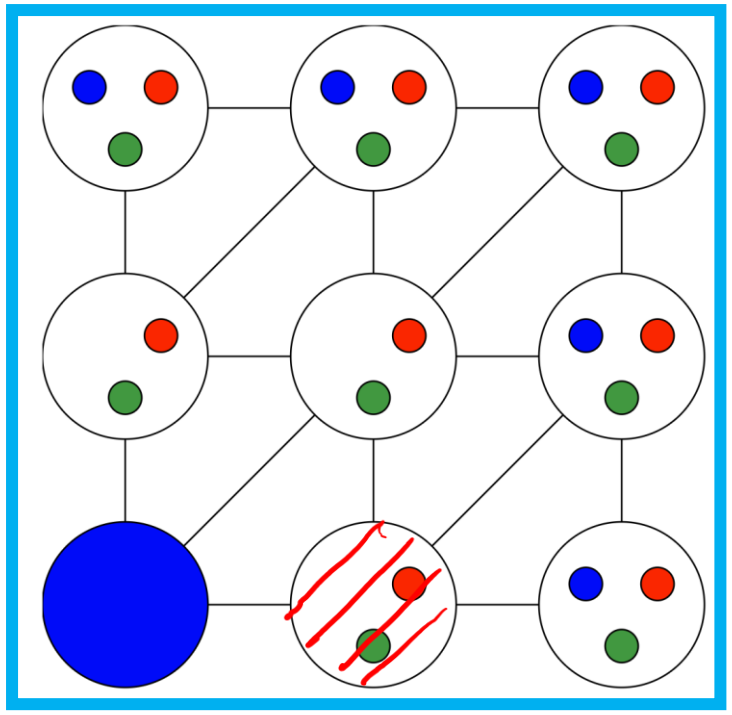
AC-3(*csp*)

- Where do you run AC-3?

Demo – Backtracking with AC-3

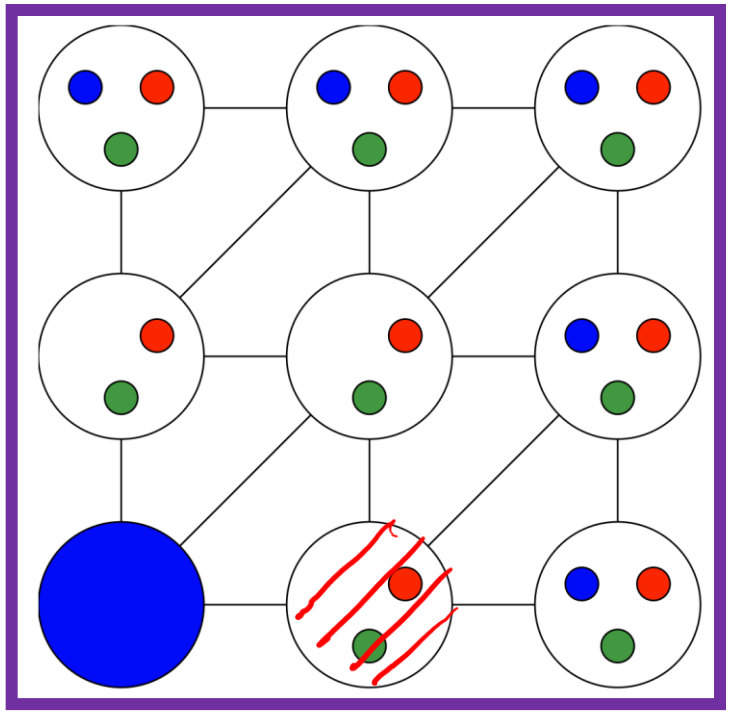
Compare

- Backtracking with Forward Checking
- Backtracking with AC-3



Forward checking only check arcs connecting variables a variable that we just assigned.

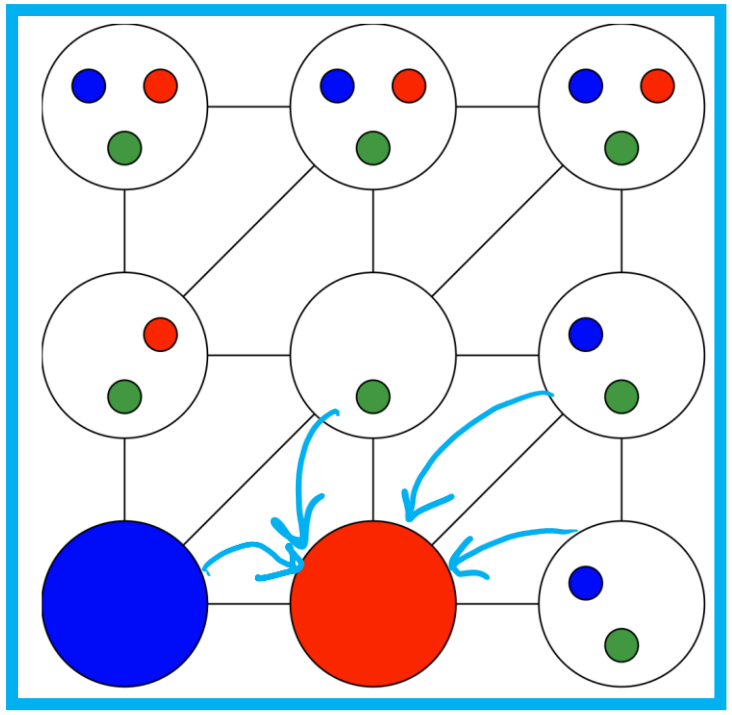
With AC-3, we make sure all arcs are consistent



Demo – Backtracking with AC-3

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3



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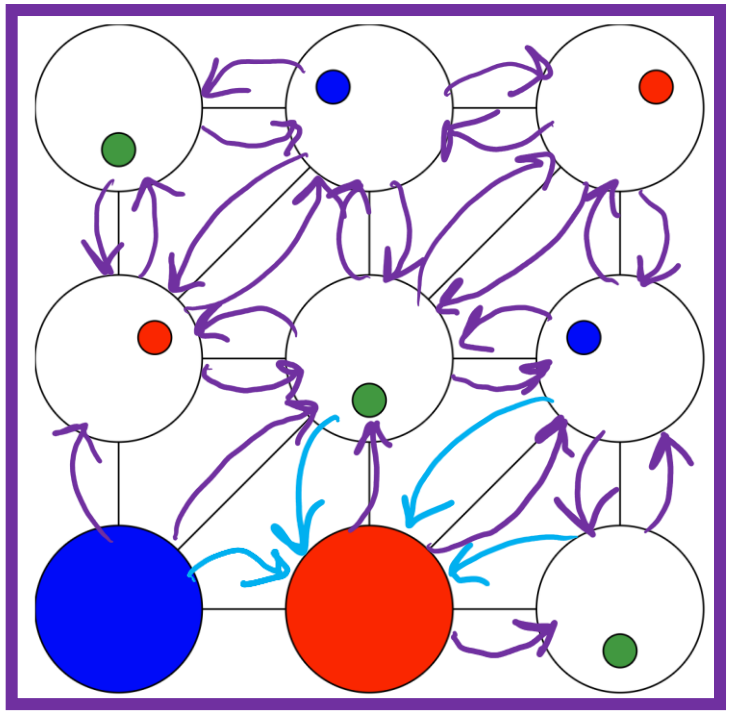
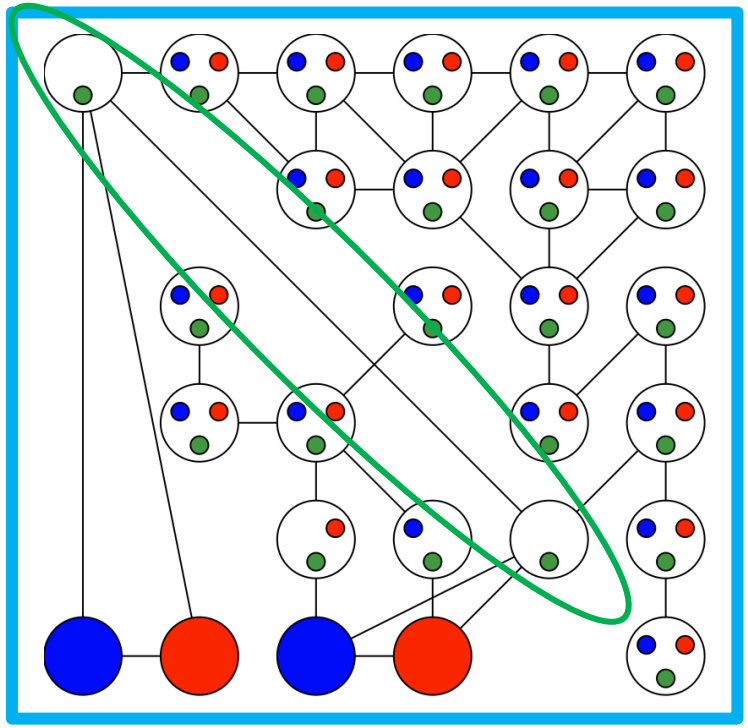


Demo – Backtracking with AC-3

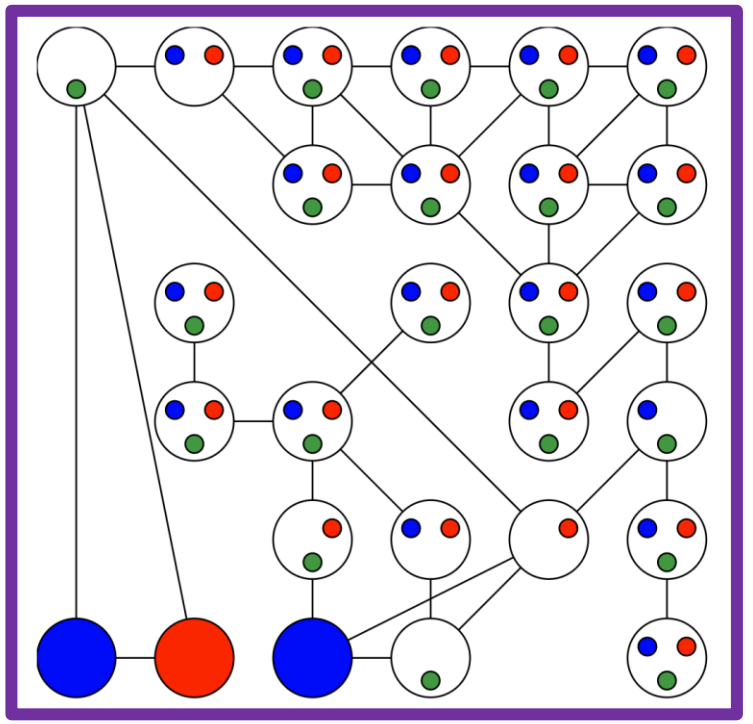
Compare

- Backtracking with Forward Checking
- Backtracking with AC-3



Forward checking only check arcs connecting variables a variable that we just assigned.

With AC-3, we can find existing problems, such as the arc between these two variables with only green left.



Complexity of a single run of AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

Recall that the whole backtracking algorithm with AC-3 will call AC-3 many times

Complexity of a single run of AC-3

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```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$

Complexity of a single run of AC-3

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- n node in total, each has $\leq d$ values
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- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

Complexity of a single run of AC-3

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function AC-3( csp) returns the CSP, possibly with reduced domains
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---

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```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2 d)$

- Check arc consistency per arc: $O(d^2)$

Complexity of a single run of AC-3 is at most $O(n^2 d^3)$

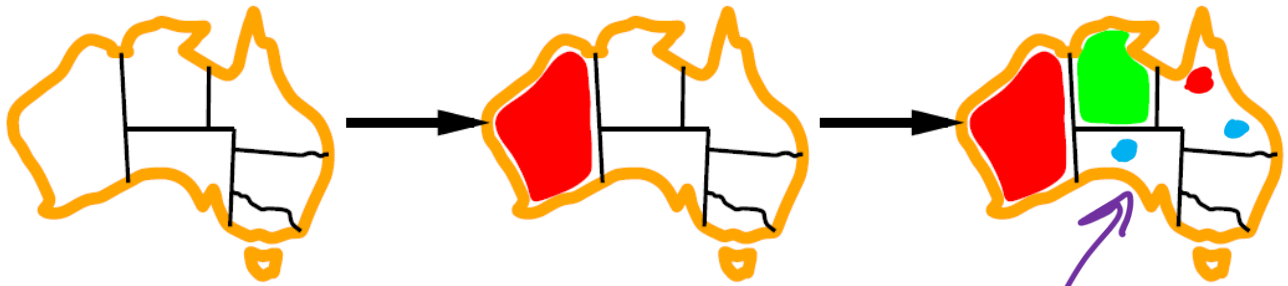
(Not required) Zhang&Yap (2001) show that its complexity is $O(n^2 d^2)$

Ordering

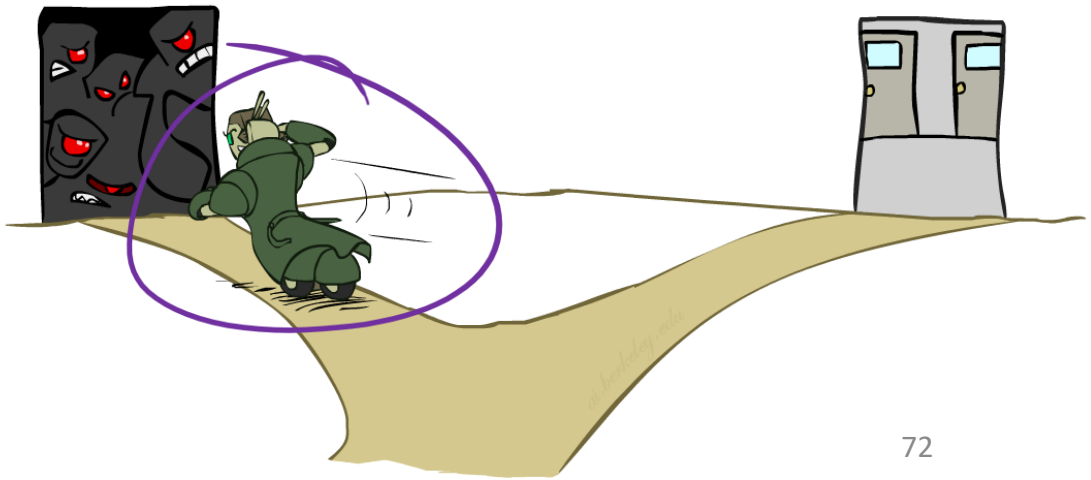
Ordering: Minimum Remaining Values

LCV alues

- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain



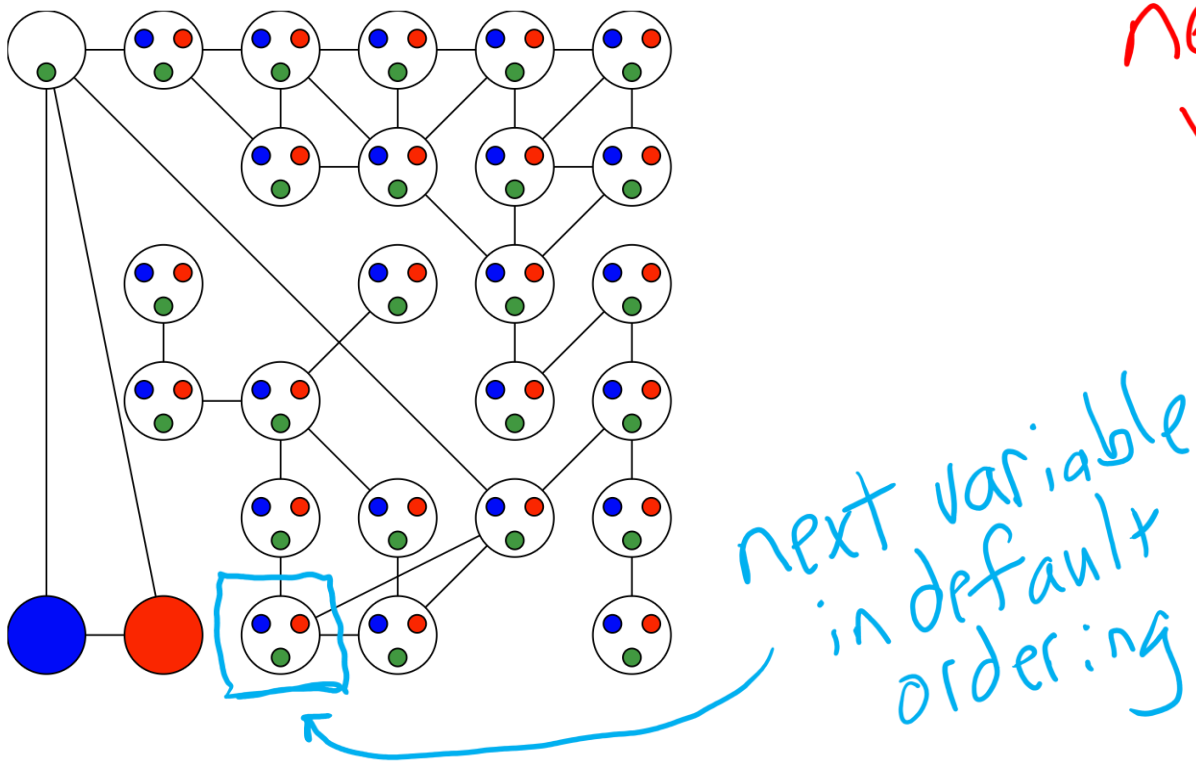
- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering

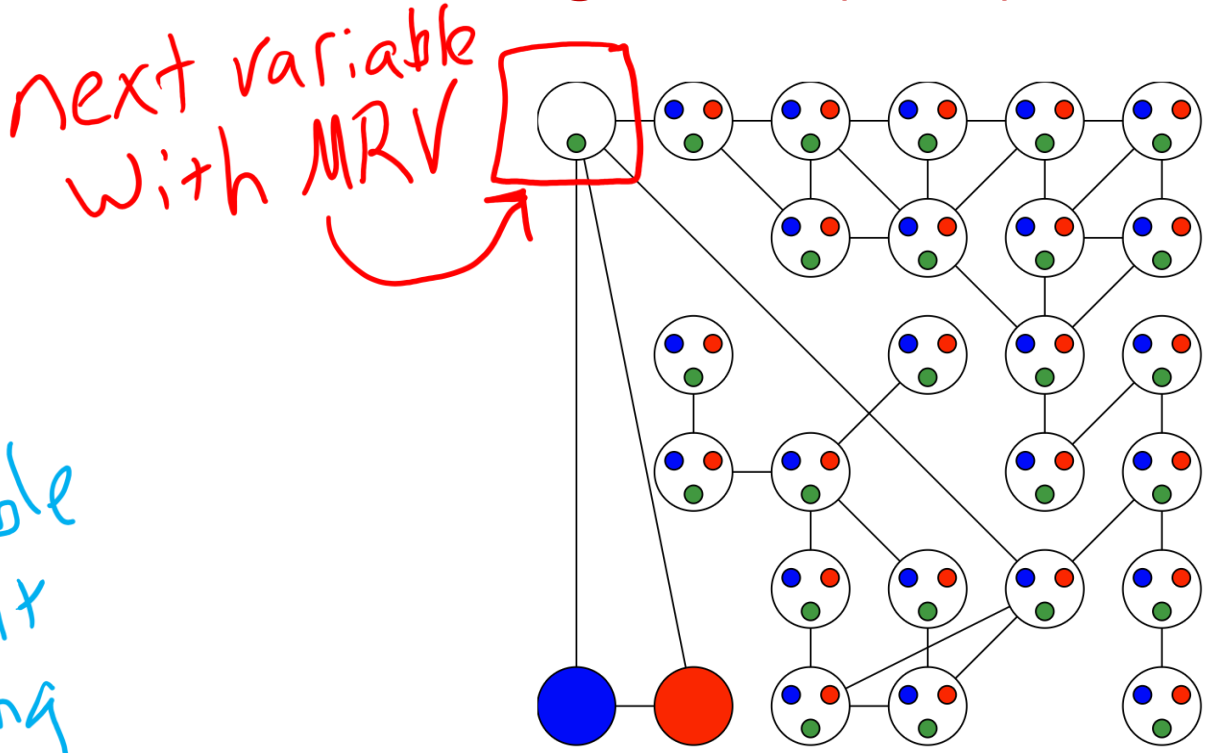


Demo – Coloring with a Complex Graph

Compare

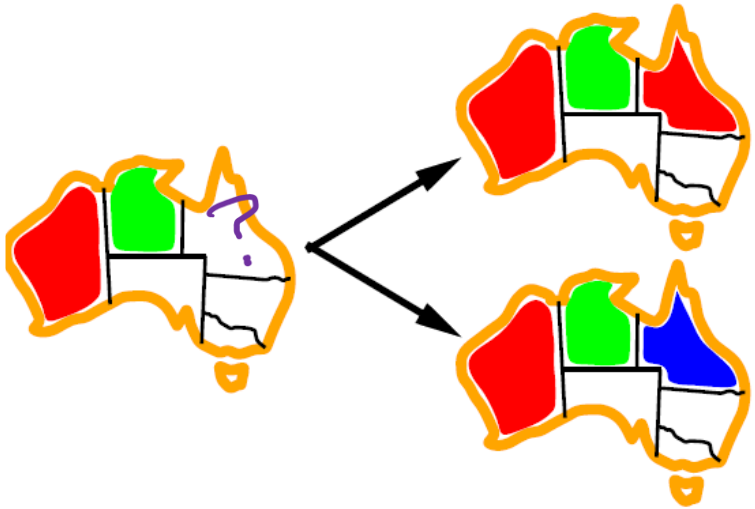
- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)

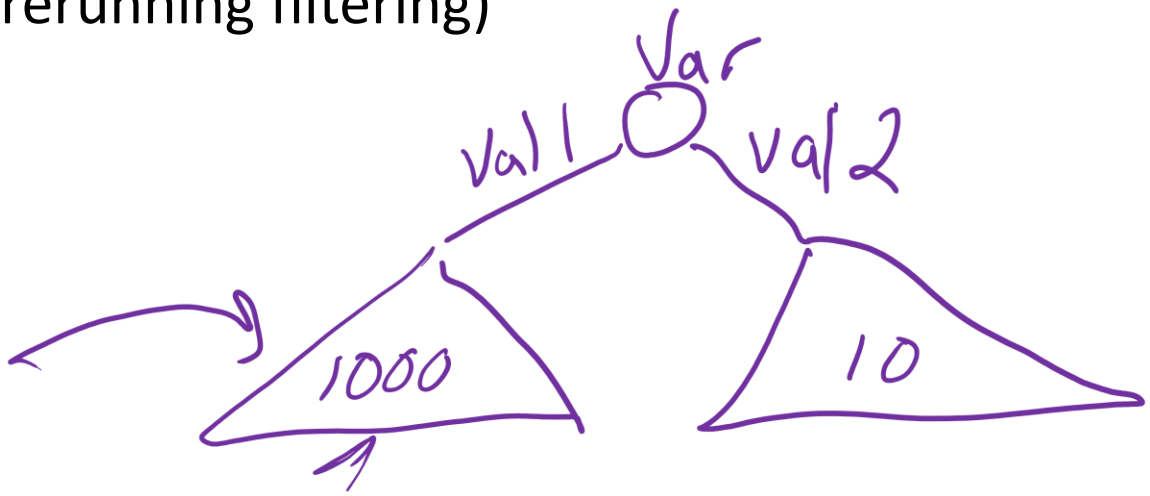




Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - i.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)





Ordering: Least Constraining Value

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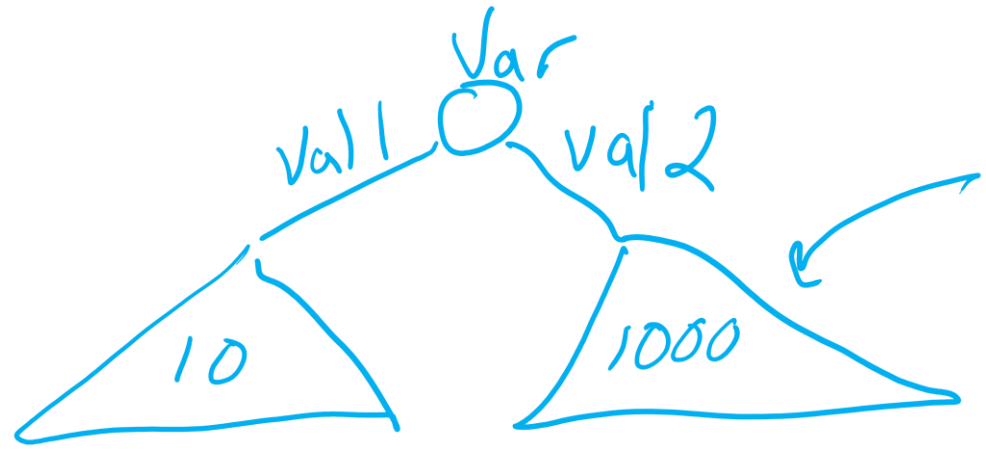
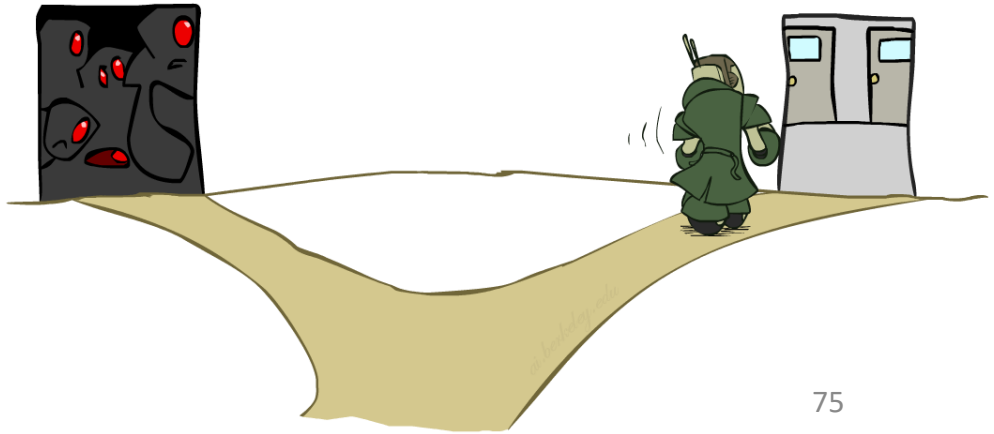
  - Given a choice of variable, choose the *least constraining value*
  - i.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)

- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible





Demo – Coloring with a Complex Graph

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)
- Backtracking + AC-3 + MRV + LCV

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - (Structure)