As you come in…

Consider an LP with no constraints

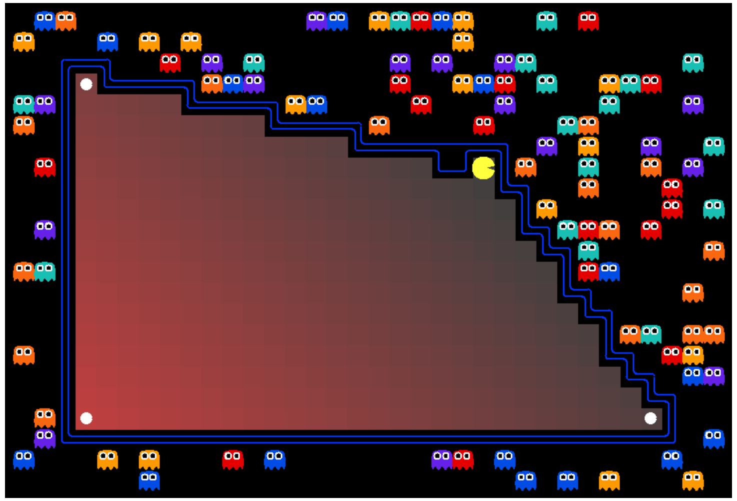
$$\min_{\boldsymbol{x}} . \qquad \boldsymbol{c}^T \boldsymbol{x}$$

Suppose vector c is not all 0s.

Then what is the minimum value of the objective?

AI: Representation and Problem Solving

Solving linear programs;
Integer programs

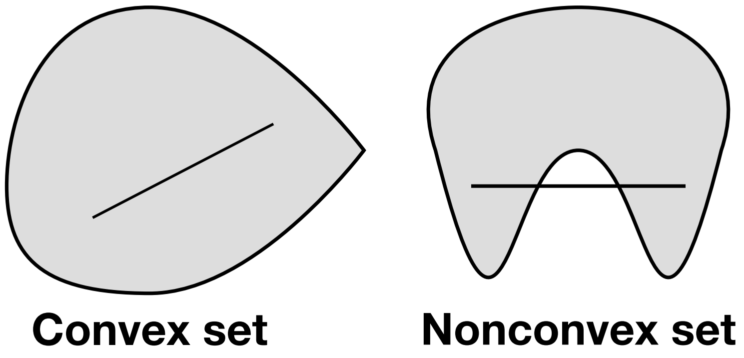


Instructors: Nihar Shah and Tuomas Sandholm

Slide credits: CMU AI with some drawings from ai.berkeley.edu

“Convex” set

A set in R^d is said to be a convex set if for every pair of points in the set, all the points on the line joining these two points are also in the set.

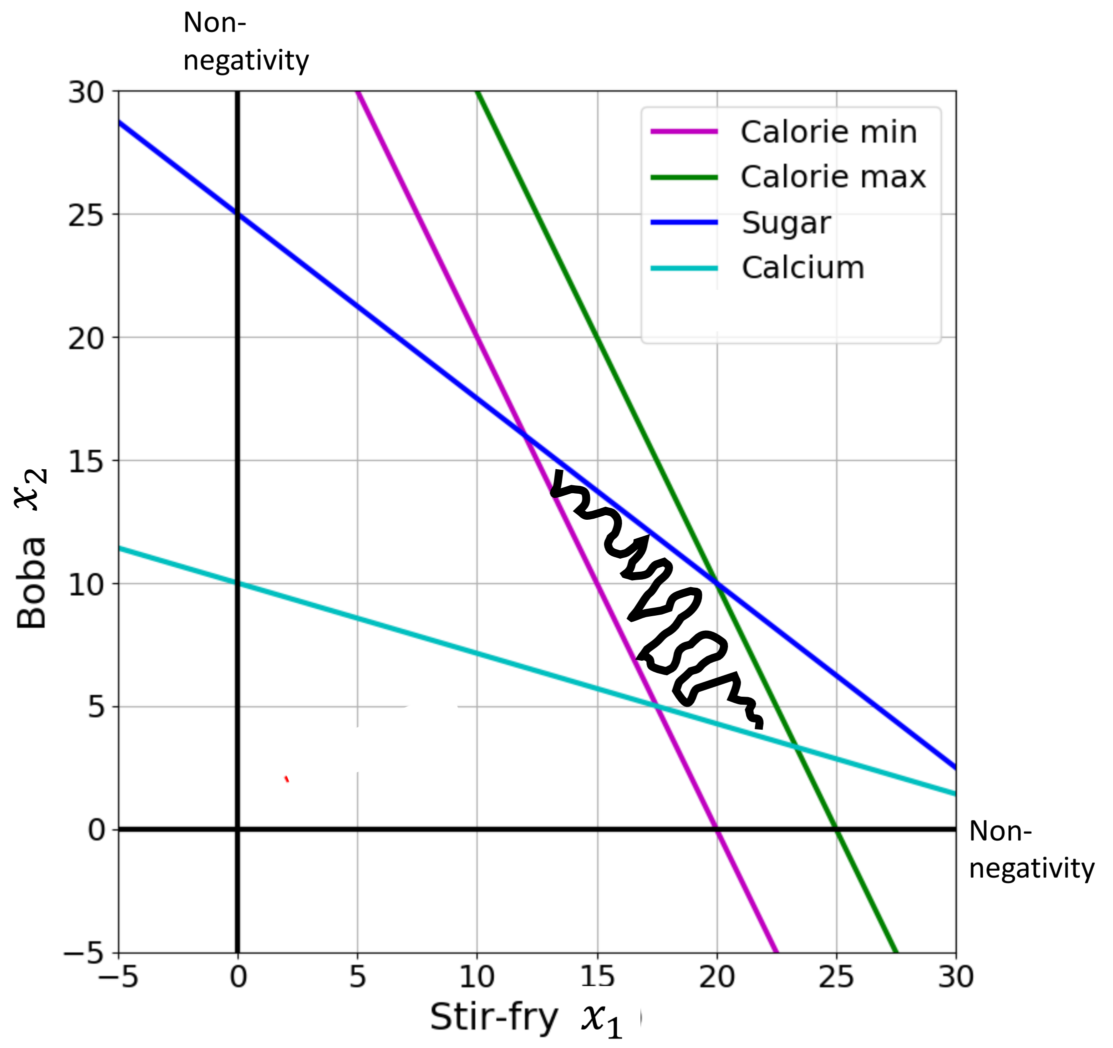


Very useful for many types of problems!

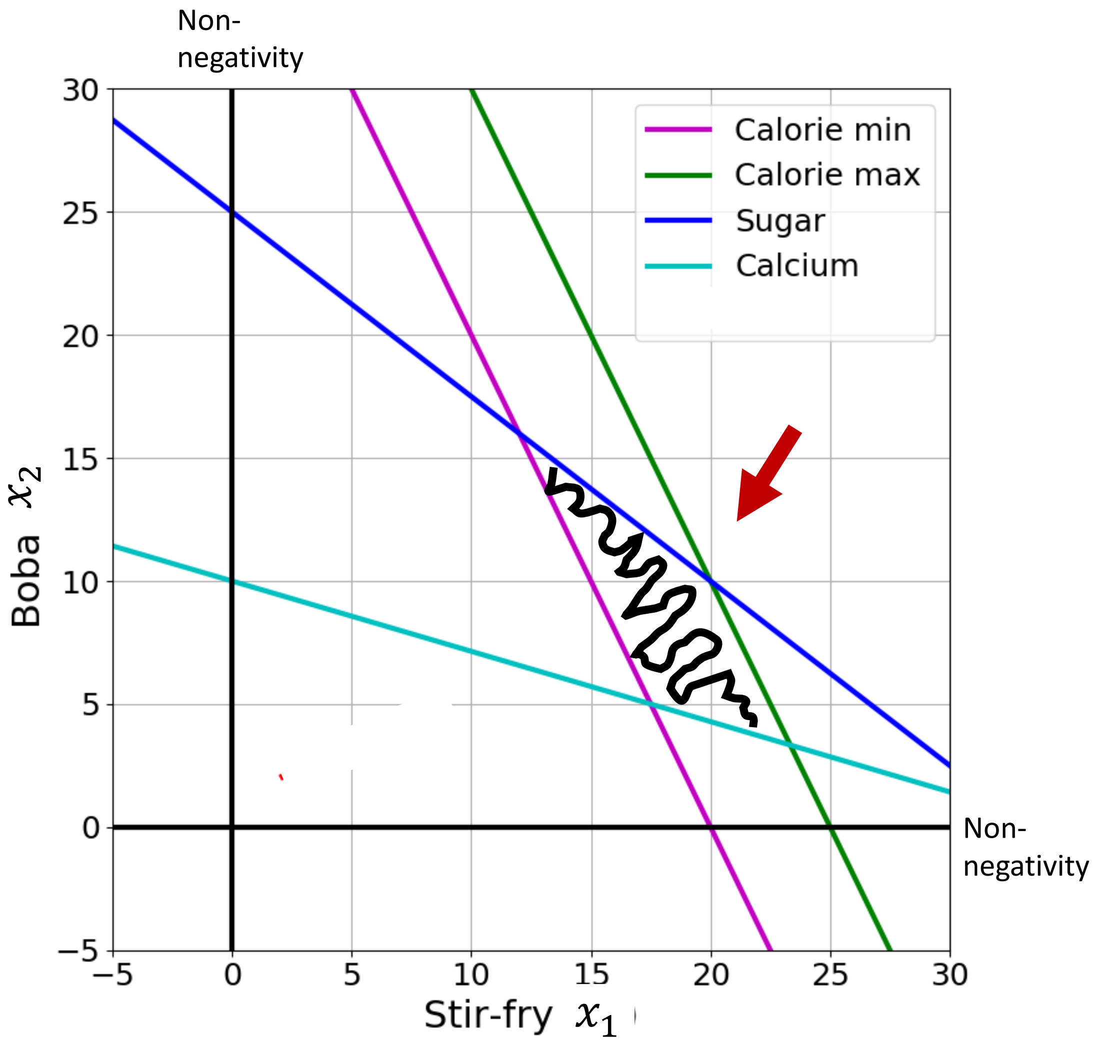
Recall from previous lecture

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
$$3\,x_1 + 4\,x_2 \leq 100$$
$$20\,x_1 + 70\,x_2 \geq 700$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

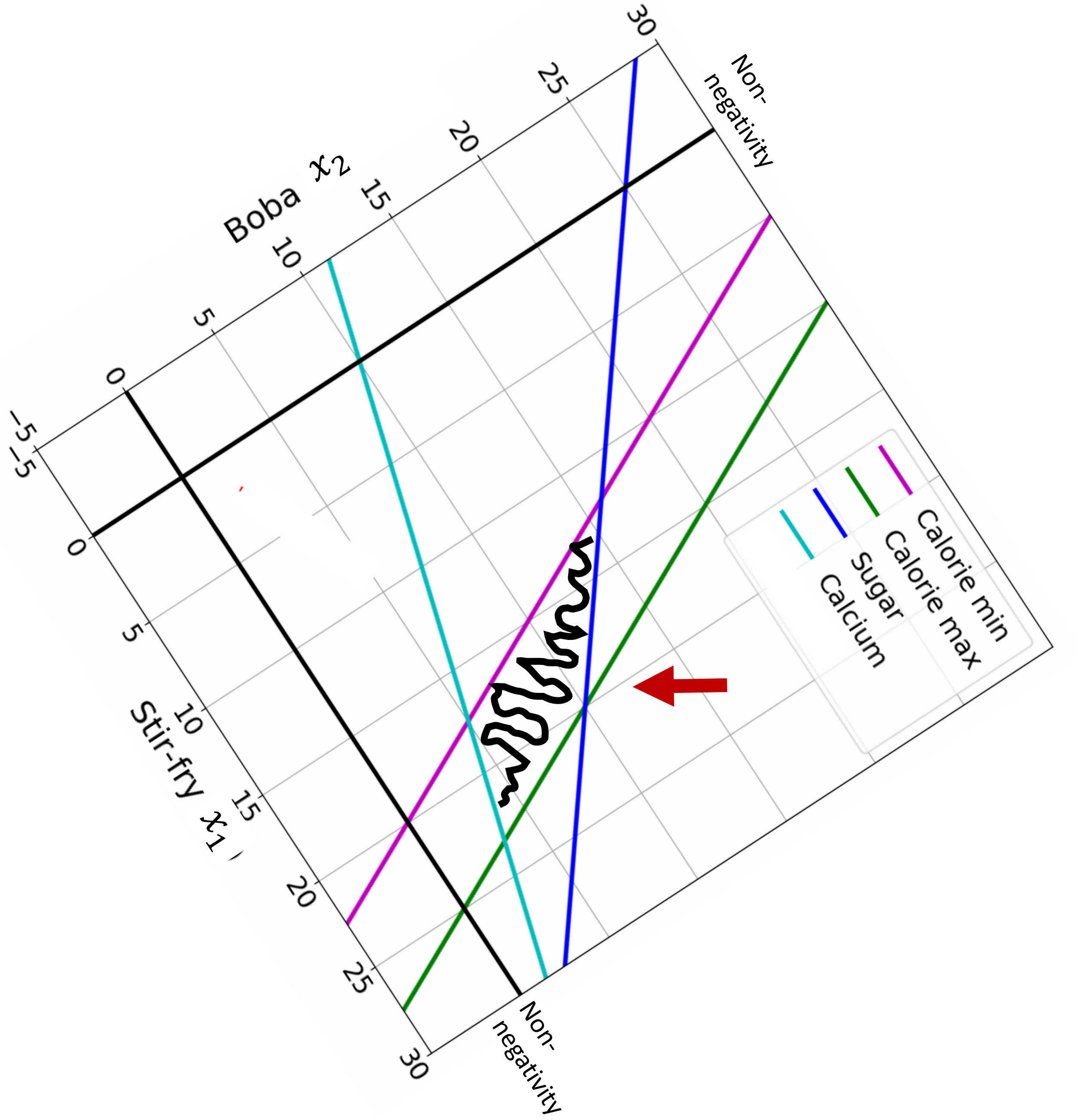


Is the feasible set convex?

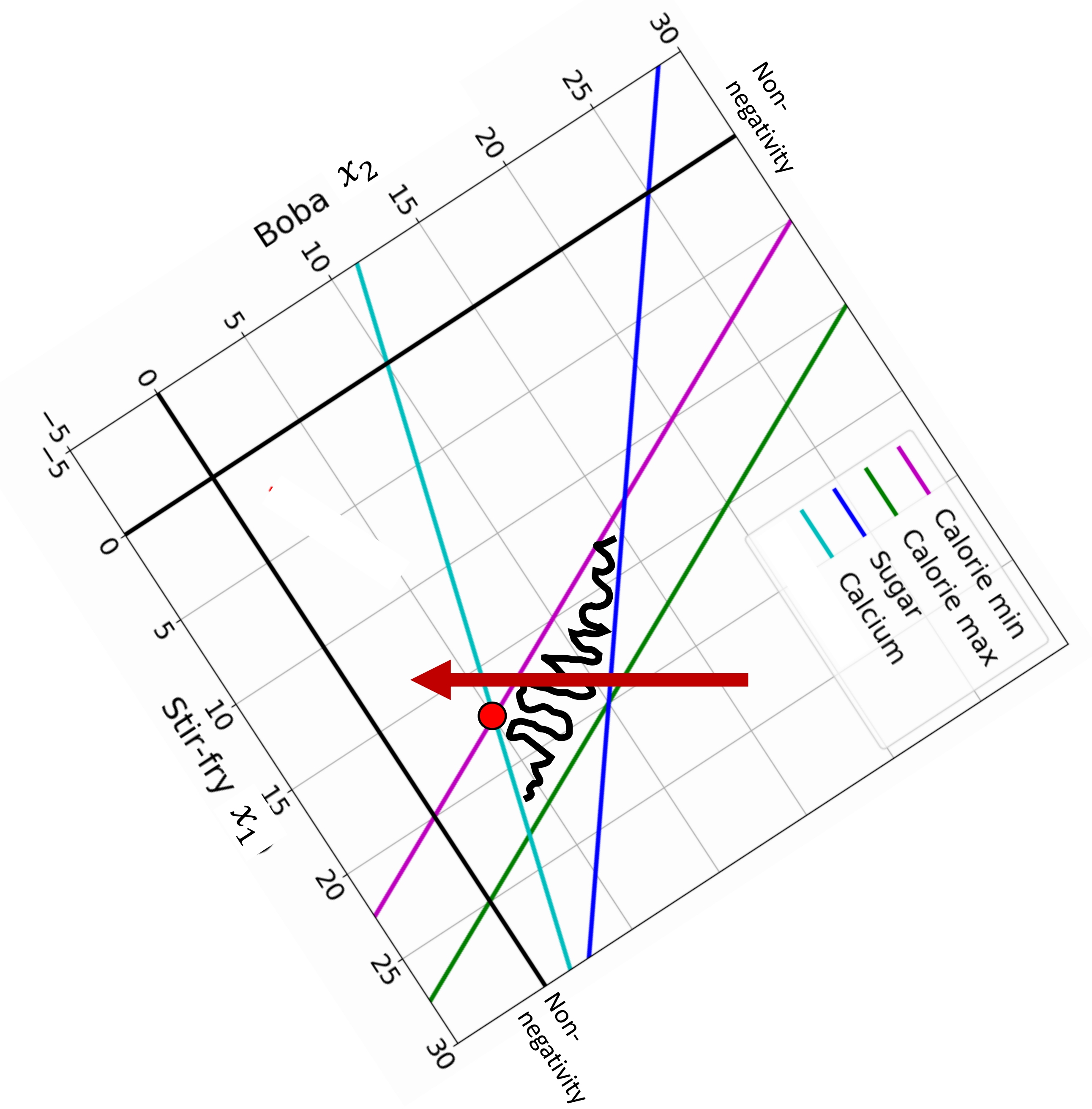


Non-
negativity
Calorie min
Calorie max
Sugar
Calcium
30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30
Boba x_2
Stir-fry x_1
Non-
negativity

What is the leftmost point in the feasible set?

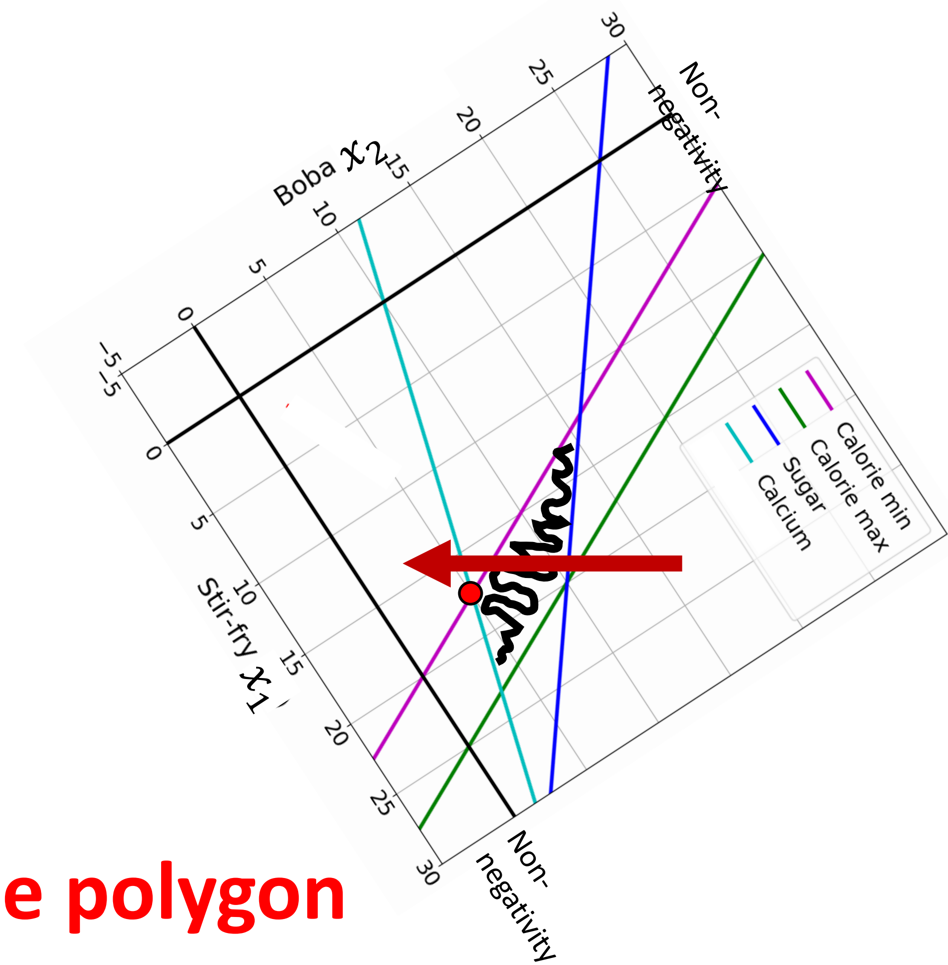


What is the leftmost point in the feasible set?



Generalizing this example

- Consider any linear program
- Suppose x = [x_1, x_2] (i.e., it is 2 dimensional)
- Assume feasible set is closed
- ∴ Feasible set is a convex polygon
- Consider any direction of minimization

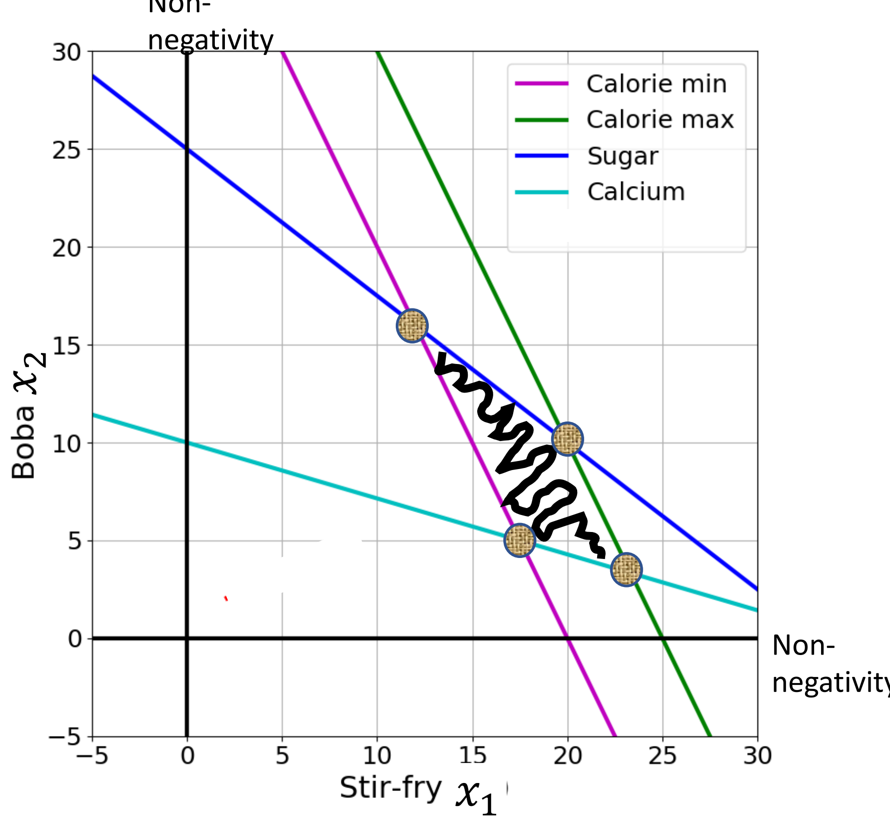


There will be a minimizer at a corner of the polygon

But.. what exactly is a “corner”?

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
$$3\,x_1 + 4\,x_2 \leq 100$$
$$20\,x_1 + 70\,x_2 \geq 700$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



In two dimensions: A corner is a point where two constraints are met with equality. In other words, it is a point at the intersection of constraint boundaries.

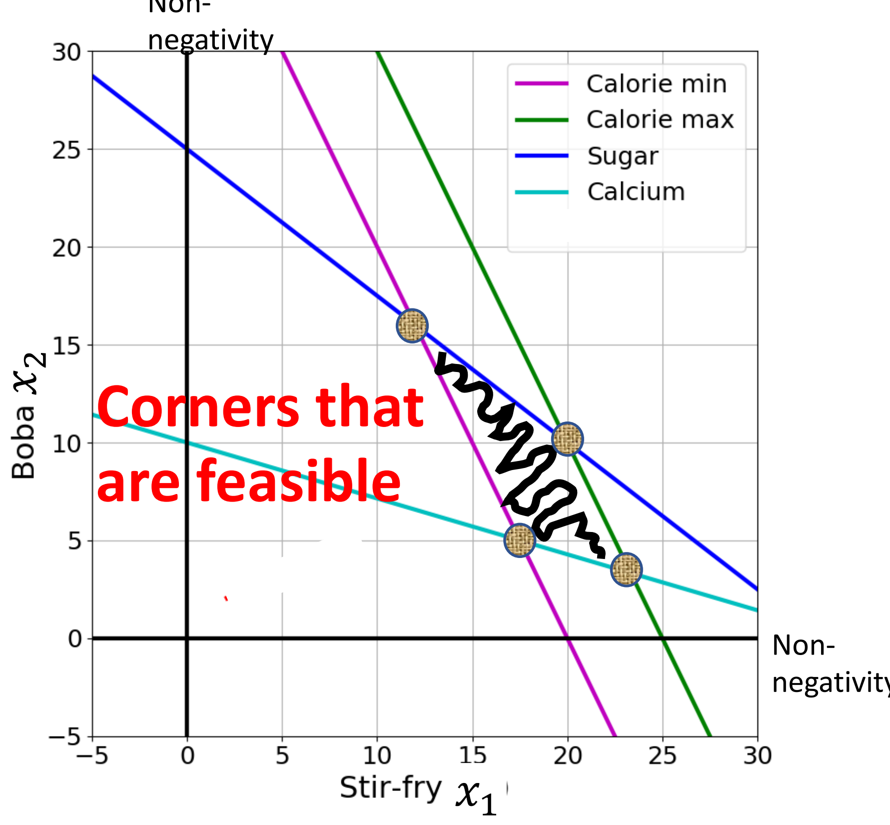
Bu

But there are 6 constraints in this problem, so there should be 6 choose 2 corners. You are showing us only 4 corners?!

$$20\,x_1 + 70\,x_2 \geq 700$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$



In two dimensions: A corner is a point where two constraints are met with equality. In other words, it is a point at the intersection of constraint boundaries.

What about higher dimensions?

- Consider constraint: $Ax \leq b$, where $x \in R^d$
- In words, a corner is a point where d constraints are met with equality
- Consider any subset of d rows of A, and call it $\widetilde{A}$.
- If $\widetilde{A}$ is of full rank then $\widetilde{A}x=b$ has a unique solution.
- This is a corner.

Solving an LP

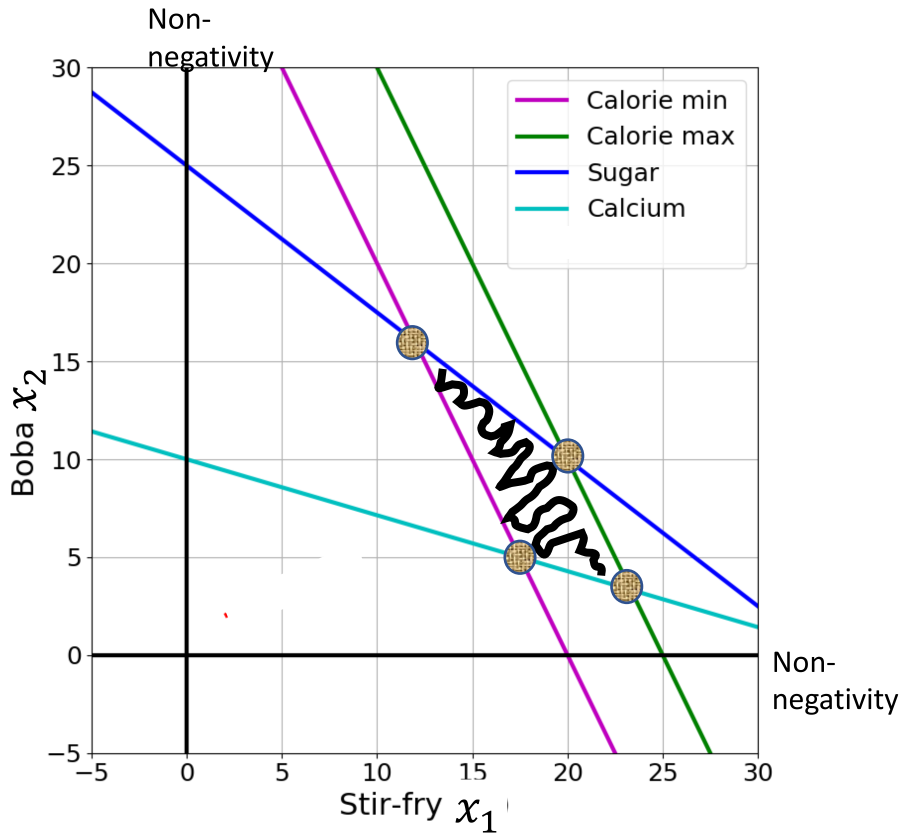
Algorithm

1. Enumerate all intersections (corners).
2. Keep only those that are feasible (i.e., satisfy *all* inequalities).
3. Return feasible intersection with the lowest objective value.

Problem: There may be too many feasible intersections.

Simplex algorithm (intuition)

- Start at a feasible intersection (if not trivial, can solve another LP to find one)
- Define successors as “neighbors” of current intersection
  - i.e., remove one row from our square subset of A, and add another row not in the subset; then check feasibility
- Move to any successor with lower objective than current intersection
  - If no such successors, we are done



Greedy local hill-climbing search! … but always finds *optimal* solution (if defined right)

Solving an LP

Remember: Solutions are at feasible intersections of constraint boundaries

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior point methods

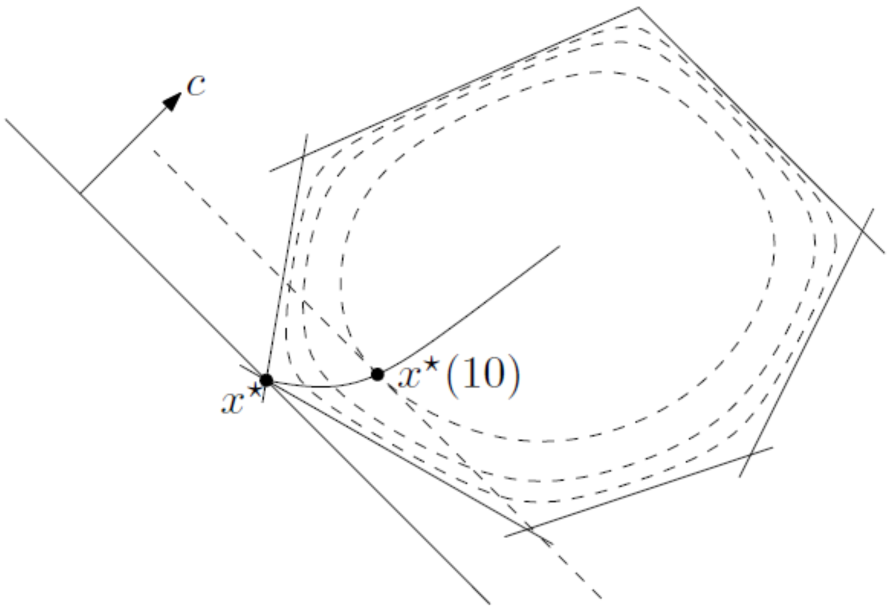


Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

Integer Programs

Another representation

Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Healthiness goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (**bowls**) and boba (**glasses**).

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per **bowl**)	1	100	3	20
Boba (per **glass**)	0.5	50	4	70

Healthiness goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (**bowls**) and boba (**glasses**) should we buy?

Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- *Binary Integer Programming:* Constraints restrict each entry of x to take values in {0,1}
- *Mixed Integer Linear Programming:* Some variables have integer constraints and some don't

Solving integer programs

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$
$$\sout{\boldsymbol{x} \in \mathbb{Z}^N}$$

Relax the integer program to a linear program by simply dropping integer constraints

We know how to solve linear programming problems.

Can we just solve this LP and use the output as our solution for the IP?

Problem: The solution to the LP may not be integer valued.

Branch and Bound algorithm

- Push current LP (with its solution) into priority queue, ordered by objective value of LP solution
- Repeat:
  - If queue is empty, output "IP is infeasible"
  - Pop candidate solution $\boldsymbol{x}_{LP}^{\star}$ from priority queue
  - If $\boldsymbol{x}_{LP}^{\star}$ is all integer valued, return solution
  - Select a coordinate x_i that is not integer valued.
  - Solve two additional LPs, each of which has one additional constraint on the current LP:

    (i) Added constraint $x_i \leq floor(x_i)$

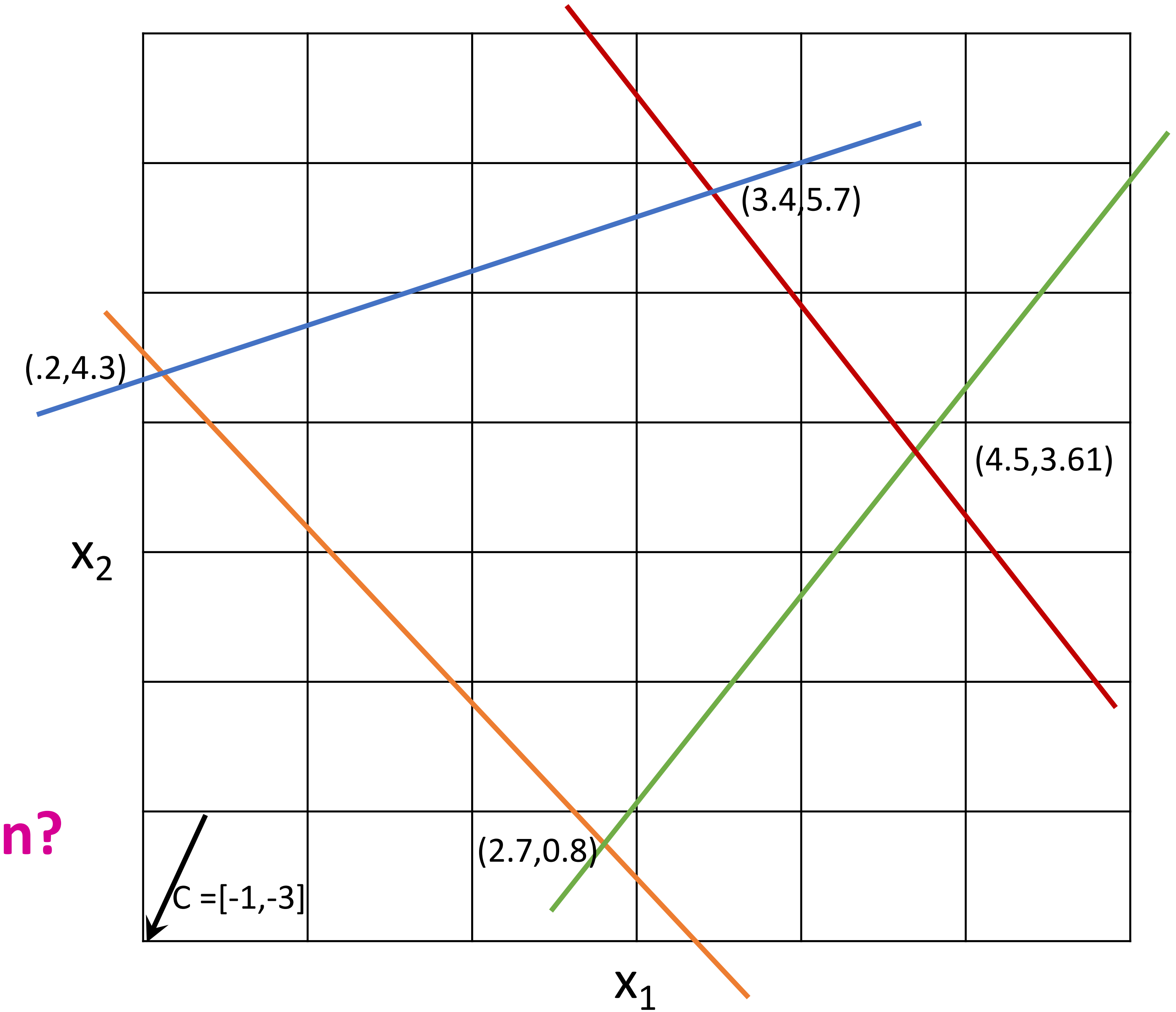
    (ii) Added constraint $x_i \geq ceil(x_i)$
  - To the priority queue, add whichever of these LPs are feasible

Example

min. $-x_1 - 3x_2$

subject to

$$x_2 = -1.4x_1 + 4.58$$
$$x_2 = 1.56x_1 + 3.41$$
$$x_2 = -1.9x_1 + 12.16$$
$$x_2 = .44x_1 + 4.21$$



What is the LP solution?

min. $-x_1 - 3x_2$

subject to

$x_2 = -1.4x_1 + 4.58$

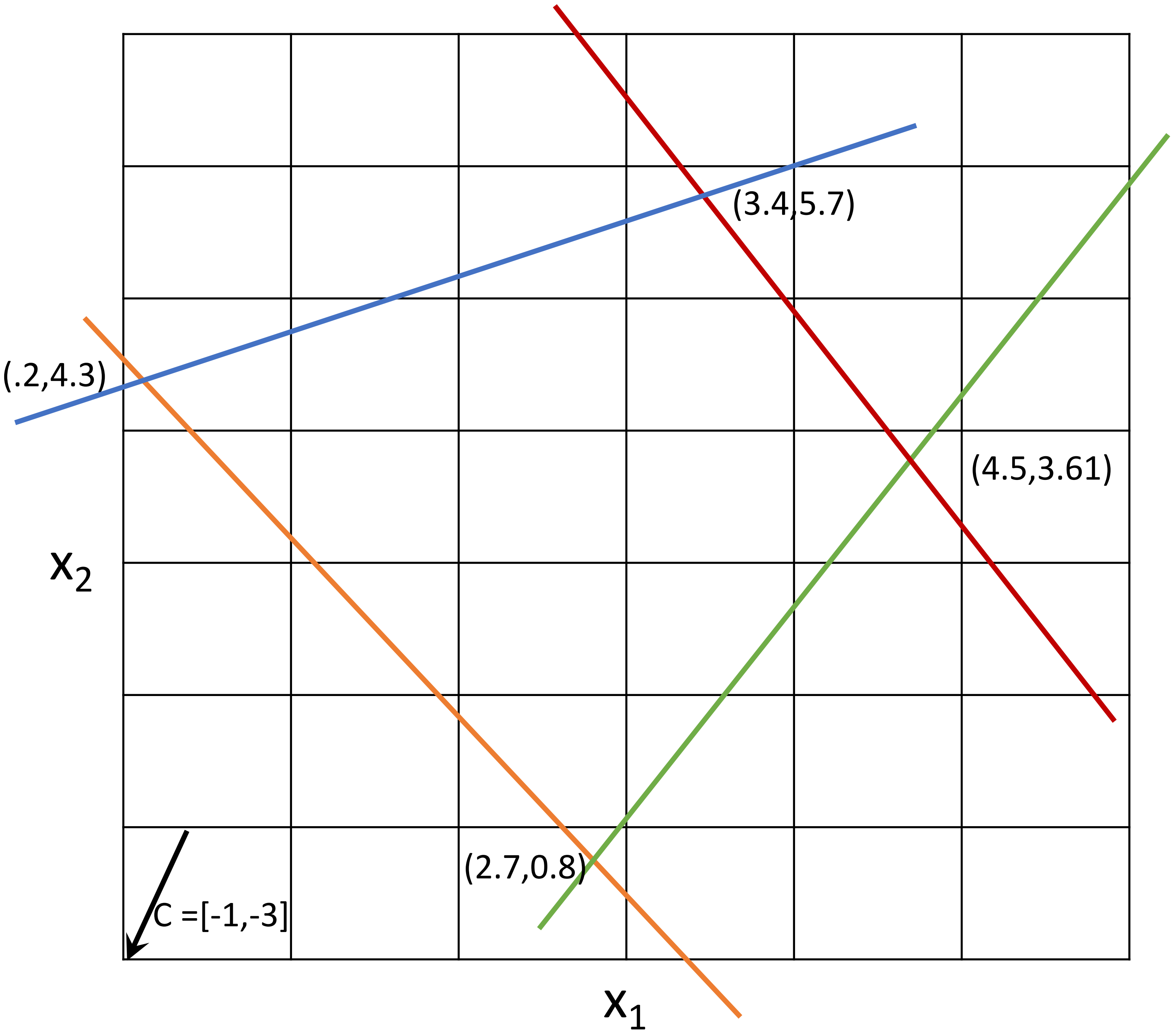
$x_2 = 1.56x_1 + 3.41$

$x_2 = -1.9x_1 + 12.16$

$x_2 = .44x_1 + 4.21$

Priority Queue:

-20.5: (3.4,5.7)



min. $-x_1 - 3x_2$

subject to

$x_2 = -1.4x_1 + 4.58$

$x_2 = 1.56x_1 + 3.41$

$x_2 = -1.9x_1 + 12.16$

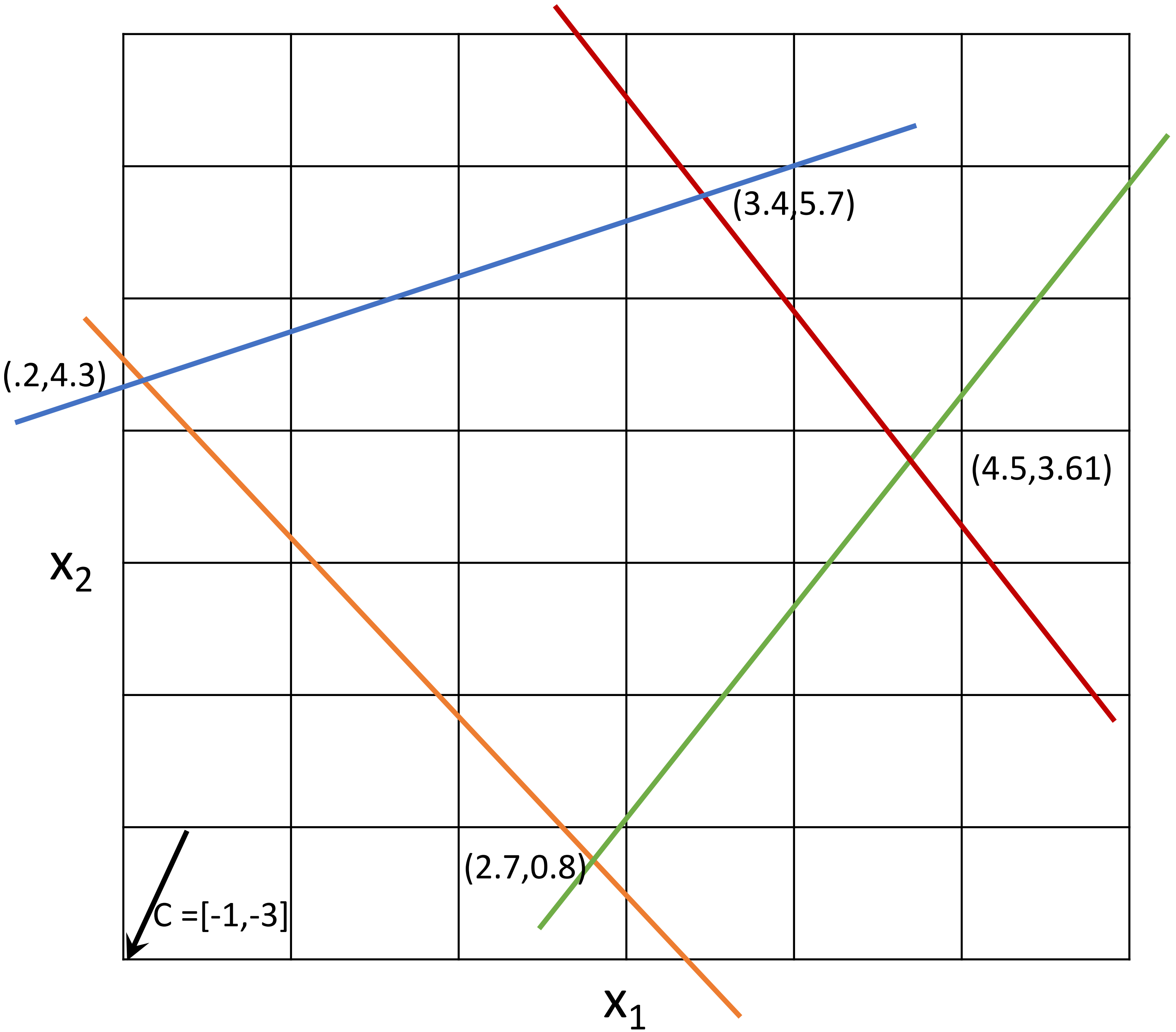
$x_2 = .44x_1 + 4.21$

Priority Queue:

~~-20.5: (3.4,5.7)~~

-19.6: (3,5.53) (x <= 3)

-17.7: (4,4.56) (x >=4)



min. $-x_1 - 3x_2$

subject to

$x_2 = -1.4x_1 + 4.58$

$x_2 = 1.56x_1 + 3.41$

$x_2 = -1.9x_1 + 12.16$

$x_2 = .44x_1 + 4.21$

Priority Queue:

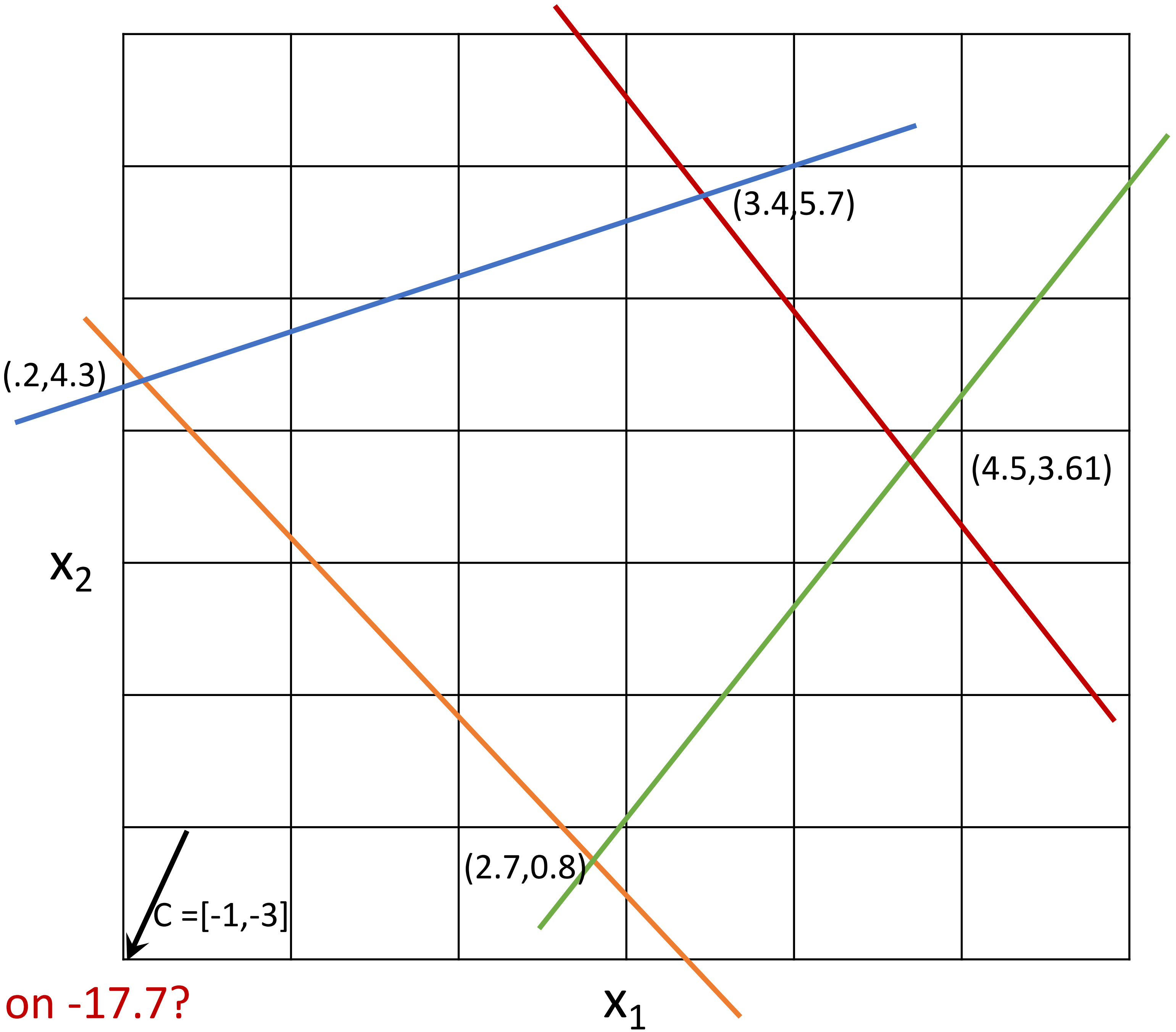
~~-20.5: (3.4,5.7)~~

~~-19.6: (3,5.53) (x <= 3)~~

-17.7: (4,4.56) (x >=4)

-18.0: (3,5) (x<=3,y<=5)

Inf: (x<=3,y>=6)



Why do we not need to recurse on -17.7?

Convex optimization

Another representation…there is an entire course on this! 10-725

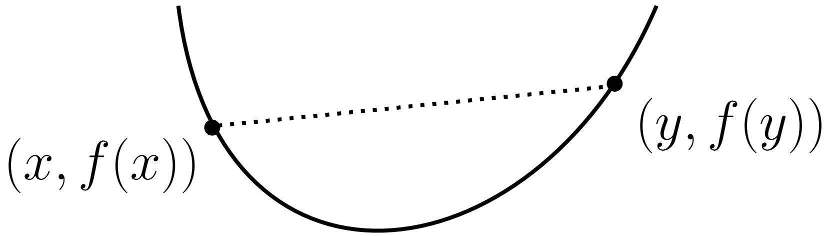
Convex functions

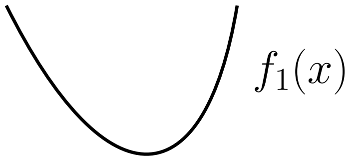
A function $f: R^d \to R$ is convex if for every pair of points $x \in R^d$ and $y \in R^d$, and every value $\theta \in [0,1]$:

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

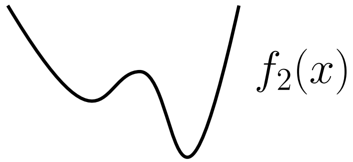
In words, the line joining two points is never below the function.

Linear functions are convex!





Convex function



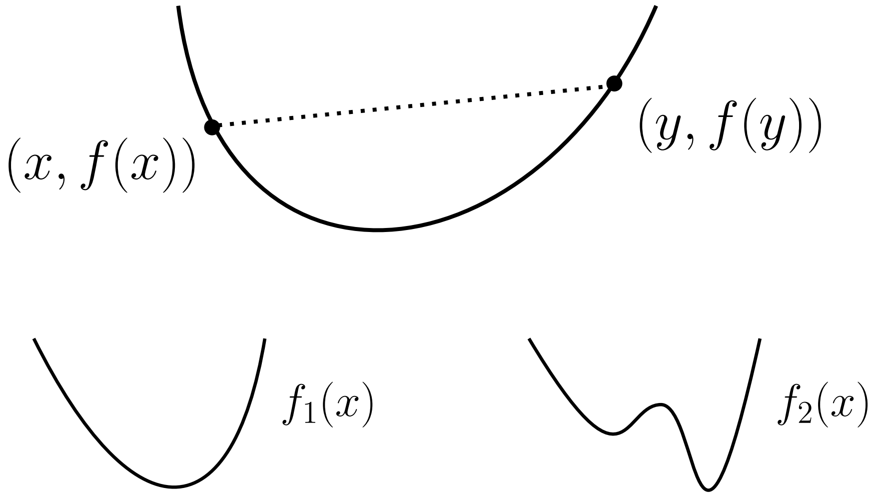
Nonconvex function

Convex functions

A function $f: R^d \to R$ is convex if for every pair of points $x \in R^d$ and $y \in R^d$, and every value $\theta \in [0,1]$:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

In words, the line joining two points is never below the function.



Convex function **Nonconvex function**

Linear functions are:

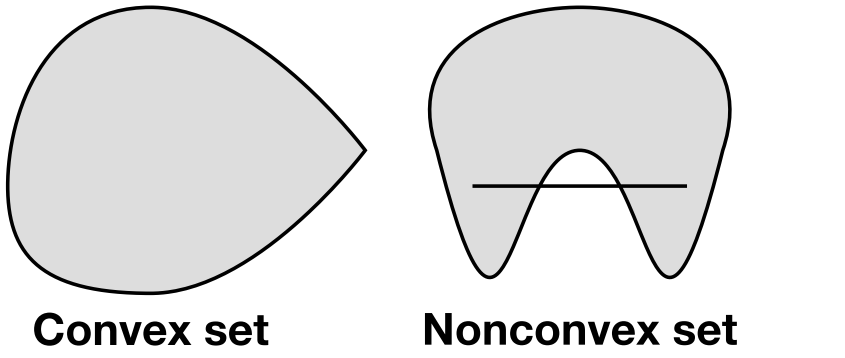
(a) always convex

(b) may or may not be convex

(c) never convex

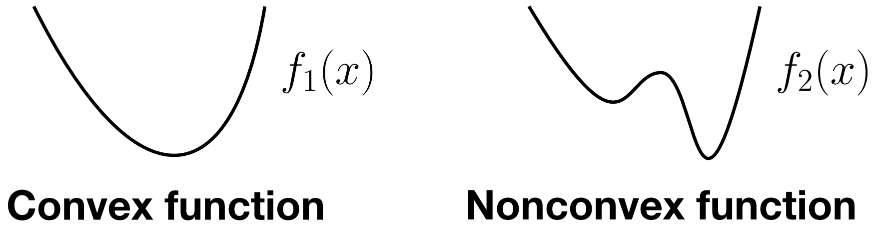
Convex optimization

An optimization problem is a convex optimization problem if the objective is a convex function and the feasible set is a convex set.



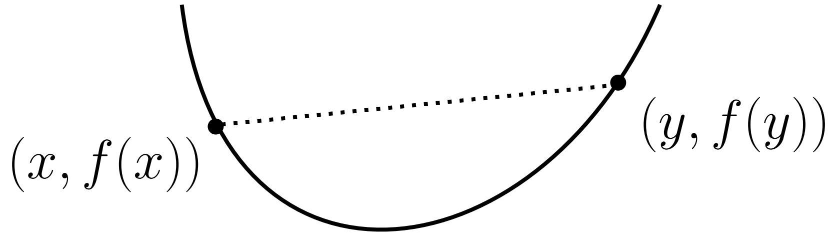
Are linear programming problems also convex optimization problems?

Yes.



Are integer programming problems also convex optimization problems?

No.



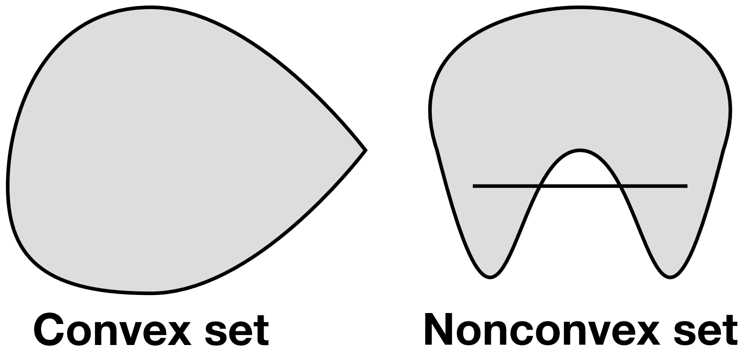
Very useful property of convex optimization: any local minimum is also the global minimum

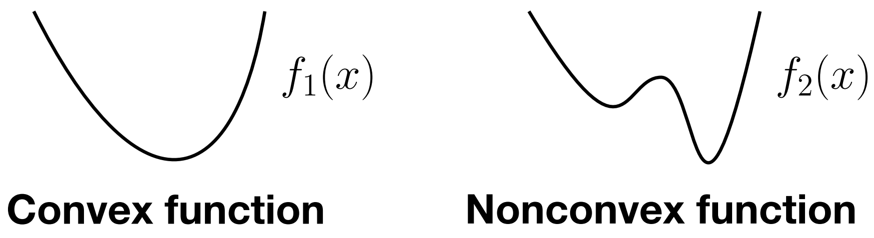
Convex optimization

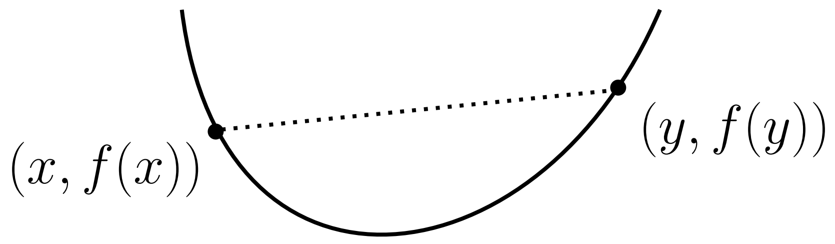
An optimization problem is a convex optimization problem if the objective is a convex function and the feasible set is a convex set.

Very useful property of convex optimization: any local minimum is also the global minimum

Greedy ("gradient descent") works!







Poll 1

Consider a linear program of the following form with exactly one constraint, and assume c is not 0. Then it will always have a minimum objective value of $-\infty$.

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \le b \end{aligned}$$

(True or False)

Poll 2:

Let y_{IP}^* be the optimal objective of an integer program P.

Let $\boldsymbol{x}_{IP}^*$ be an optimal point of the integer program P.

Let y_{LP}^* be the optimal objective of the LP-relaxed version of P.

Let $\boldsymbol{x}_{LP}^*$ be an optimal point of the LP-relaxed version of P.

Assume that P is a minimization problem.

Which of the following must always be true? Select all that apply.

A) $\boldsymbol{x}_{IP}^* = \boldsymbol{x}_{LP}^*$

B) $y_{IP}^* \leq y_{LP}^*$

C) $y_{IP}^* \geq y_{LP}^*$