Plan

Last time

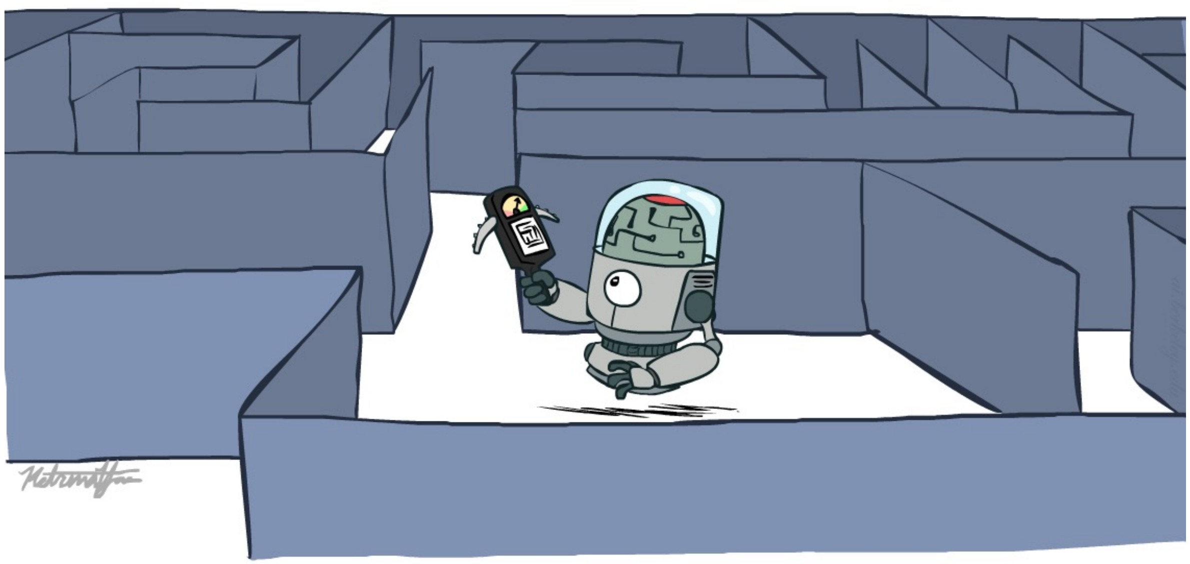
- Tree search vs graph search
- BFS, DFS, Uniform cost search, iterative deepening search

Today

- Heuristics
- Greedy search
- A* search
  - Optimality
- More on heuristics

AI: Representation and Problem Solving

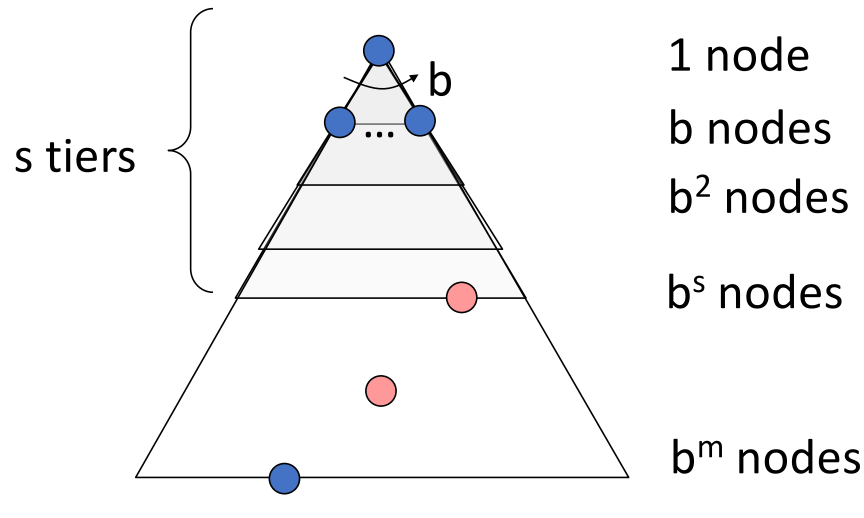
Informed Search



Instructor: Tuomas Sandholm and Nihar Shah

Slide credits: CMU AI, http://ai.berkeley.edu

Breadth-First Search (BFS) Properties



What nodes does BFS expand?

- Processes all nodes above shallowest solution
- Let depth of shallowest solution be s
- Search takes time $O(b^s)$

How much space does the frontier take?

- Has roughly the last tier, so $O(b^s)$

Is it complete?

- s must be finite if a solution exists, so yes!

Is it optimal?

- Only if costs are all the same (more on costs later)

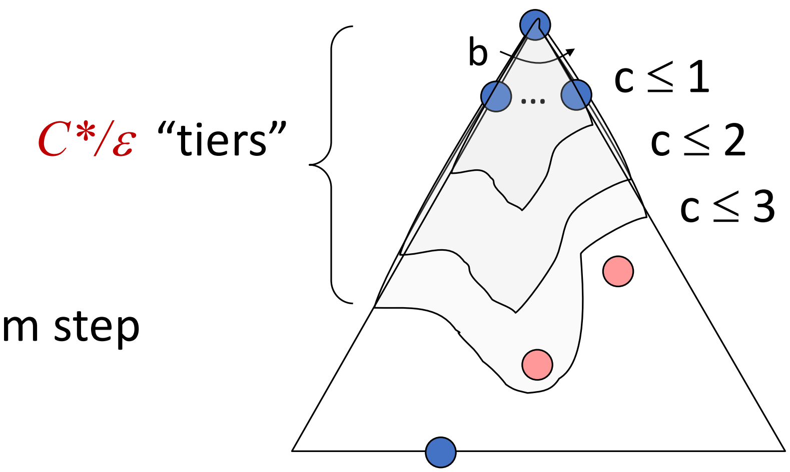
Uniform Cost Search (UCS) Properties

What nodes does UCS expand?

- Processes all nodes with cost less than cheapest solution
- If that solution costs $C*$ and step costs are at least ε , then the "effective depth" is roughly $C*/\varepsilon$
- Takes time O($b^{C*/\varepsilon}$) (exponential in effective depth)

How much space does the frontier take?

- Has roughly the last tier, so O($b^{C*/\varepsilon}$)



Is it complete?

- Assuming best solution has a finite cost and minimum step cost is positive, yes!

Is it optimal?

- Yes! (Proof via A*)

Uniform Cost Issues

Strategy:

- Explore (expand) the lowest path cost on frontier

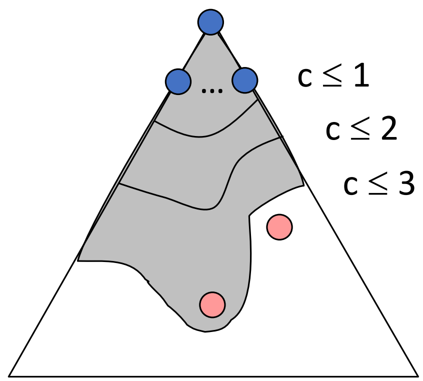
The good:

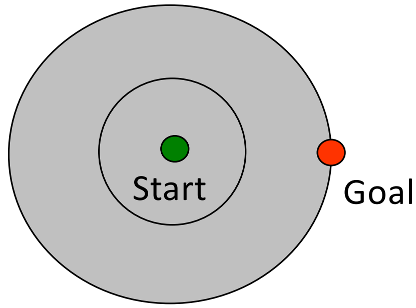
- UCS is complete and optimal!

The bad:

- Explores options in every “direction”
- No information about goal location

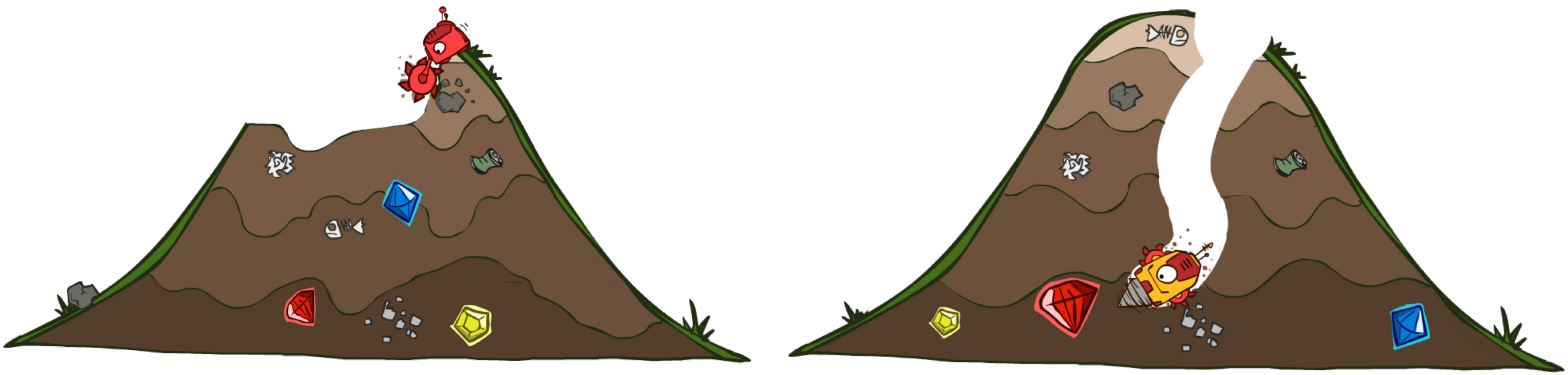
We’ll fix that today!





```
function GRAPH-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using some metric as the priority
    add initial state of problem to frontier with initial metric = 0
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with worse metric then
                replace that frontier node with child
```

Uninformed vs Informed Search



Today

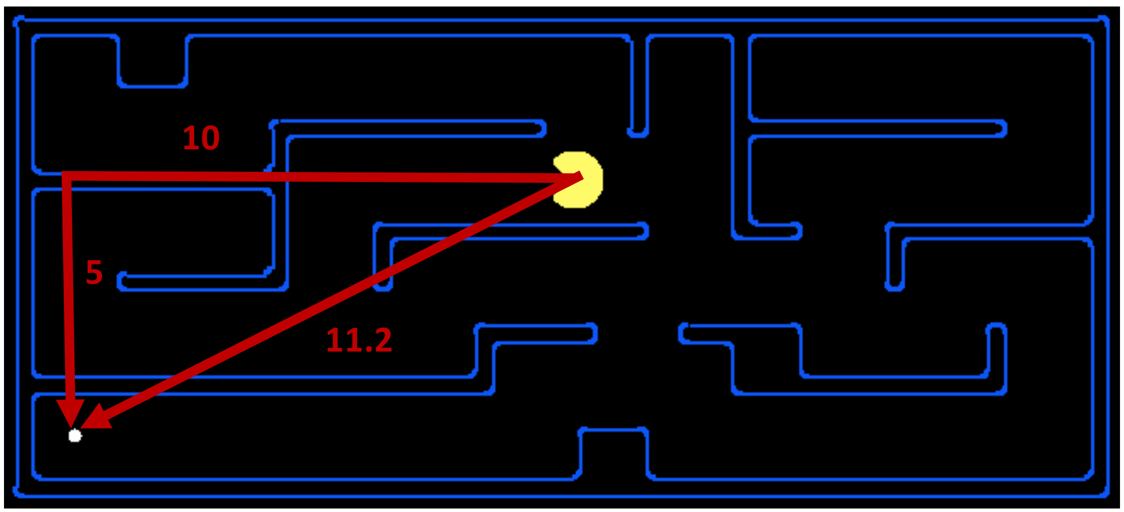
Informed Search

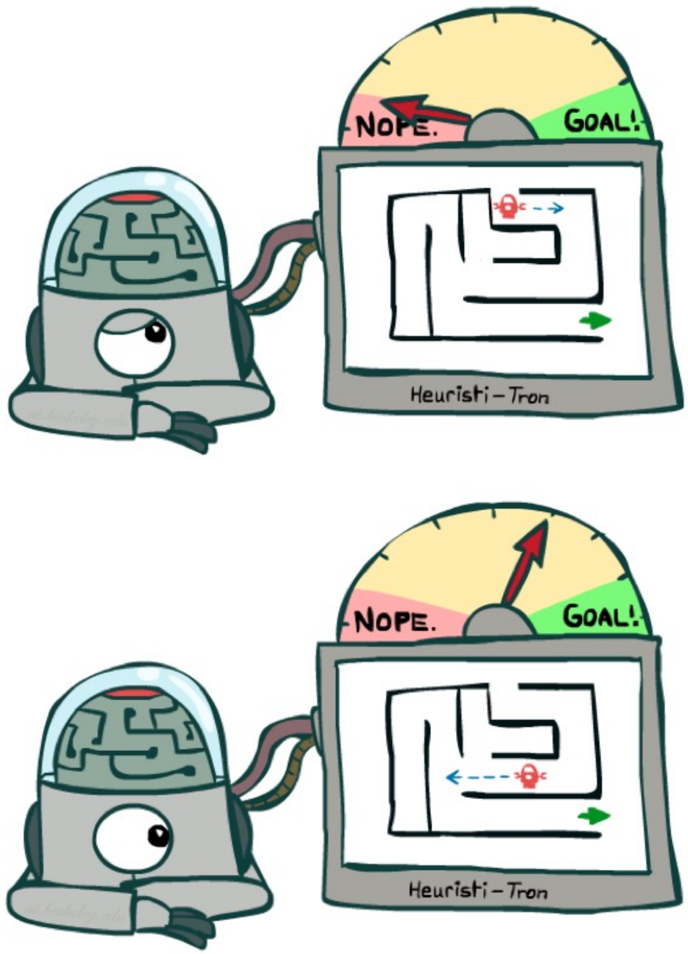
- Heuristics
- Greedy Search
- A* Search

Search Heuristics

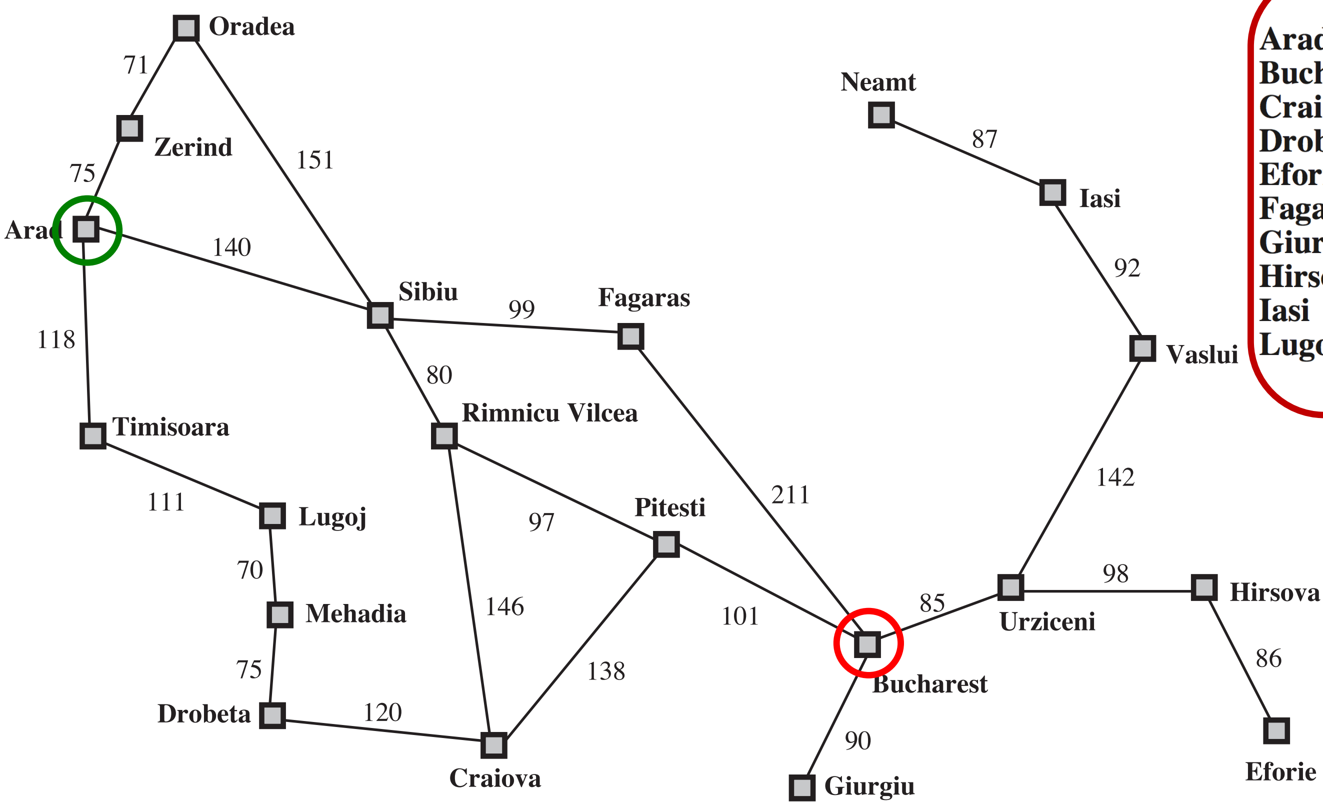
A heuristic is:

- A function that *estimates* how close a state is to a goal
- Designed for a particular search problem
- Examples: Manhattan distance, Euclidean distance for pathing





Example: Euclidean distance to Bucharest

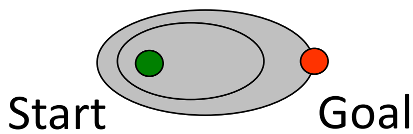


Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

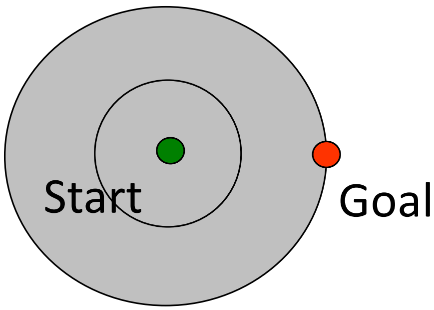
h(state) → value

Effect of heuristics

Guide search ***towards the goal*** instead of ***all over the place***



Informed



Uninformed

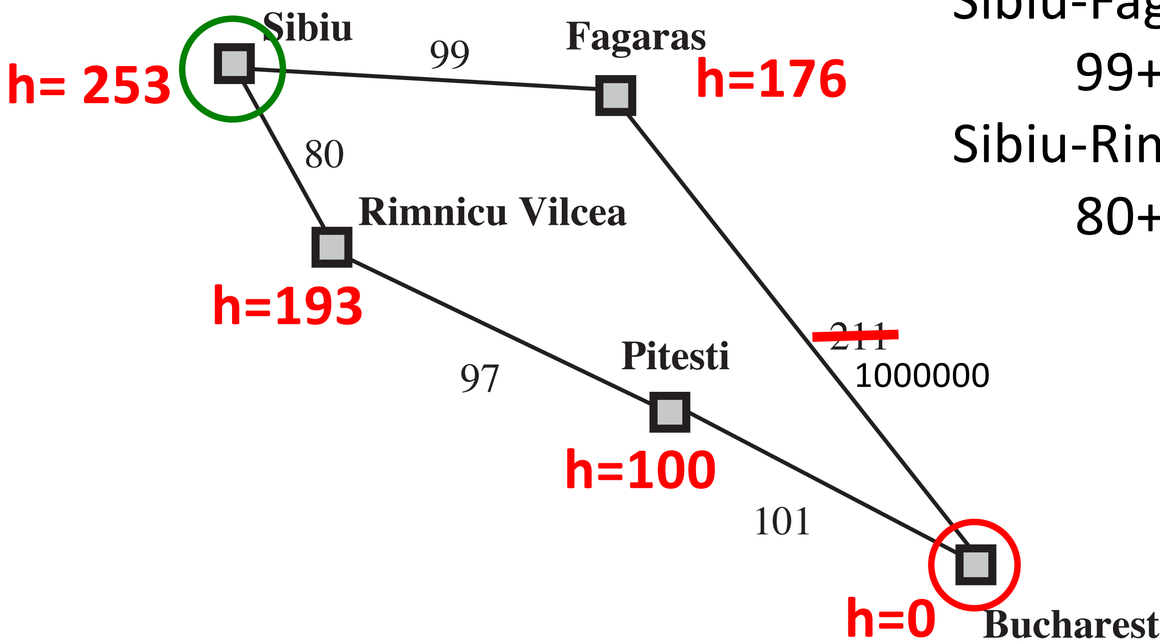
Greedy Search

Greedy Search

(An aside: greedy search is not a greedy algorithm.
The latter, viewed through the lense of search algorithms, is just one branch of a tree.)

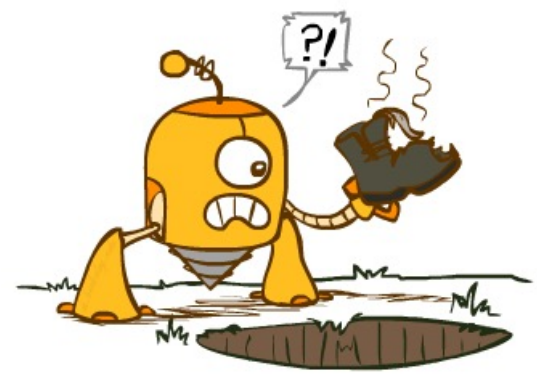
Expand the node that seems closest…(order frontier by h)

What can possibly go wrong?



Sibiu-Fagaras-Bucharest =
99+211 = **310**

Sibiu-Rimnicu Vilcea-Pitesti-Bucharest =
80+97+101 = **278**

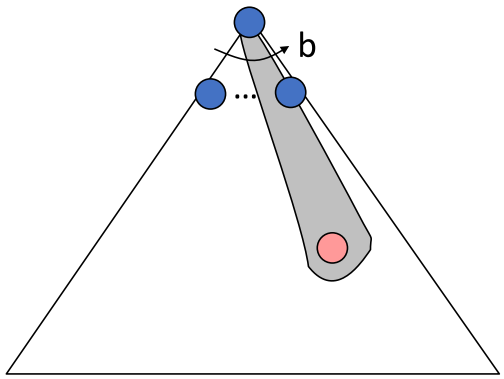


Greedy Search

Strategy: expand a node that ***seems*** closest to a goal state, according to h

Problem 1: it chooses a node even if it's at the end of a very long and winding road

Problem 2: it takes h literally even if it's completely wrong



A* Search

A* Search

UCS

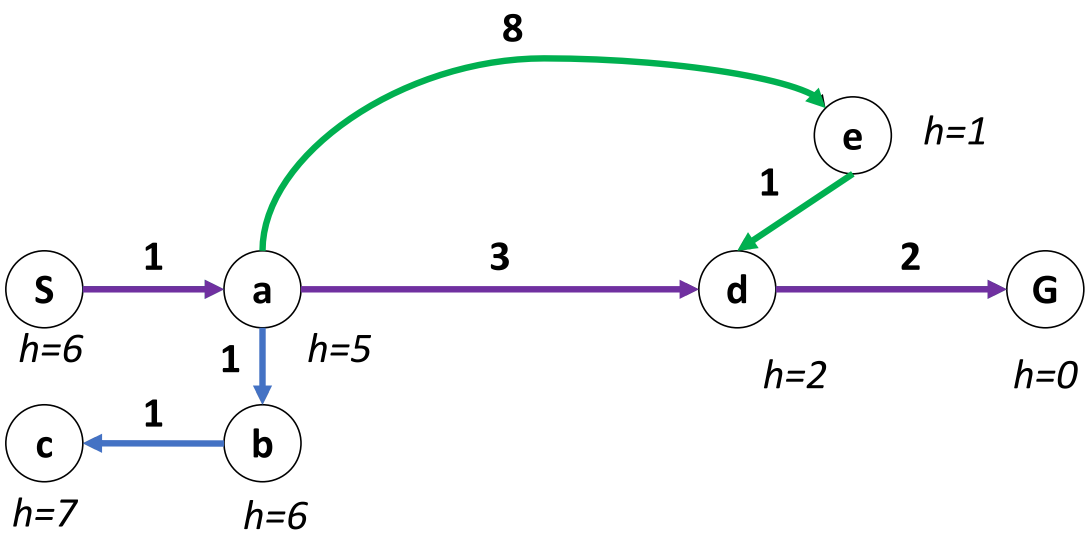
Greedy

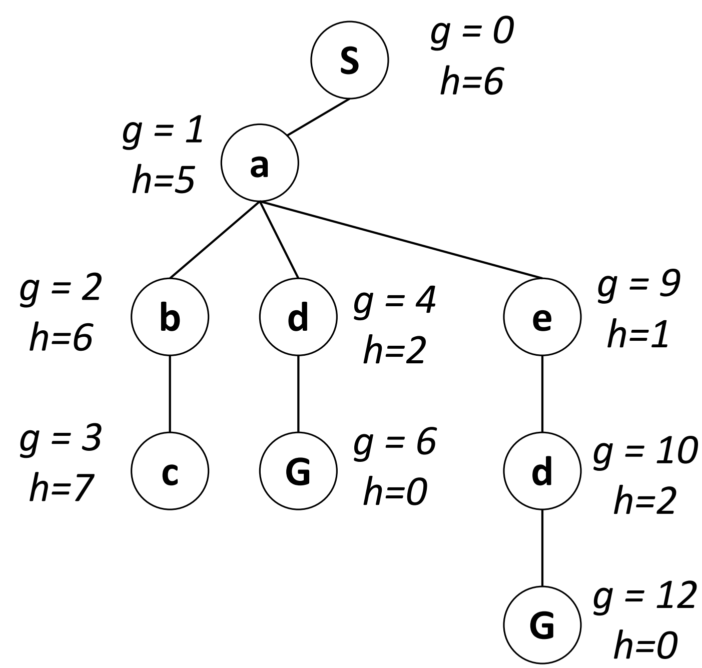
A*

Combining UCS and Greedy

Uniform-cost orders by path cost, or *backward cost* $g(n)$

Greedy orders by goal proximity, or *forward cost* $h(n)$





A* Search orders by the sum: $f(n) = g(n) + h(n)$

Example: Teg Grenager

```
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using g(n) as the priority
    add initial state of problem to frontier with priority g(S) = 0
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher g(n) then
                replace that frontier node with child
```

```
function A-STAR-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using f(n) = g(n) + h(n) as the priority
    add initial state of problem to frontier with priority f(S) = 0 + h(S)
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher f(n) then
                replace that frontier node with child
```

A* Search Algorithms

A* Tree Search

- Same tree search algorithm but with a frontier that is a priority queue using priority $f(n) = g(n) + h(n)$

A* Search Algorithms

A* Tree Search

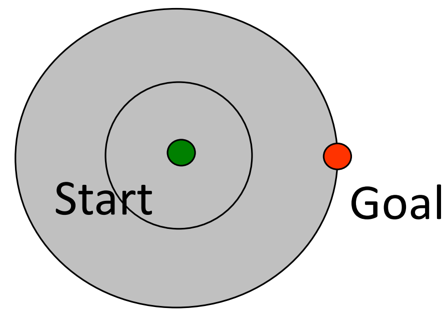
- Same tree search algorithm but with a frontier that is a priority queue using priority $f(n) = g(n) + h(n)$

A* Graph Search

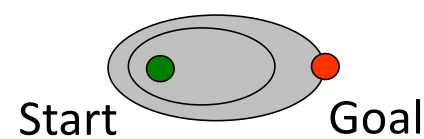
- Same as **UCS** graph search algorithm but with a frontier that is a priority queue using priority $f(n) = g(n) + h(n)$

UCS vs A* Contours

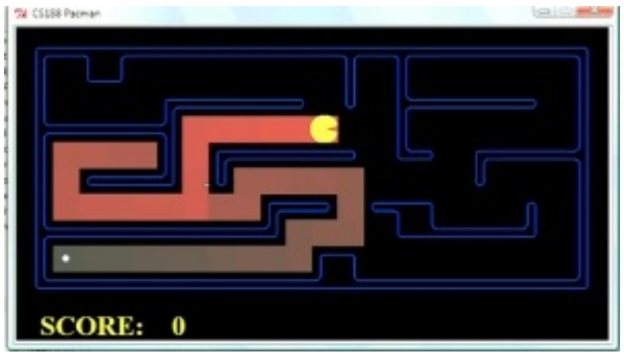
Uniform-cost expands equally in all "directions"



A* expands mainly toward the goal, but does hedge its bets to ensure optimality



Comparison

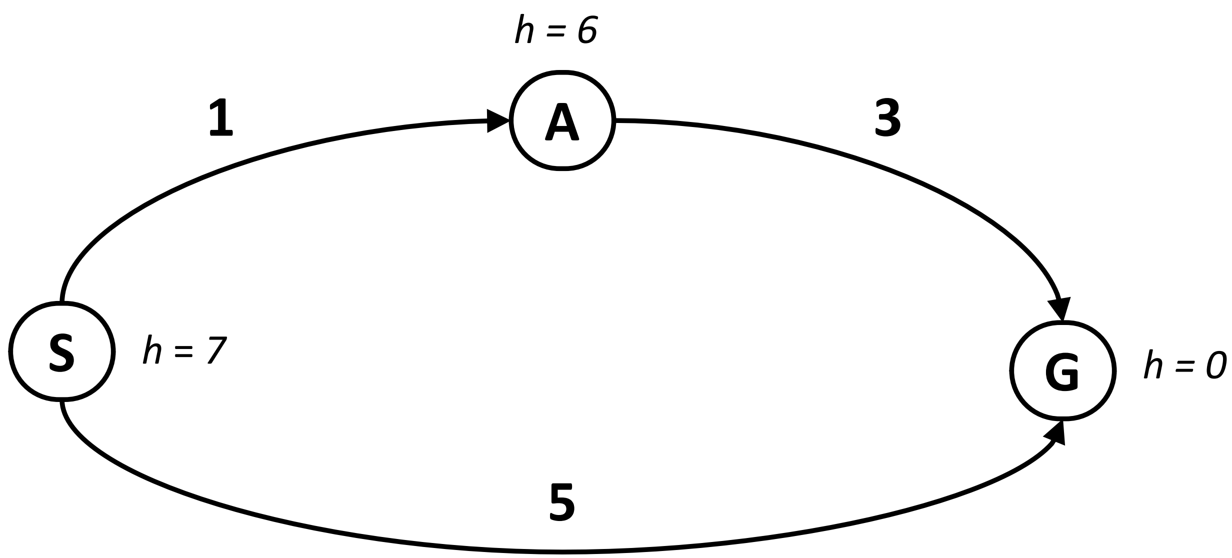


Greedy

Uniform Cost

A*

Is A* Optimal?

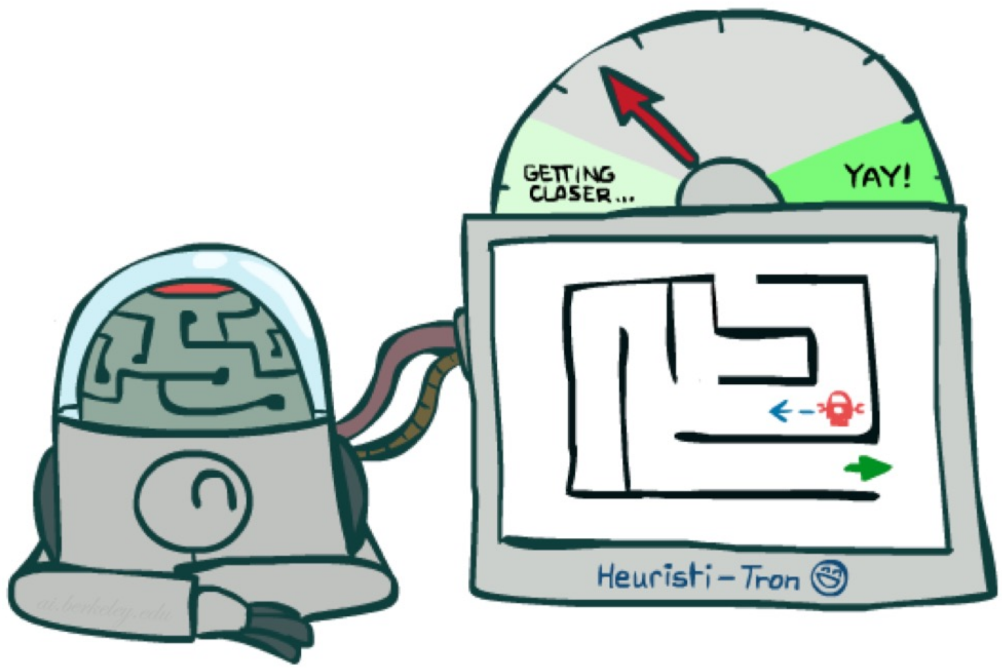


What went wrong?

Actual bad goal cost < ***estimated*** good goal cost

We need estimates to be less than actual costs!

Admissible Heuristics



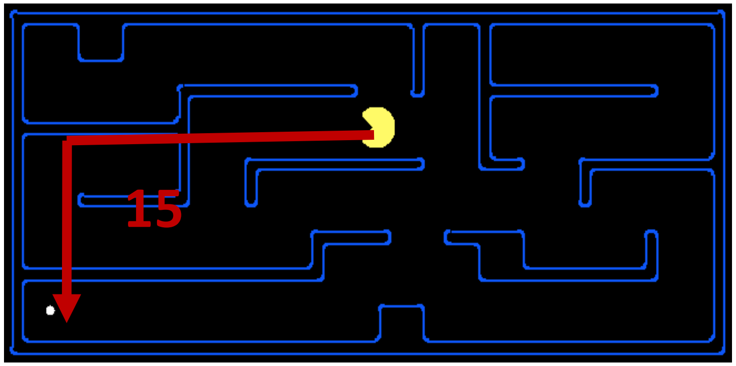
Admissible Heuristics

A heuristic h is *admissible* (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

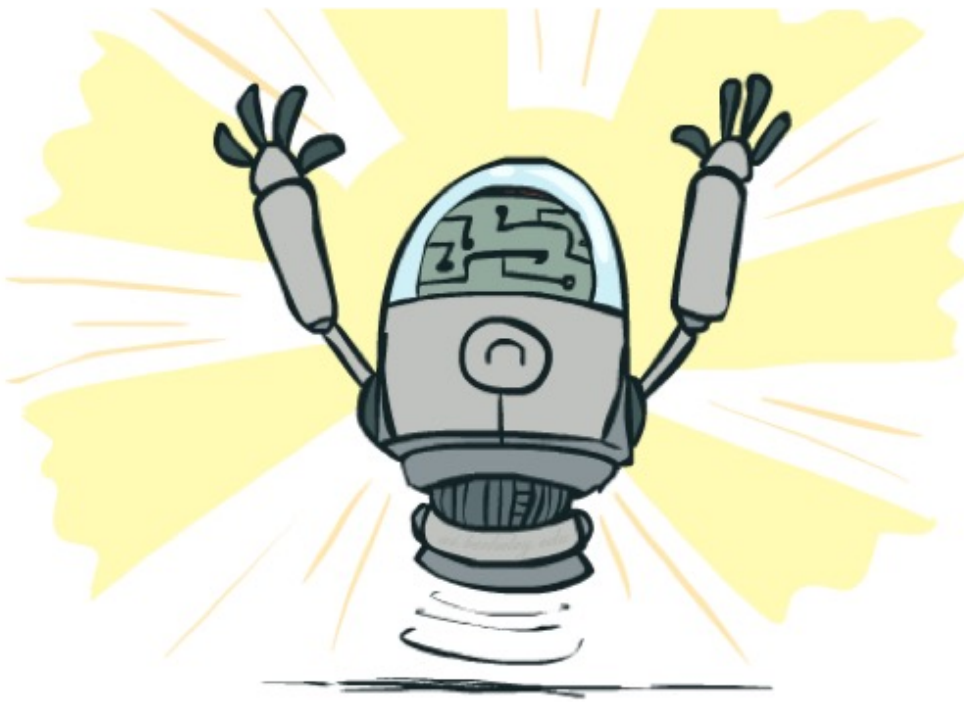
where $h^*(n)$ is the true cost to a nearest goal

Example:



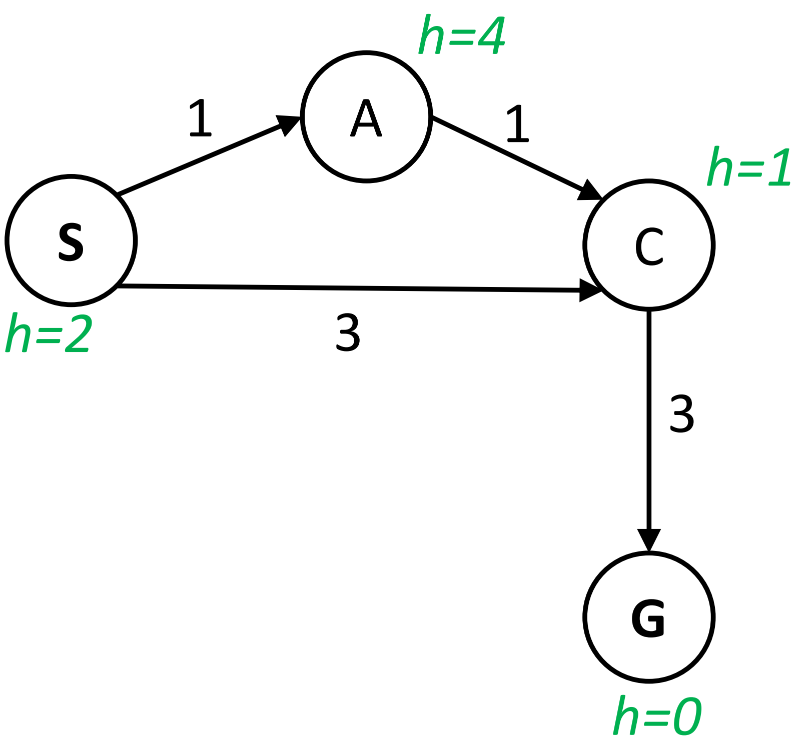
Coming up with admissible heuristics is most of what's involved in using A* in practice.

Optimality of A* Tree Search

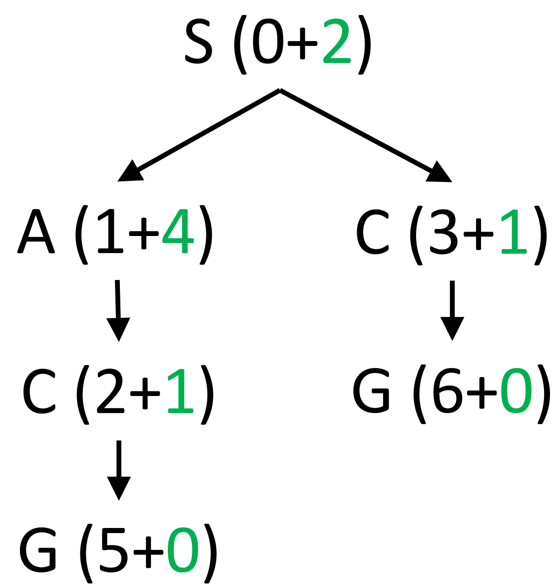


A* Tree Search

State space graph



Search tree



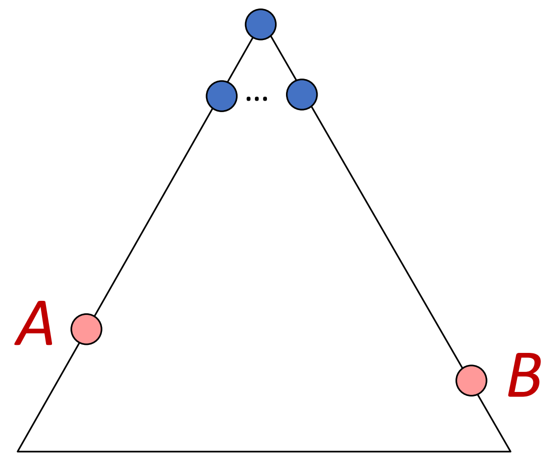
Optimality of A* Tree Search

Assume:

A is an optimal goal node

B is a suboptimal goal node

h is admissible



Claim:

A will be chosen for exploration (popped off the frontier) before *B*

Optimality of A* Tree Search: Blocking

$f(x) = g(x) + h(x)$
$h(x) \leq h^*(x)$

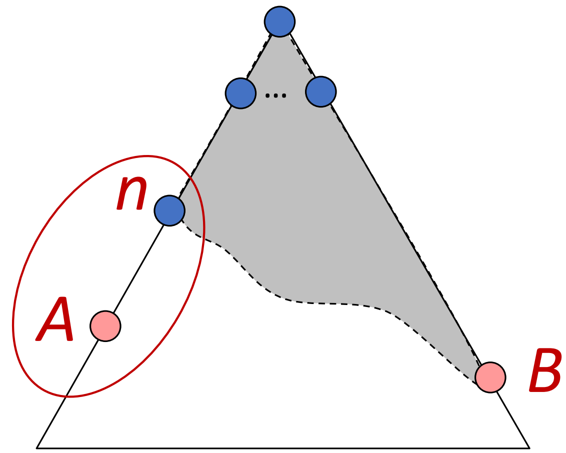
Proof:

Imagine *B* is on the frontier

Some ancestor *n* of *A* is on the frontier, too
(Maybe the start state; maybe *A* itself!)

Claim: *n* will be explored before *B*

1. $f(n)$ is less than or equal to $f(A)$



$f(n) = g(n) + h(n)$	Definition of *f*-cost
$f(n) \leq g(A)$	Admissibility of *h*
$g(A) = f(A)$	$h = 0$ at a goal

Optimality of A* Tree Search: Blocking

$f(x) = g(x) + h(x)$
$h(x) \leq h^*(x)$

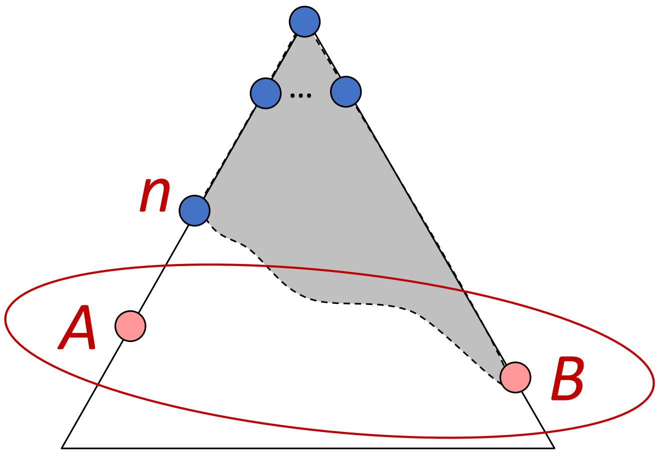
Proof:

Imagine *B* is on the frontier

Some ancestor *n* of *A* is on the frontier, too
(Maybe the start state; maybe *A* itself!)

Claim: *n* will be explored before *B*

1. $f(n)$ is less than or equal to $f(A)$
2. $f(A)$ is less than $f(B)$



$g(A) < g(B)$ — Suboptimality of *B*

$f(A) < f(B)$ — $h = 0$ at a goal

Optimality of A* Tree Search: Blocking

$f(x) = g(x) + h(x)$
$h(x) \leq h^*(x)$

Proof:

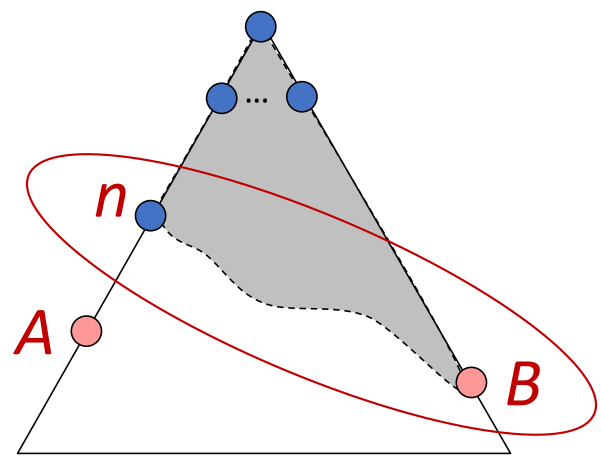
Imagine *B* is on the frontier

Some ancestor *n* of *A* is on the frontier, too
(Maybe the start state; maybe *A* itself!)

Claim: *n* will be explored before *B*

1. $f(n)$ is less than or equal to $f(A)$
2. $f(A)$ is less than $f(B)$
3. *n* is explored before *B*

$f(n) \leq f(A) < f(B)$

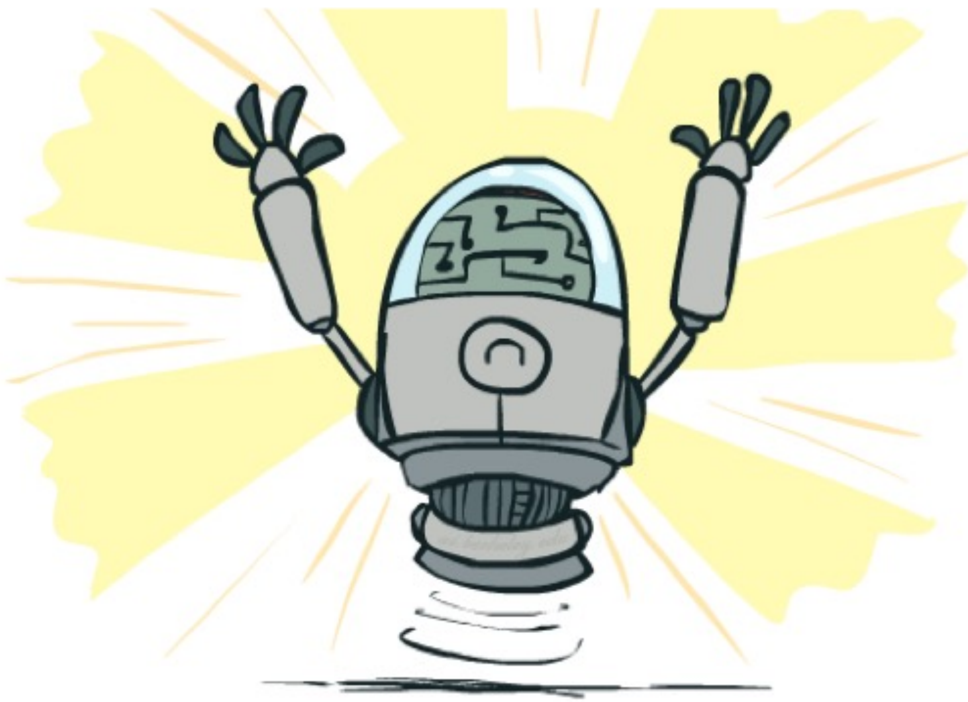


All ancestors of *A* are explored before *B*

A is explored before *B*

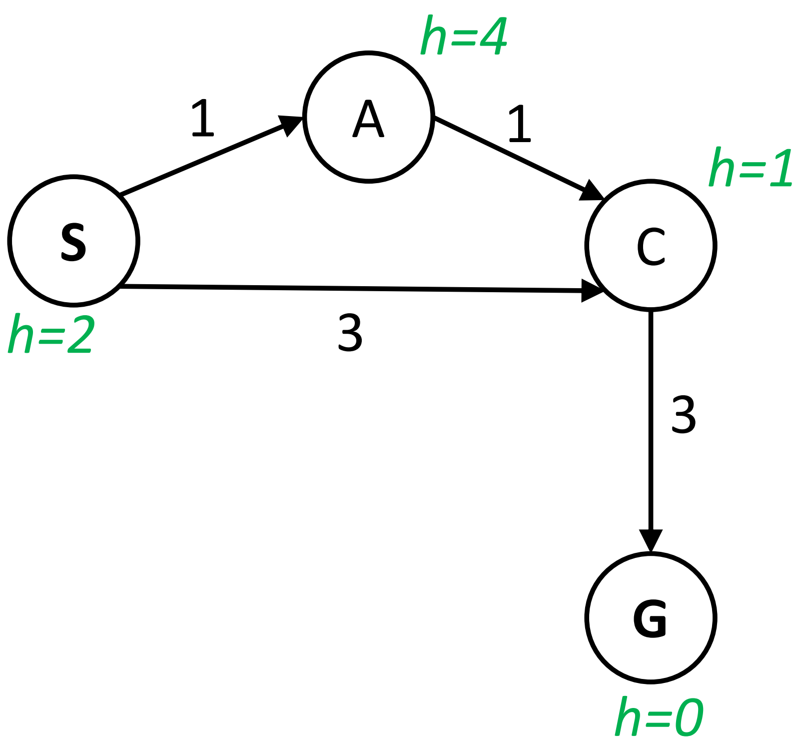
A* search is optimal

Optimality of A* Graph Search



Poll 1: A* Graph Search

What nodes does A* graph search consider during its search?



A) ~~S~~, ~~S-A~~, ~~S-C~~, S-C-G

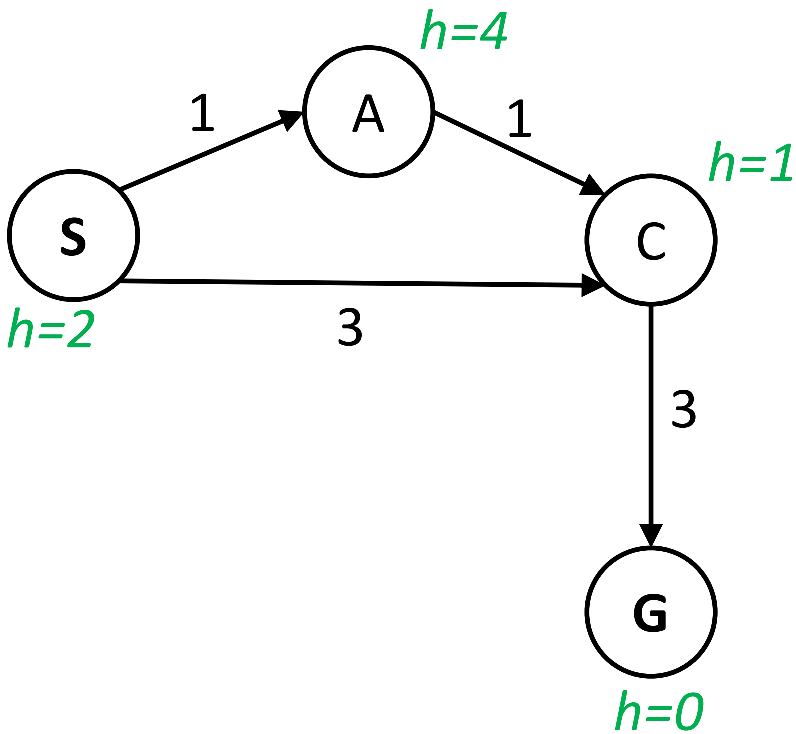
B) ~~S~~, ~~S-A~~, S-C, ~~S-A-C~~, S-C-G

C) ~~S~~, ~~S-A~~, ~~S-A-C~~, S-A-C-G

D) ~~S~~, ~~S-A~~, ~~S-C~~, ~~S-A-C~~, S-A-C-G

Poll 1: A* Graph Search

Which paths does A* graph search consider during its search?



A) S, S-A, S-C, S-C-G

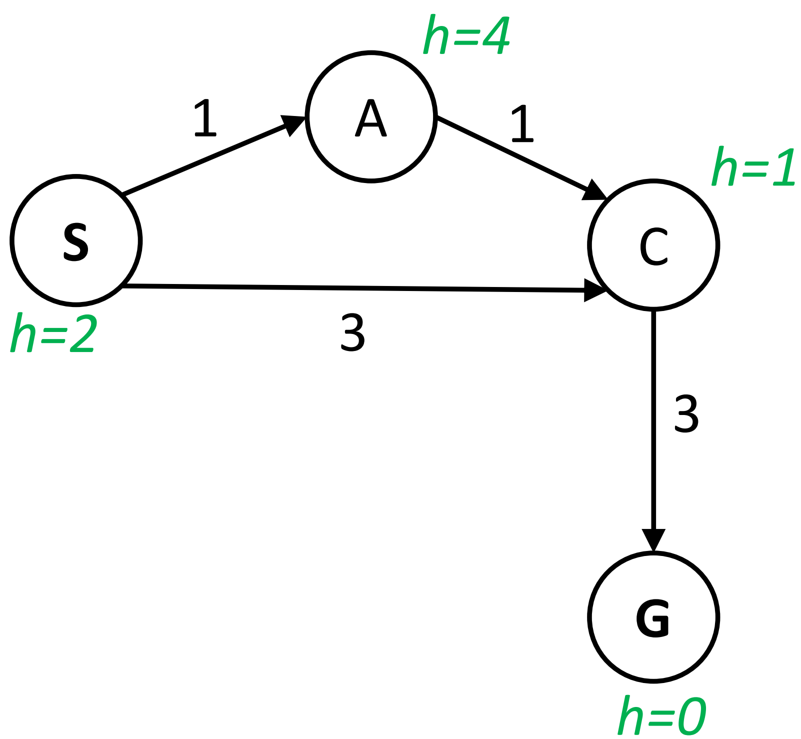
B) S, S-A, S-C, S-A-C, S-C-G

C) S, S-A, S-A-C, S-A-C-G

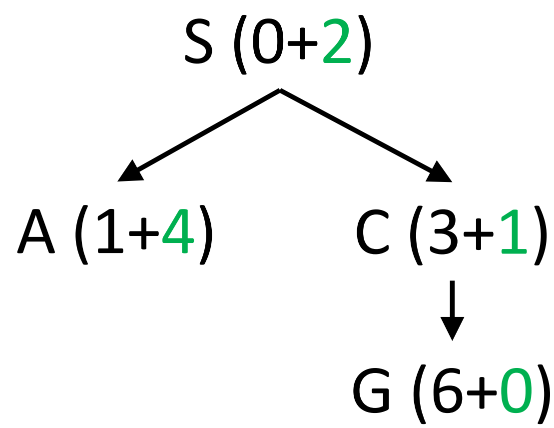
D) S, S-A, S-C, S-A-C, S-A-C-G

A* Graph Search Gone Wrong?

State space graph



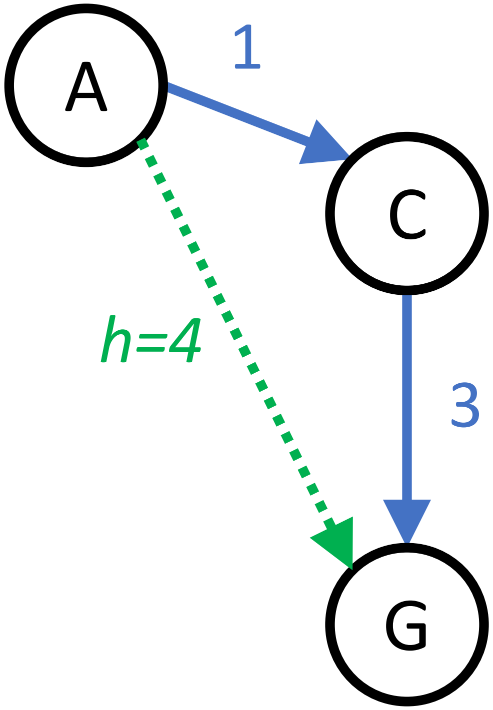
Search tree



Simple check against explored set blocks C

Fancy check allows new C if cheaper than old but requires recalculating C's descendants

Admissibility of Heuristics



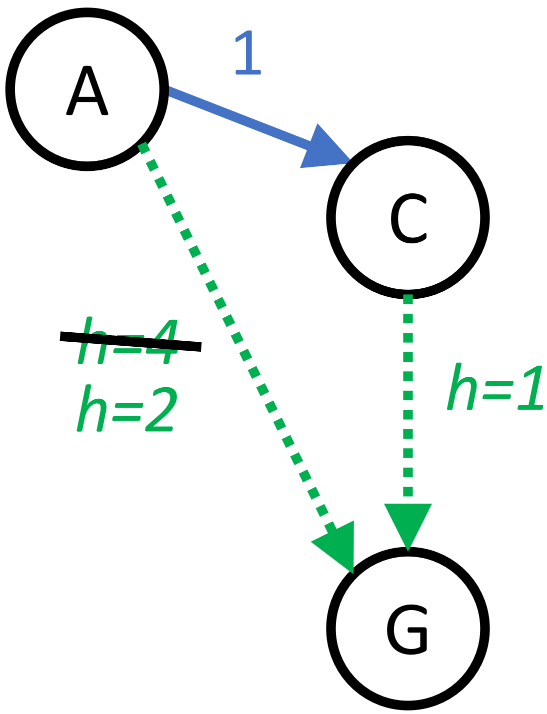
Main idea: Estimated heuristic values ≤ actual costs

- Admissibility:

  heuristic value ≤ actual cost to goal

  h(A) ≤ actual cost from A to G

Consistency of Heuristics



Main idea: Estimated heuristic costs ≤ actual costs

- Admissibility:

  heuristic cost ≤ actual cost to goal

  h(A) ≤ actual cost from A to G

- Consistency:

  "heuristic step cost" ≤ actual cost for each step

  h(A) – h(C) ≤ cost(A to C)

  triangle inequality

  h(A) ≤ cost(A to C) + h(C)

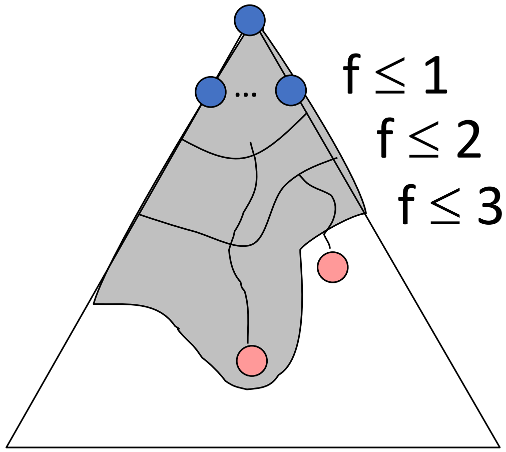
Consequences of consistency:

- The f value along a path never decreases
- A* graph search is optimal

Optimality of A* Graph Search

Sketch: consider what A* does with a consistent heuristic:

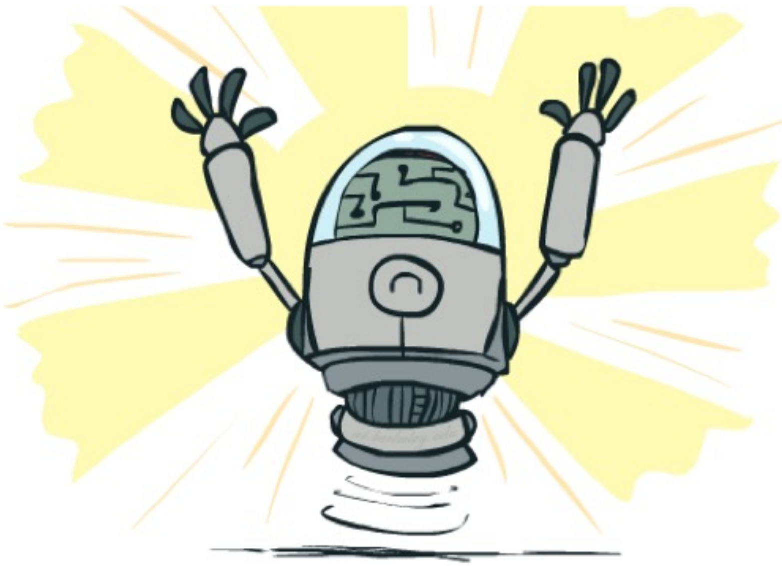
- Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)
- Fact 2: For every state s, nodes that reach s optimally are explored before nodes that reach s suboptimally
- Result: A* graph search is optimal



Optimality

Tree search:
- A* is optimal if heuristic is admissible
- UCS is a special case (h = 0)

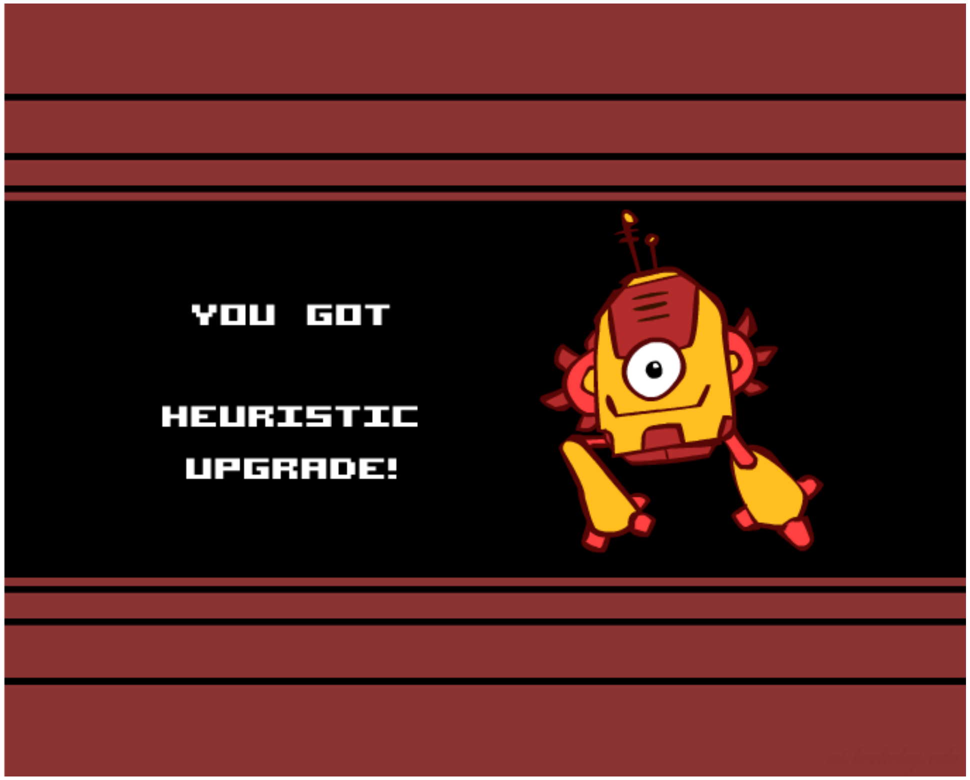
Graph search:
- A* optimal if heuristic is consistent
- UCS optimal (h = 0 is consistent)



Consistency implies admissibility

In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems

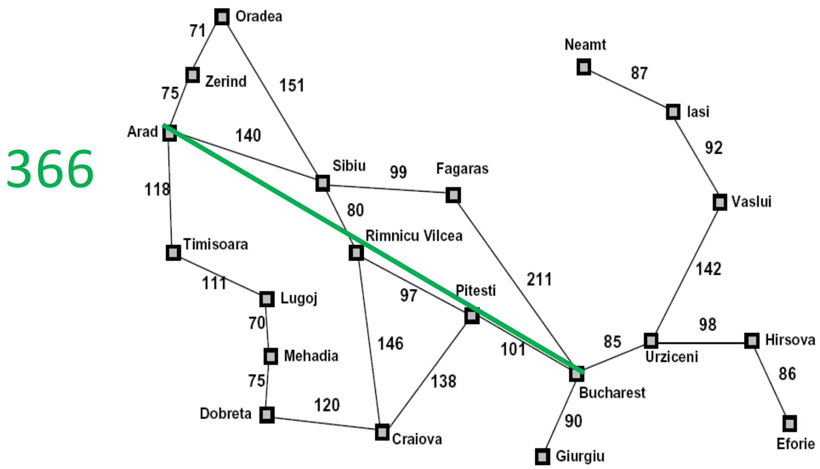
Creating Heuristics

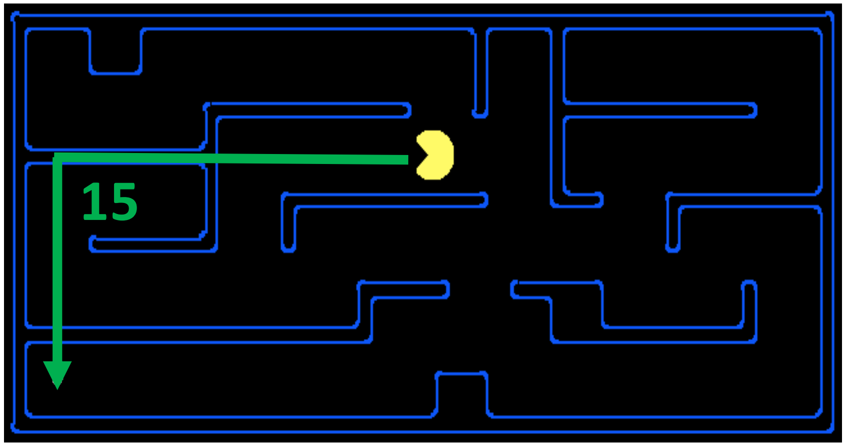


Creating Admissible Heuristics

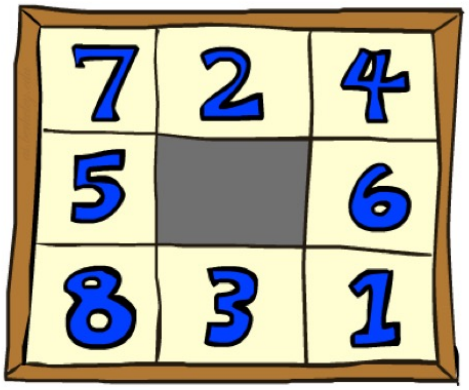
Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

Often, admissible heuristics are solutions to ***relaxed problems***, where new actions are available

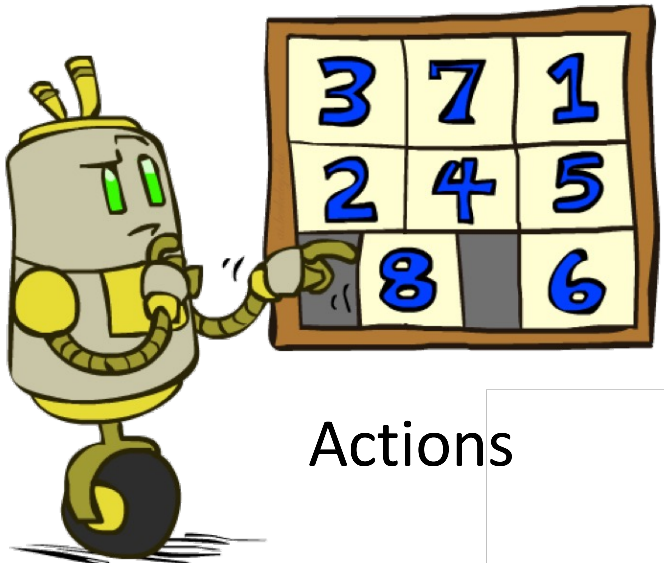




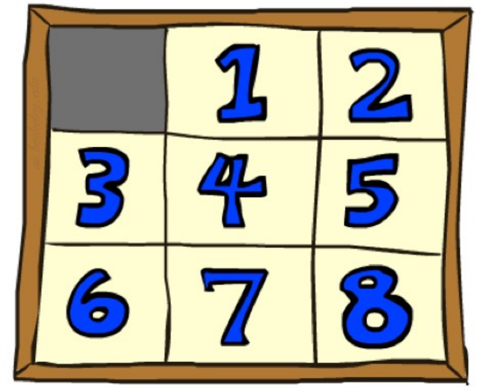
Example: 8 Puzzle



Start State



Actions



Goal State

What are the states?

How many states?

What are the actions?

How many actions from the start state?

What should the step costs be?

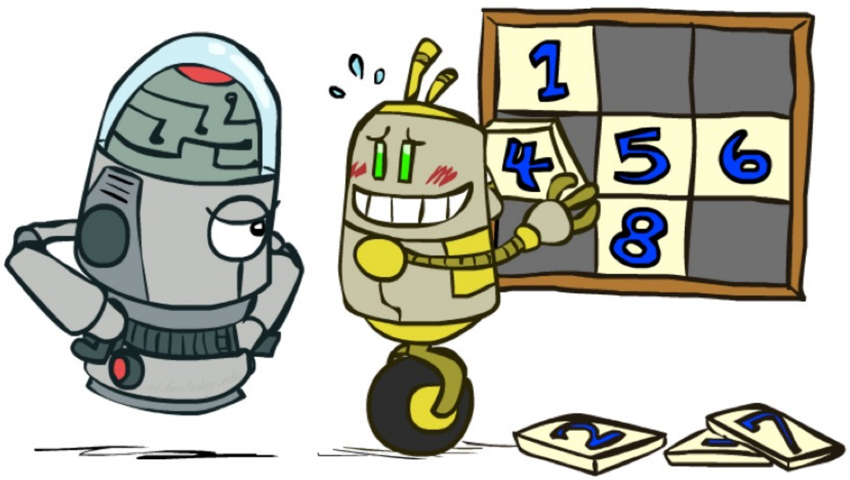
8 Puzzle I

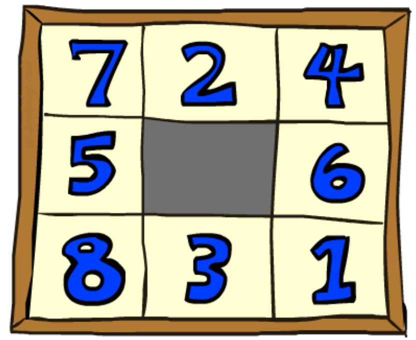
Heuristic: Number of tiles misplaced

Why is it admissible?

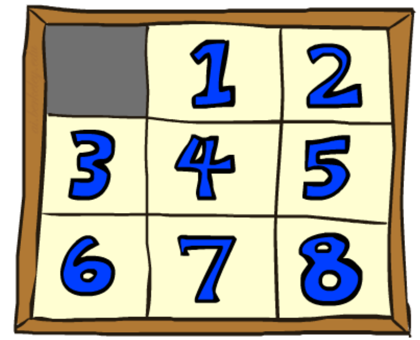
h(start) = 8

This is a *relaxed-problem* heuristic





Start State



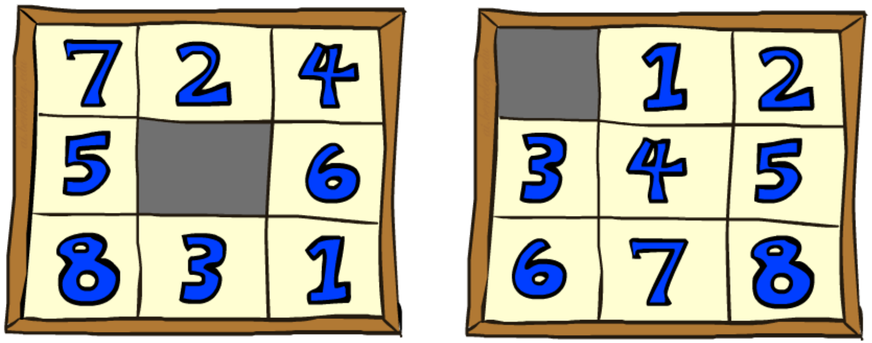
Goal State

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
UCS	112	6,300	3.6×10^6
A*TILES	13	39	227

Statistics from Andrew Moore

8 Puzzle II

What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?



Start State Goal State

Total *Manhattan* distance

Why is it admissible?

h(start) = 3 + 1 + 2 + … = 18

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
A*TILES	13	39	227
A*MANHATTAN	12	25	73

Combining heuristics

Dominance: $h_a \geq h_c$ if

$$\forall n \;\; h_a(n) \geq h_c(n)$$

- Roughly speaking, larger is better as long as both are admissible
- The zero heuristic is pretty bad (what does A* do with h=0?)
- The exact heuristic is pretty good, but usually too expensive!

What if we have two heuristics, neither dominates the other?

- Form a new heuristic by taking the max of both:

$$h(n) = \max(\, h_a(n), h_b(n) \,)$$

- Max of admissible heuristics is admissible and dominates both!

A*: Summary

A*: Summary

A* uses both cost so far ("backward cost") and (estimates of) cost to go ("forward cost")

A* is optimal with admissible / consistent heuristics

Heuristic design is key: often use relaxed problems