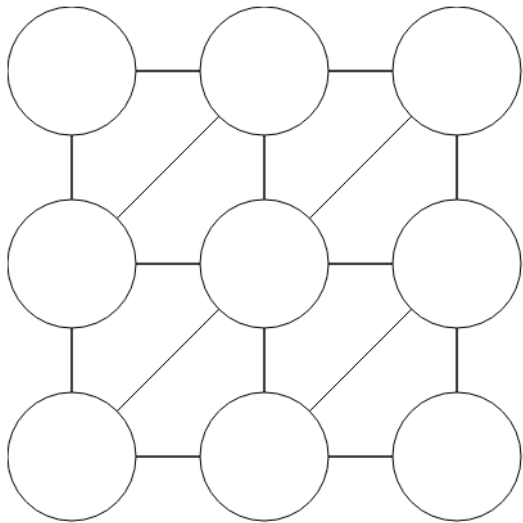


Warm-up as You Walk In

Assign Red, Green, or Blue
Neighbors must be different



Sudoku

1			
	2	1	
		3	
			4

- 1) What is your brain doing to solve these?
- 2) How would you solve these with search (BFS, DFS, etc.)?

Announcements

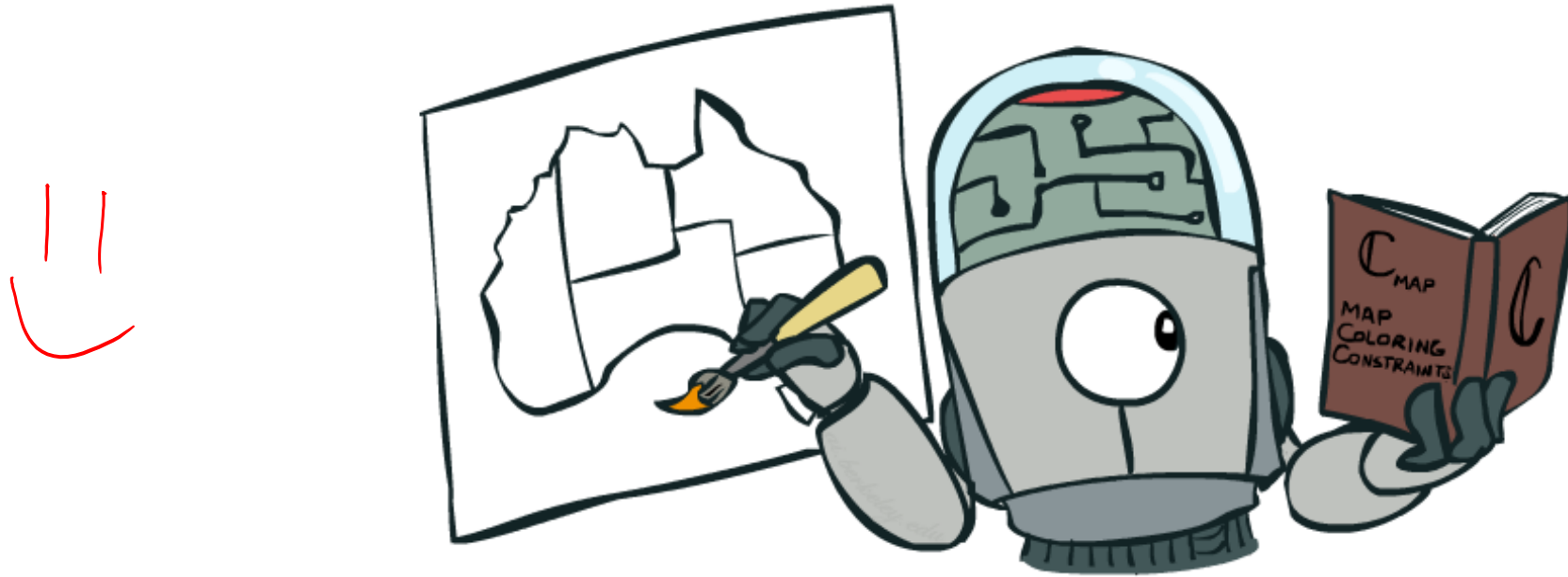
Instructors for Lecture

Assignments

- P1: Search and Games
 - Due Thu 2/6, 10 pm
 - Recommended to work in pairs
 - Submit to Gradescope early and often
- HW3 (online) – Adversarial Search and CSPs
 - Due Tue 2/4, 10 pm

AI: Representation and Problem Solving

Constraint Satisfaction Problems (CSPs)



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, <http://ai.berkeley.edu>

Learning Objectives

- Describe definition of **CSP problems** and its connection with general search problems
- Formulate a real-world problem as a CSP
- Describe and implement **backtracking algorithm**
- Define **arc consistency**
- Describe and implement **forward checking** and **AC-3**
- Explain the differences between **MRV** and **LCV heuristics**
- Understand the complexity of general binary CSP and **tree-structured binary CSP**

What is Search For?

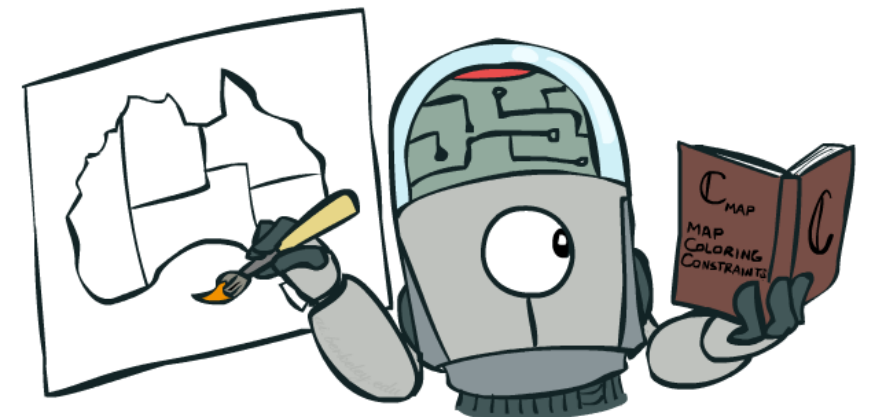
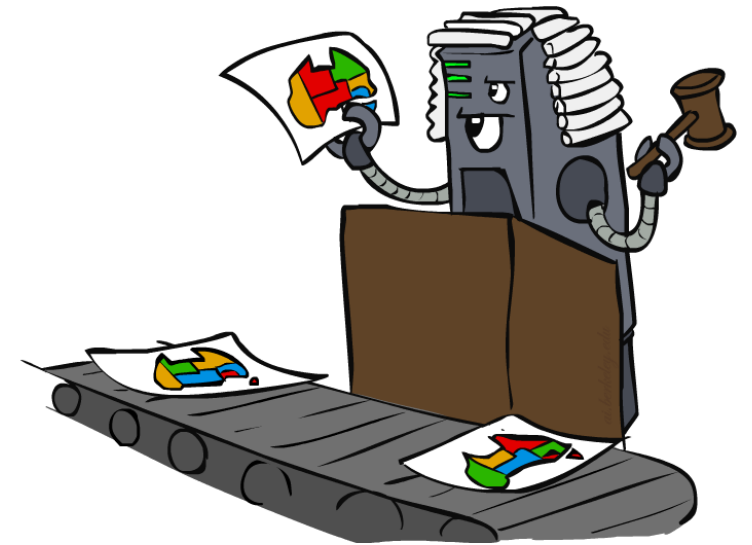
- Planning: sequences of actions
 - The path to the goal is the important thing
 - Paths have various costs, depths
 - Heuristics give problem-specific guidance
- Identification: assignments to variables
 - The goal itself is important, not the path
 - All paths at the same depth (for some formulations)

Are the warm-up assignments
planning or identification problems?



Constraint Satisfaction Problems

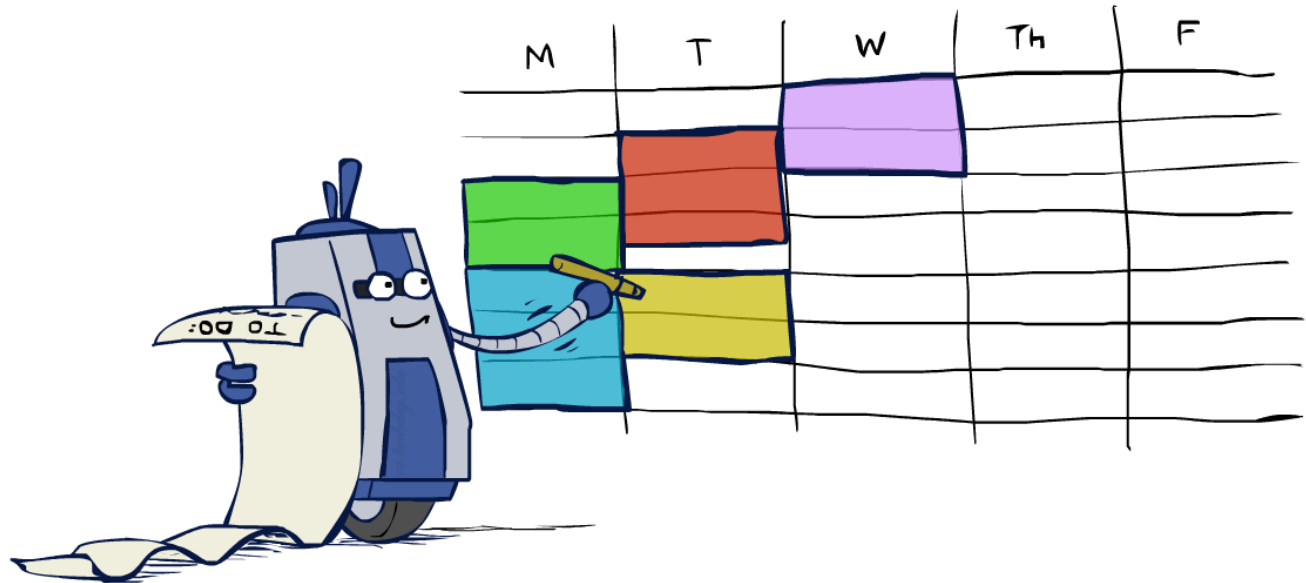
- CSP is a special class of search problems
 - Mostly identification problems
 - Have specialized algorithms for them
- Standard search problems:
 - State is a “black box”: arbitrary data structure
 - Goal test can be any function over states
 - Successor function can also be anything
- Constraint satisfaction problems (CSPs):
 - State is defined by **variables X_i** with values from a **domain D** (sometimes D depends on i)
 - Goal test is a **set of constraints** specifying allowable combinations of values for subsets of variables



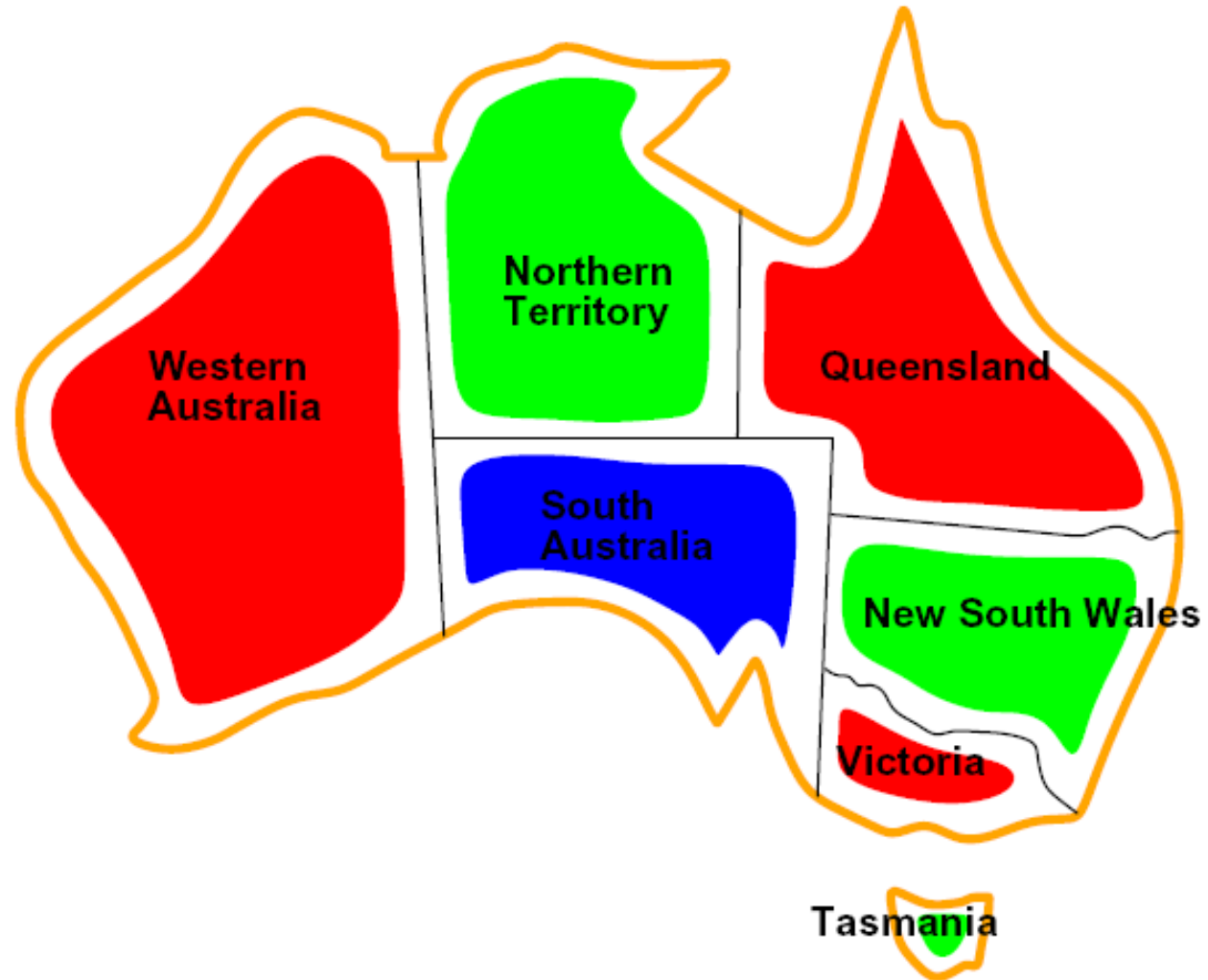
Why study CSPs?

Many real-world problems can be formulated as CSPs

- Assignment problems: e.g., who teaches what class
 - Timetabling problems: e.g., which class is offered when and where?
 - Hardware configuration
 - Transportation scheduling
 - Factory scheduling
 - Circuit layout
 - Fault diagnosis
 - ... lots more!
-
- Sometimes involve real-valued variables...



CSP Examples



Example: Map Coloring

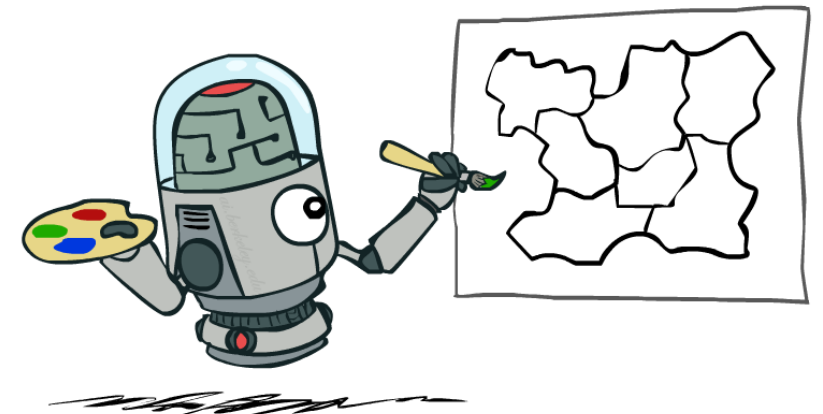
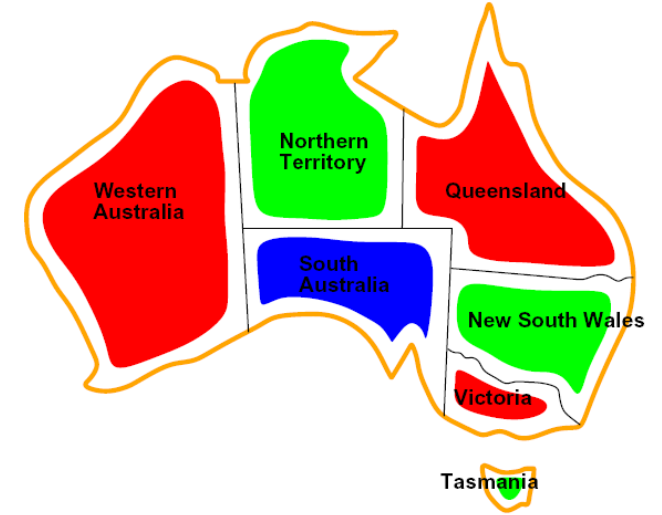
- Variables: WA, NT, Q, NSW, V, SA, T
- Domains: $D = \{\text{red, green, blue}\}$
- Constraints: adjacent regions must have different colors

Implicit: $WA \neq NT$

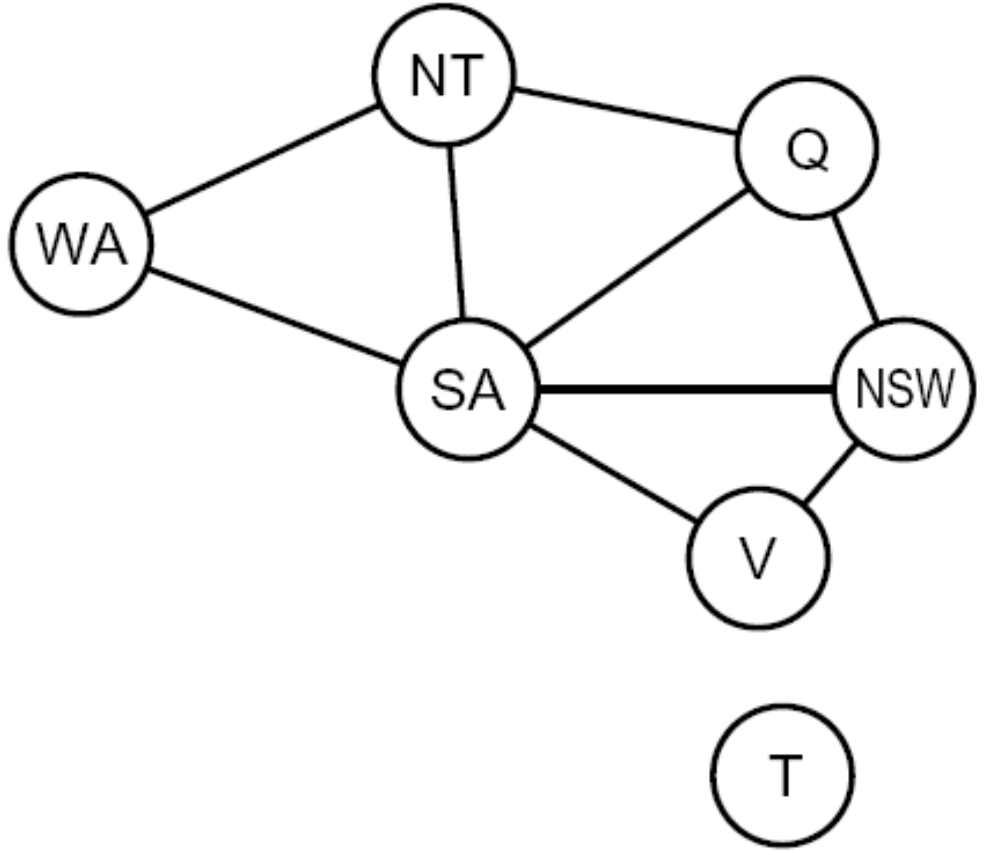
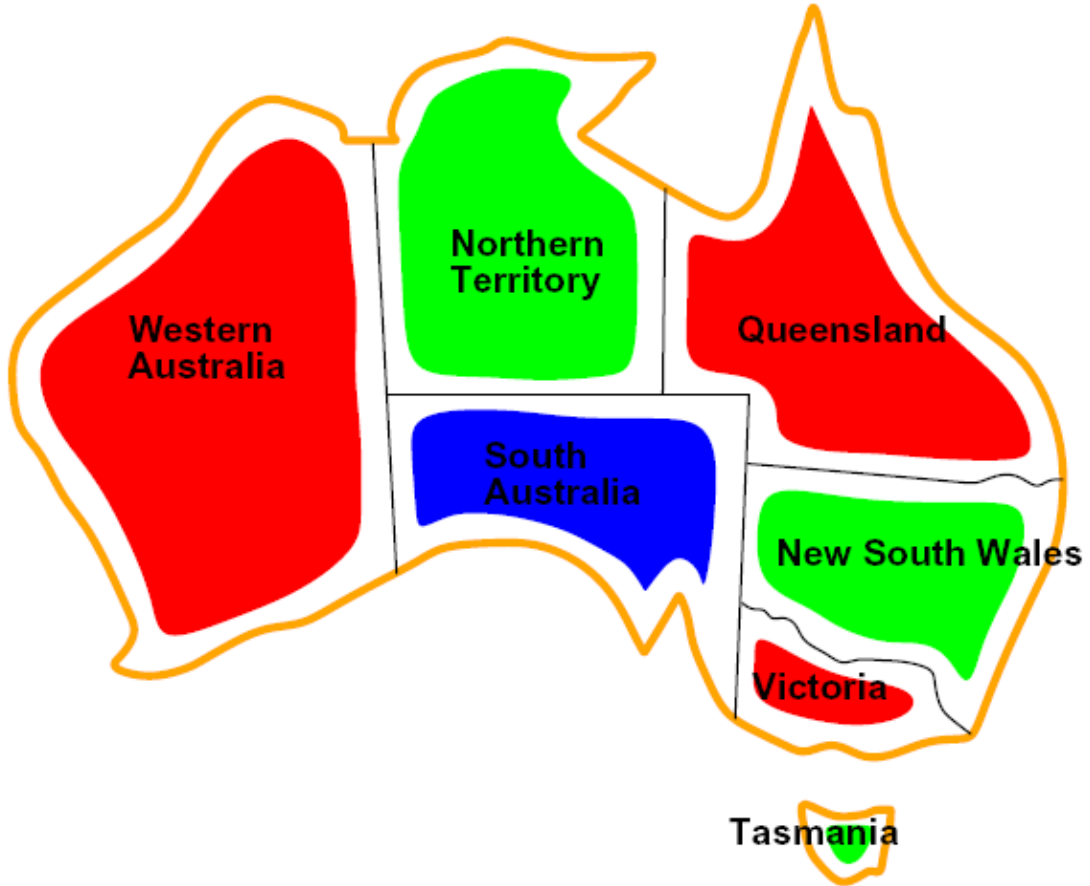
Explicit: $(WA, NT) \in \{(\text{red, green}), (\text{red, blue}), \dots\}$

- Solutions are assignments satisfying all constraints, e.g.:

$\{WA=\text{red}, NT=\text{green}, Q=\text{red}, NSW=\text{green},$
 $V=\text{red}, SA=\text{blue}, T=\text{green}\}$

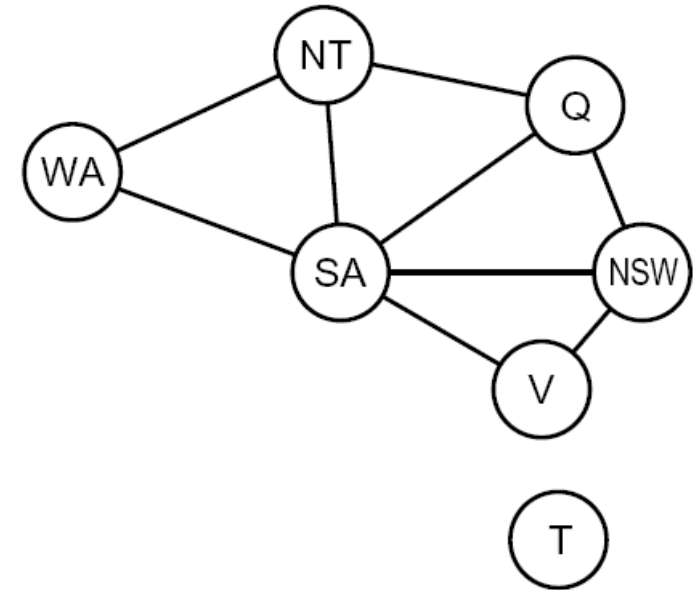


Constraint Graphs

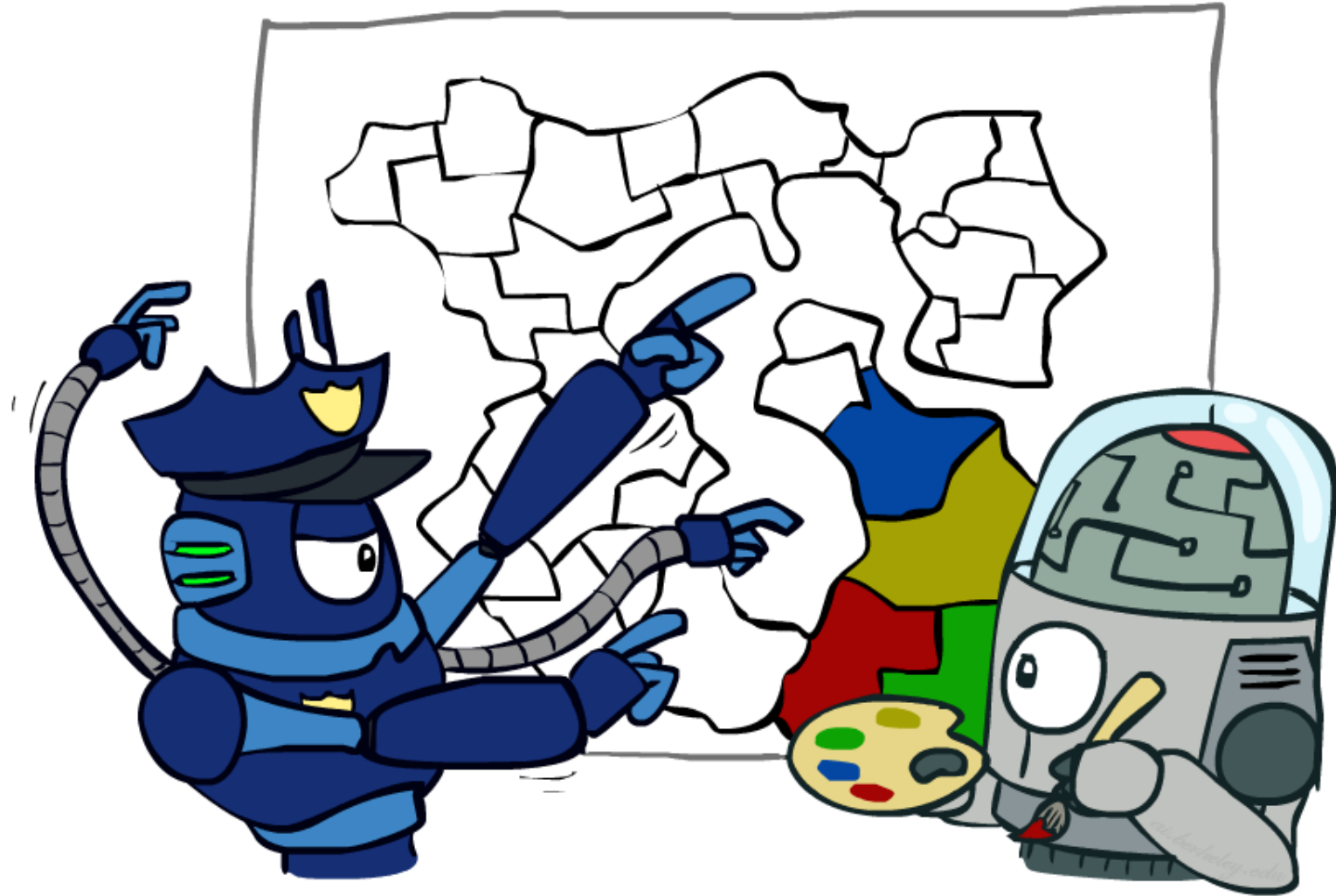


Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



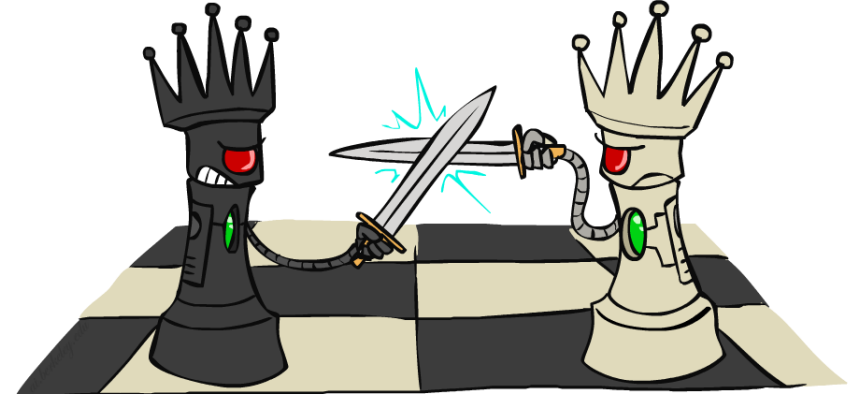
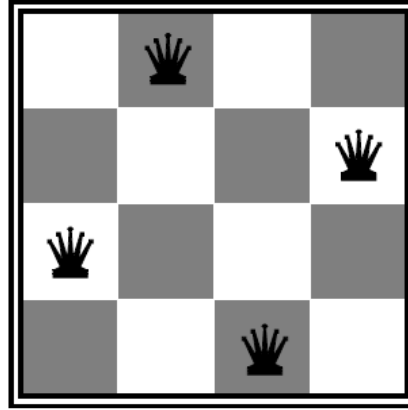
Varieties of CSPs and Constraints



Example: N-Queens

- Formulation 1:

- Variables: X_{ij}
- Domains: $\{0, 1\}$
- Constraints



$$\forall i, j, k \quad (X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$\forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$\forall i, j, k \quad (X_{ij}, X_{i+k, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$\forall i, j, k \quad (X_{ij}, X_{i+k, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:

- Variables: Q_k

- Domains: $\{1, 2, 3, \dots, N\}$

- Constraints:

Implicit: $\forall i, j \text{ non-threatening}(Q_i, Q_j)$

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \dots\}$

...

$Q_1 = 2$

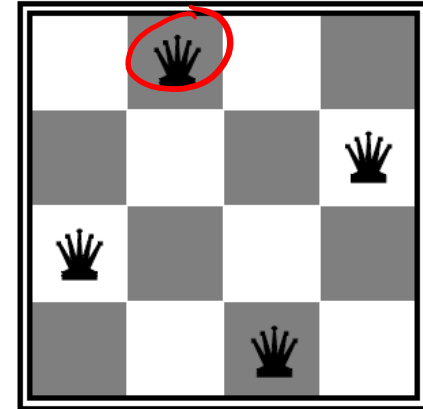


Q_1

Q_2

Q_3

Q_4



Example: Cryptarithmic

- Variables:

$F\ T\ U\ W\ R\ O\ \underline{X_1\ X_2\ X_3}$

- Domains:

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Constraints:

$\text{alldiff}(F, T, U, W, R, O)$

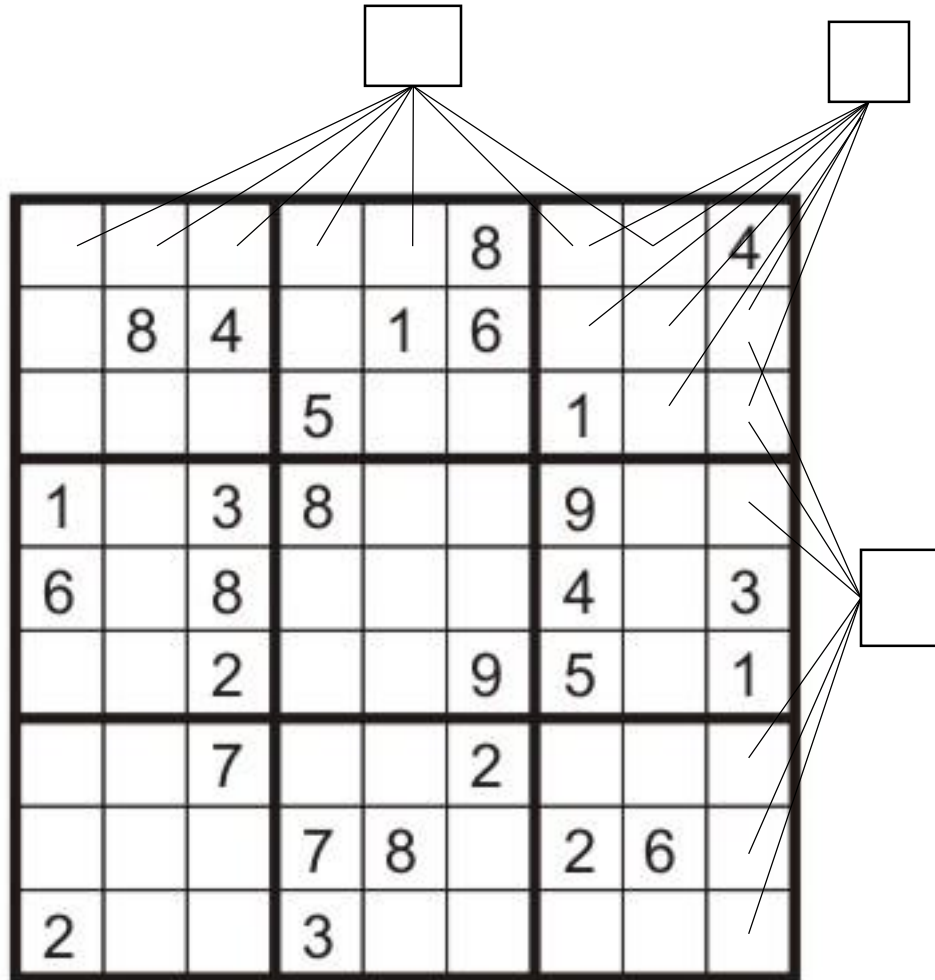
$O + O = R + 10 \cdot X_1$

...

$$\begin{array}{r} X_3 X_2 X_1 \\ T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$



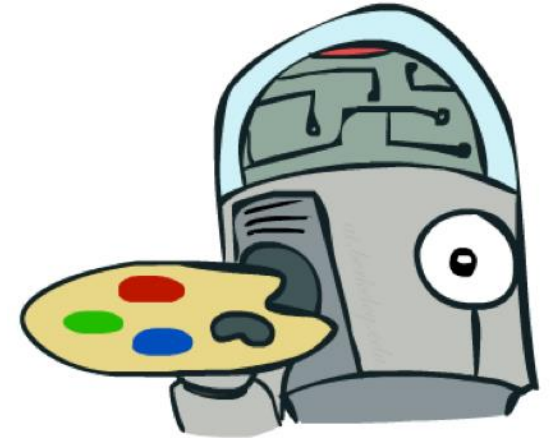
Example: Sudoku



- Variables: Each (open) square
- Domains: $\{1, 2, \dots, 9\}$
- Constraints:
 - 9-way alldiff for each column
 - 9-way alldiff for each row
 - 9-way alldiff for each region
 - (or can have a bunch of pairwise inequality constraints)

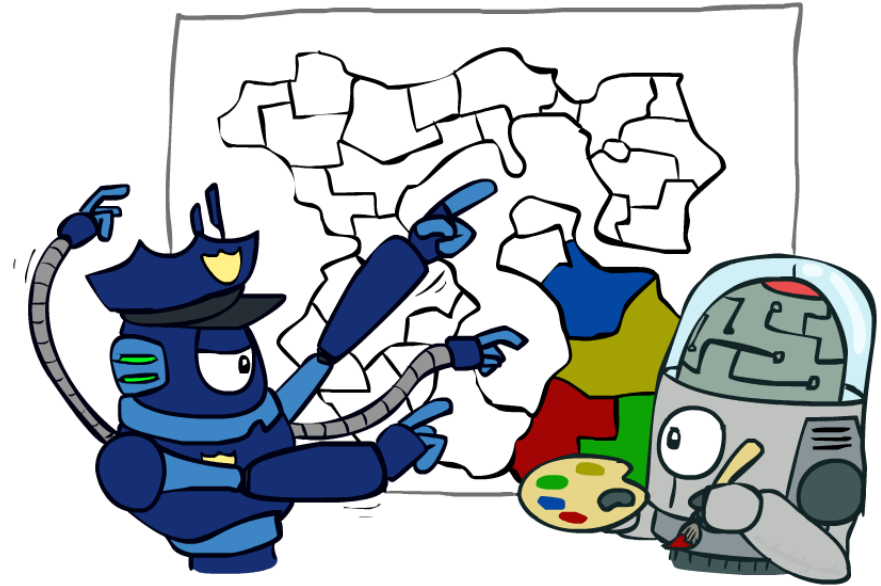
Varieties of CSPs

- Discrete Variables We will cover today
 - Finite domains
 - Size d means $O(d^n)$ complete assignments
 - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
 - Infinite domains (integers, strings, etc.)
 - E.g., job scheduling, variables are start/end times for each job
 - Linear constraints solvable, nonlinear undecidable
- Continuous variables We will cover in later lecture (linear programming)
 - E.g., start/end times for Hubble Telescope observations
 - Linear constraints solvable in polynomial time



Varieties of Constraints

- Varieties of Constraints
 - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
 $SA \neq \text{green}$ Focus of today
 - Binary constraints involve pairs of variables, e.g.:
 $SA \neq WA$
 - Higher-order constraints involve 3 or more variables:
e.g., cryptarithmic column constraints
 $O + O = R + 10 \cdot X_1$
- Preferences (soft constraints):
 - E.g., red is better than green
 - Often representable by a cost for each variable assignment
 - Gives constrained optimization problems

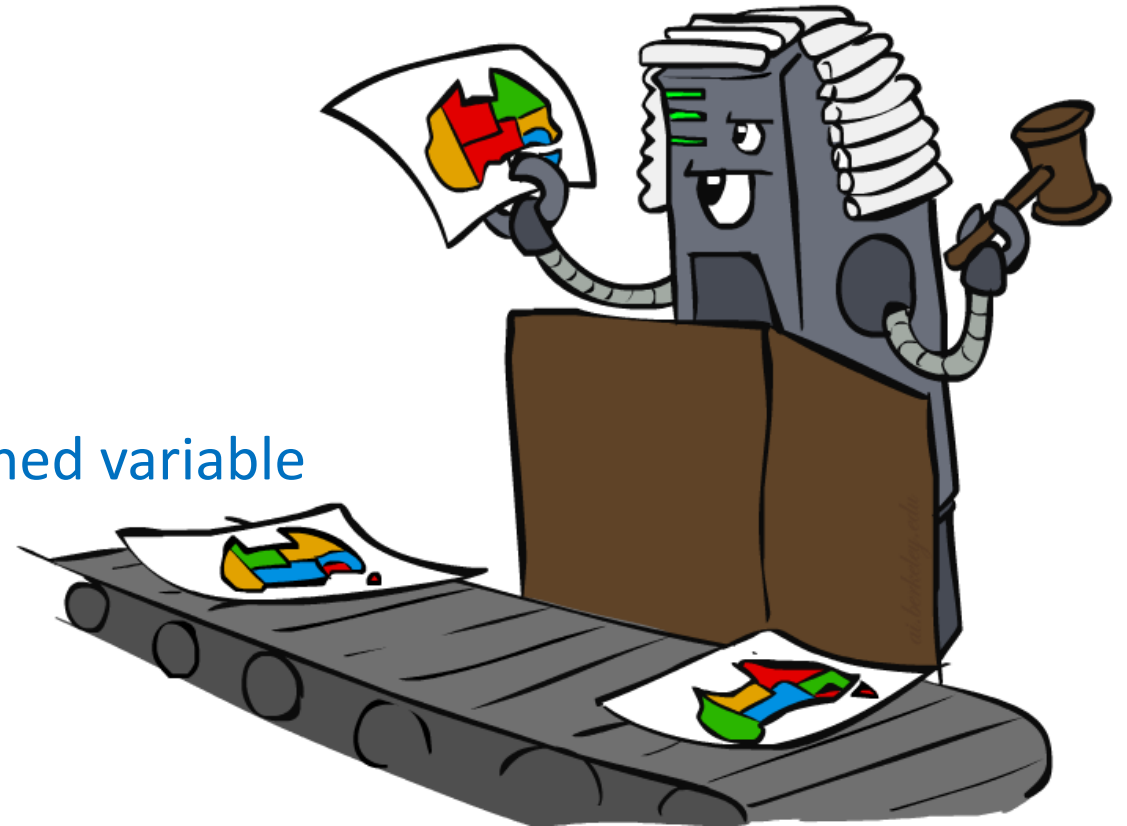


Solving CSPs



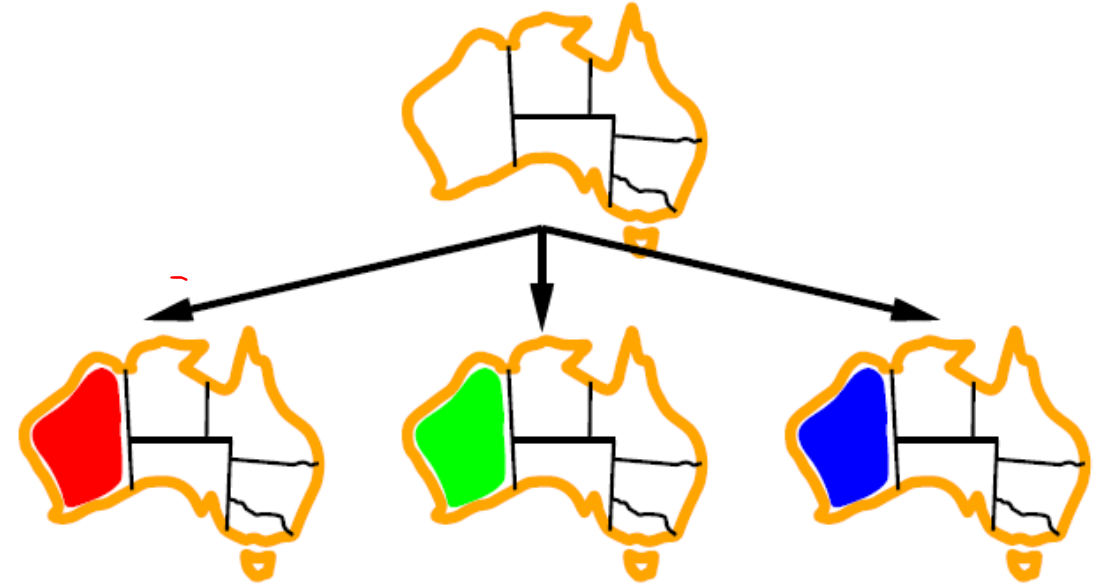
Standard Search Formulation

- Standard search formulation of CSPs
- States defined by the values assigned so far (partial assignments)
 - Initial state: the empty assignment, $\{\}$
 - Successor function: assign a value to an unassigned variable → Can be any unassigned variable
 - Goal test: the current assignment is **complete** and **satisfies all constraints**
- We'll start with the straightforward, naïve approach, then improve it



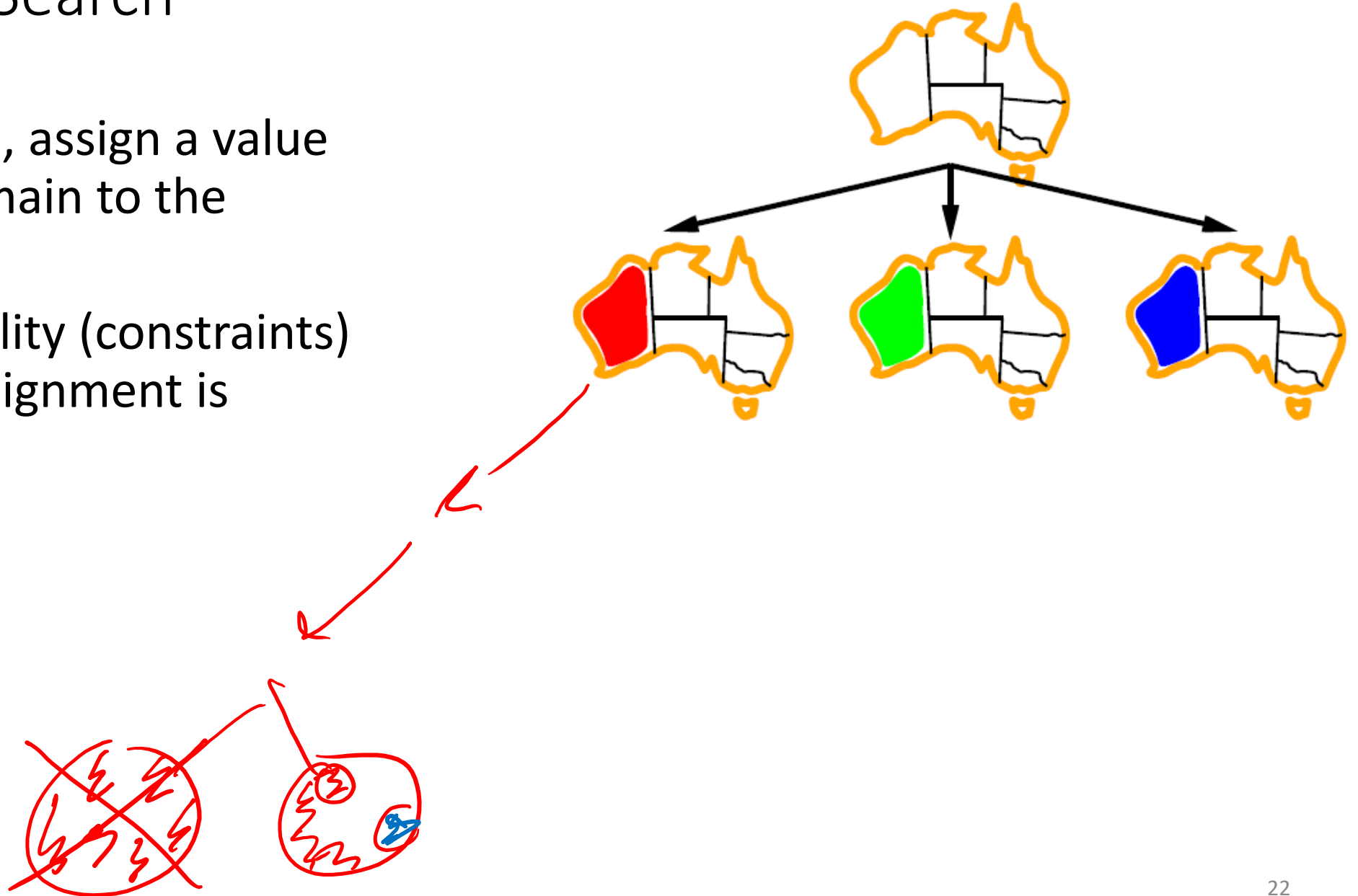
Poll 1: Search for CSPs

Should we use BFS or DFS?

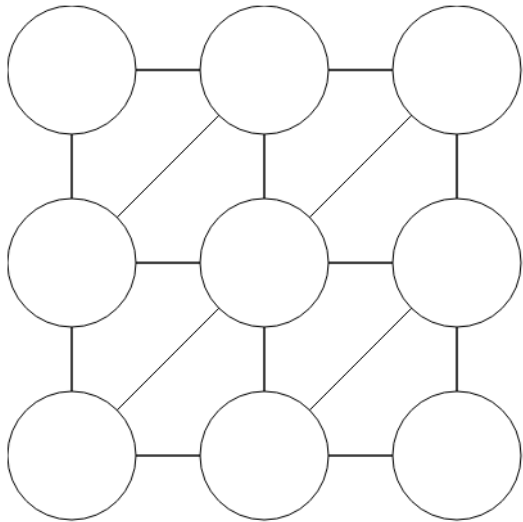


Depth First Search

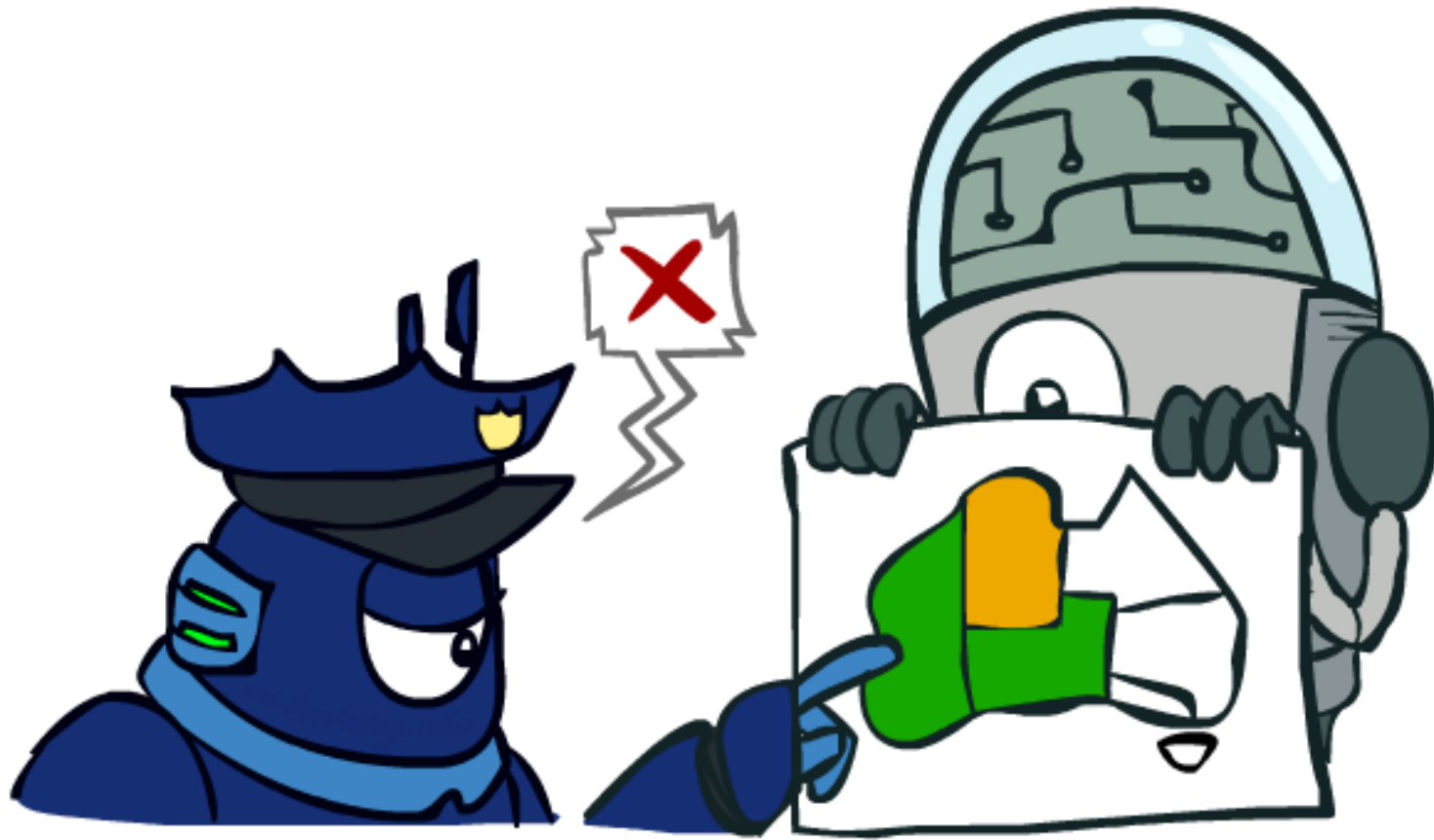
- At each node, assign a value from the domain to the variable
- Check feasibility (constraints) when the assignment is complete



Demo – Naïve Search

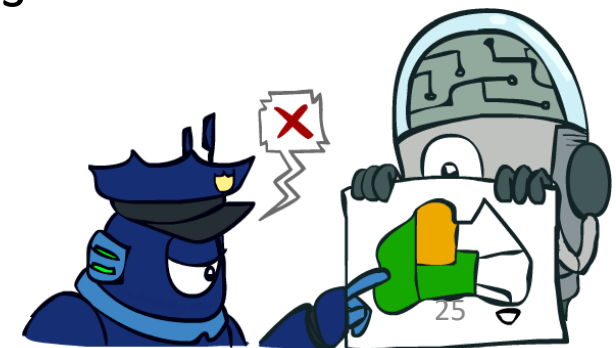


Backtracking Search

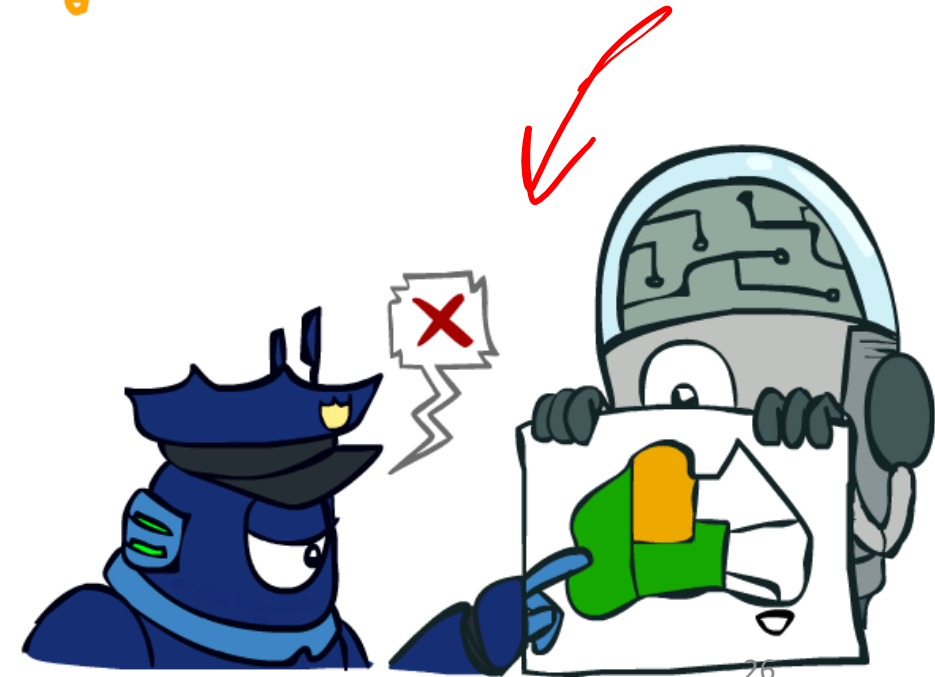
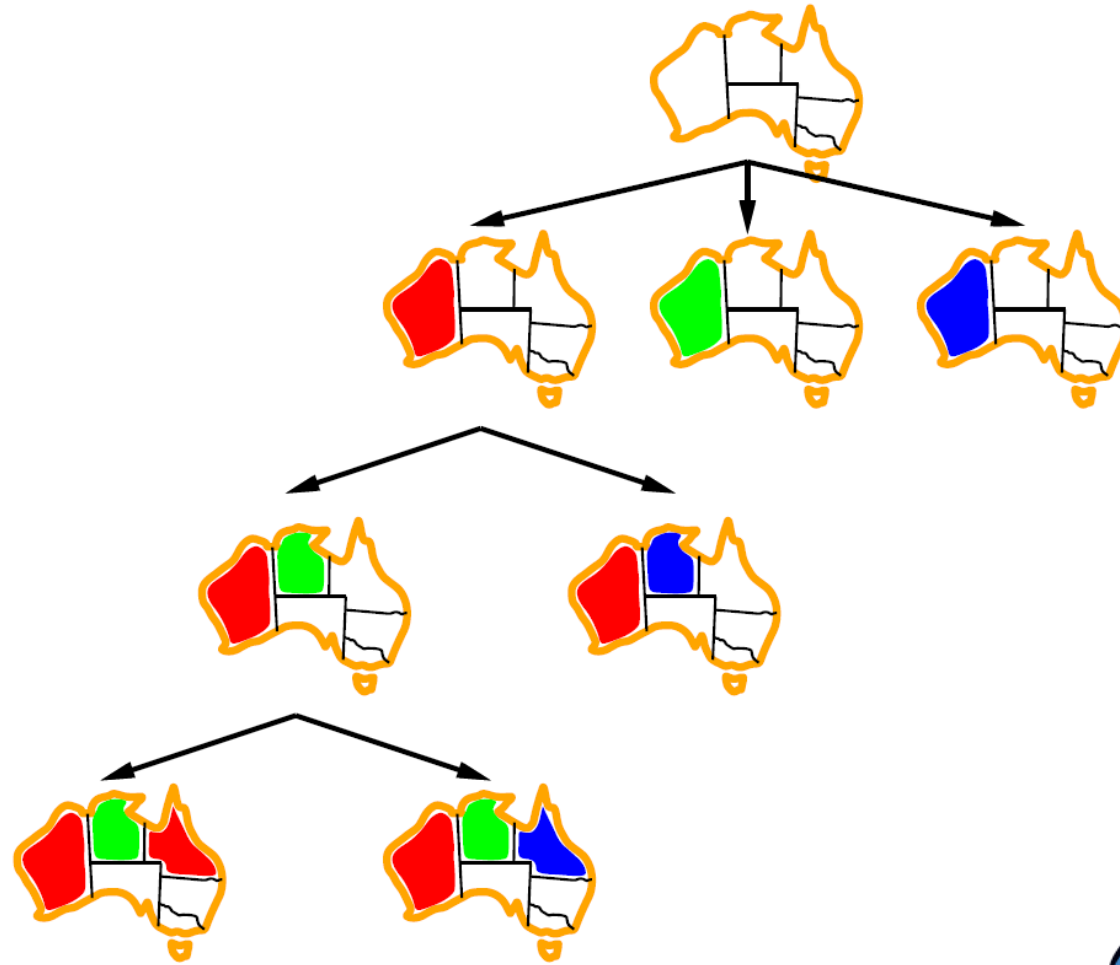


Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Backtracking search = DFS + two improvements
- Idea 1: One variable at a time
 - Variable assignments are commutative
 - [WA = red then NT = green] same as [NT = green then WA = red]
 - Only need to consider assign value to a single variable at each step
- Idea 2: Check constraints as you go
 - Consider only values which do not conflict previous assignments
 - May need some computation to check the constraints
 - “Incremental goal test”
- Can solve n-queens for $n \approx 25$



Backtracking Example



Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment given CONSTRAINTS[csp] then
      add {var = value} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove {var = value} from assignment
  return failure
```

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      remove {var = value} from assignment
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```

No need to check consistency for a complete assignment

Backtracking Search

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```

Checks consistency at each assignment

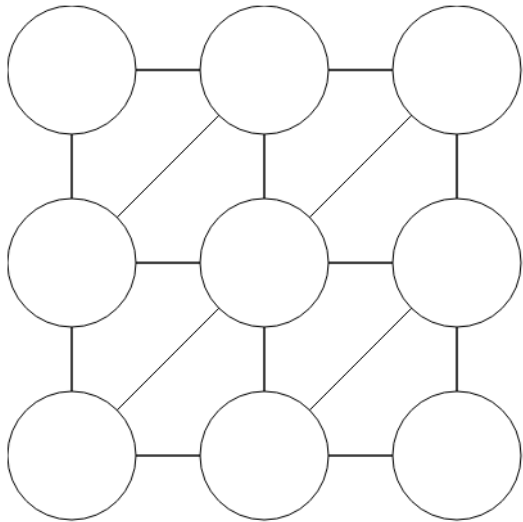
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      if result ≠ failure then return result
      remove {var = value} from assignment
  return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

Demo – Backtracking



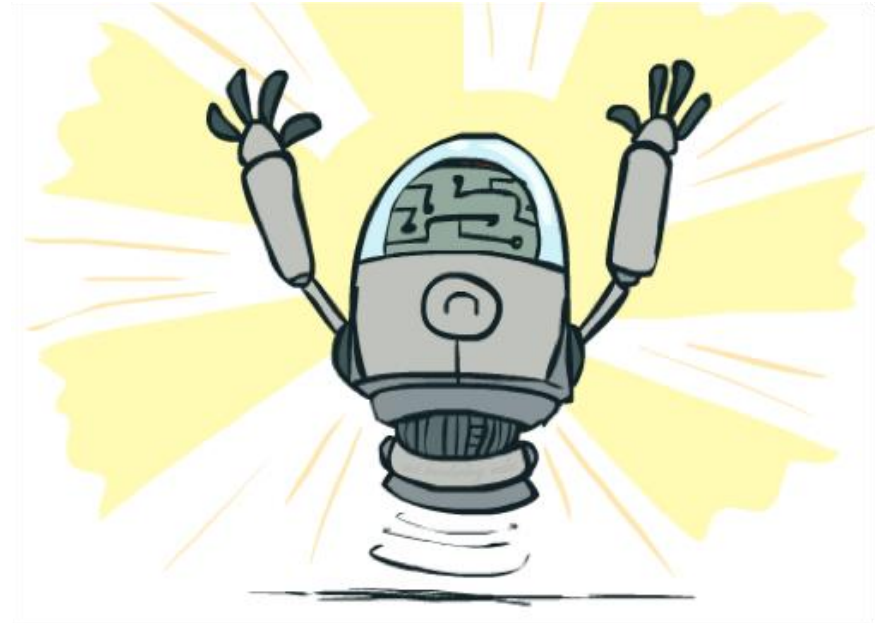
Improving Backtracking

- General-purpose ideas give huge gains in speed
- Filtering: Can we detect inevitable failure early?

- Ordering:

- Which variable should be assigned next?
- In what order should its values be tried?

- Structure: Can we exploit the problem structure?



Filtering

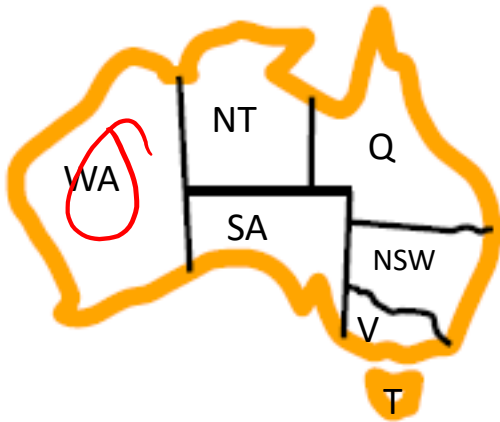


Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: A simple way for filtering
 - After a variable is assigned a value, check related constraints and cross off values of unassigned variables which violate the constraints
 - Failure detected if some variables have no values remaining

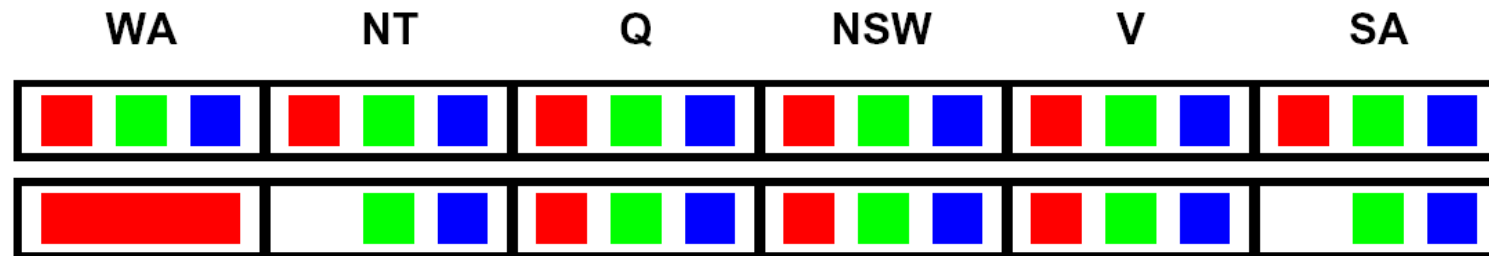
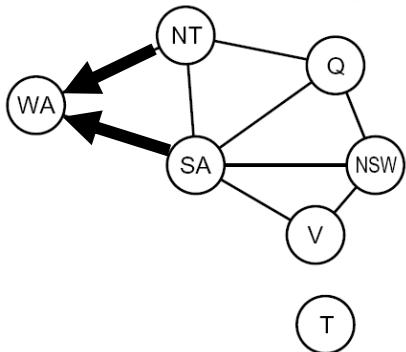
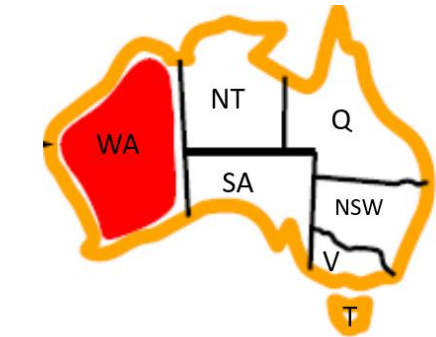
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Filtering: Forward Checking

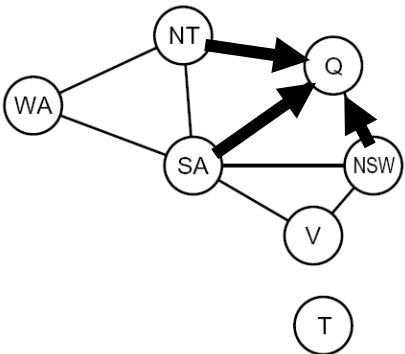
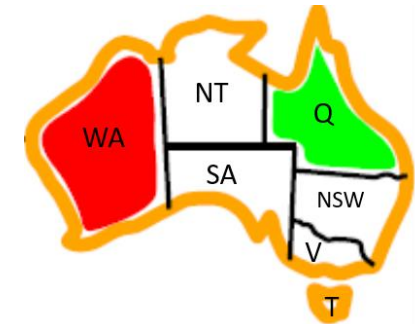
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Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

Filtering: Forward Checking

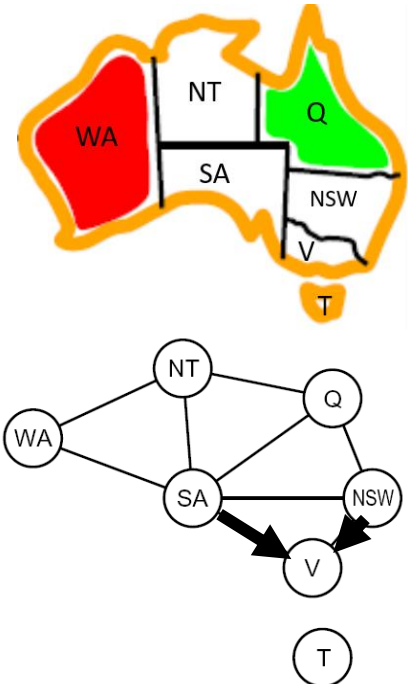
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WA	NT	Q	NSW	V	SA
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Filtering: Forward Checking

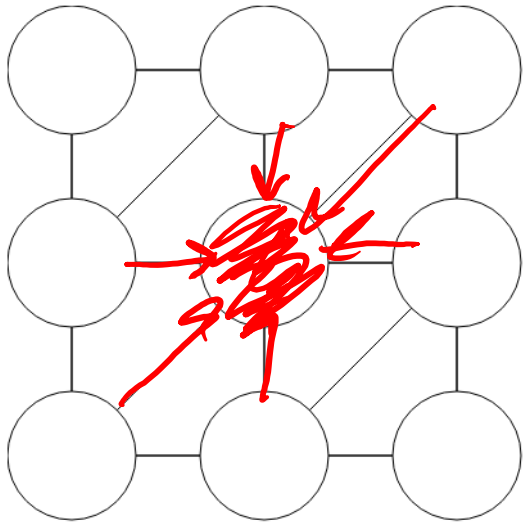
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WA	NT	Q	NSW	V	SA
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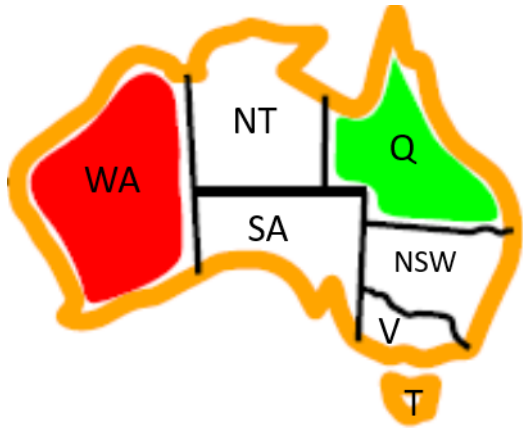
FAIL – variable with no possible values

























































Demo – Backtracking with Forward Checking



Filtering: Constraint Propagation

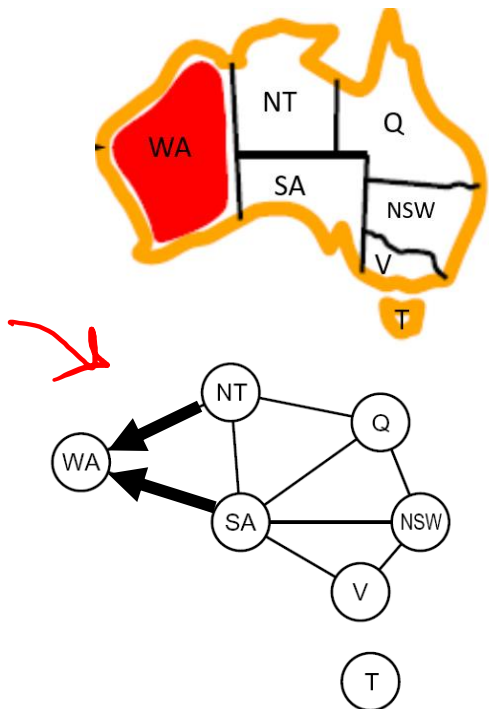
- Limitations of simple forward checking: propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures
 - NT and SA cannot both be blue! Why didn't we detect this yet?
- *Constraint propagation*: reason from constraint to constraint



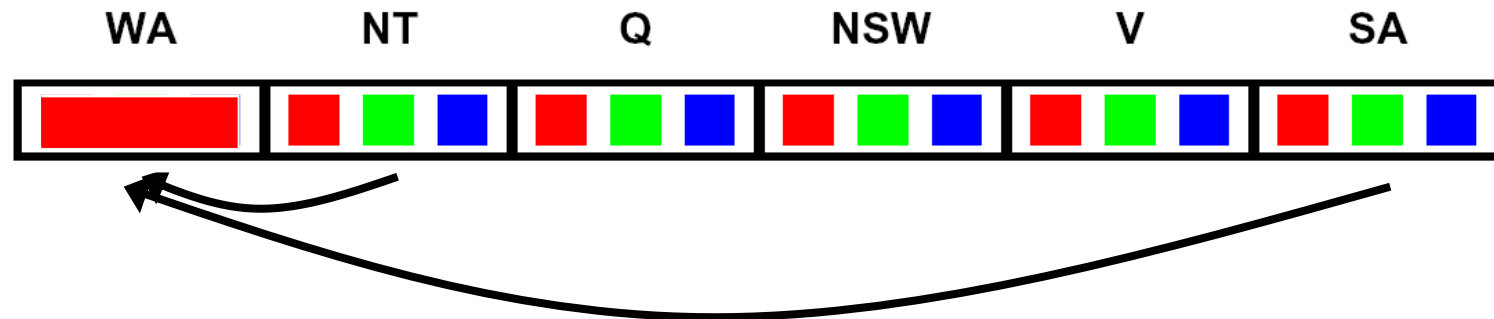
WA	NT	Q	NSW	V	SA
  	  	  	  	  	  
      	 	  	  	  	 
      		    	 	  	

Consistency of A Single Arc

- An arc $X \rightarrow Y$ is consistent iff for **every** x in the tail there is **some** y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall (X, Y) \in E$ and Y newly assigned



(Remove values from the tail!)

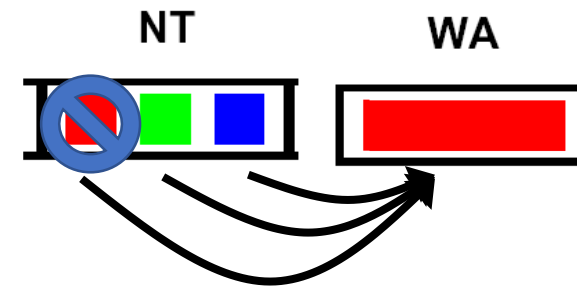
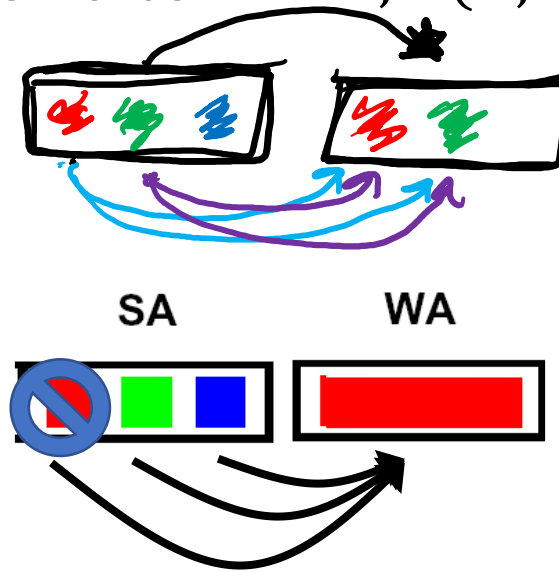
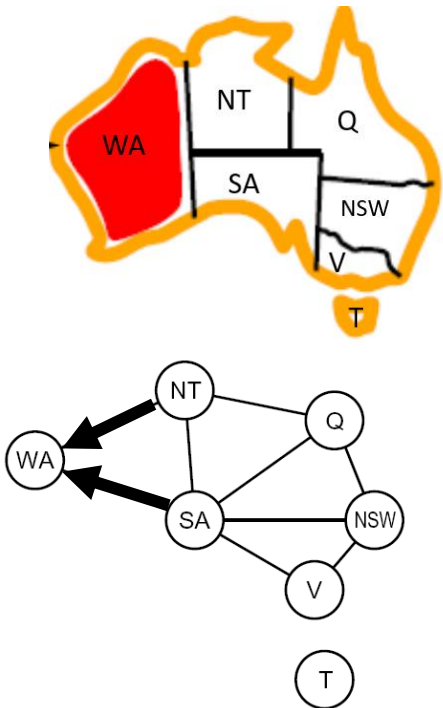


Recall: Binary constraint graph for a binary CSP (i.e., each constraint has most two variables): nodes are variables, edges show constraints

Consistency of A Single Arc

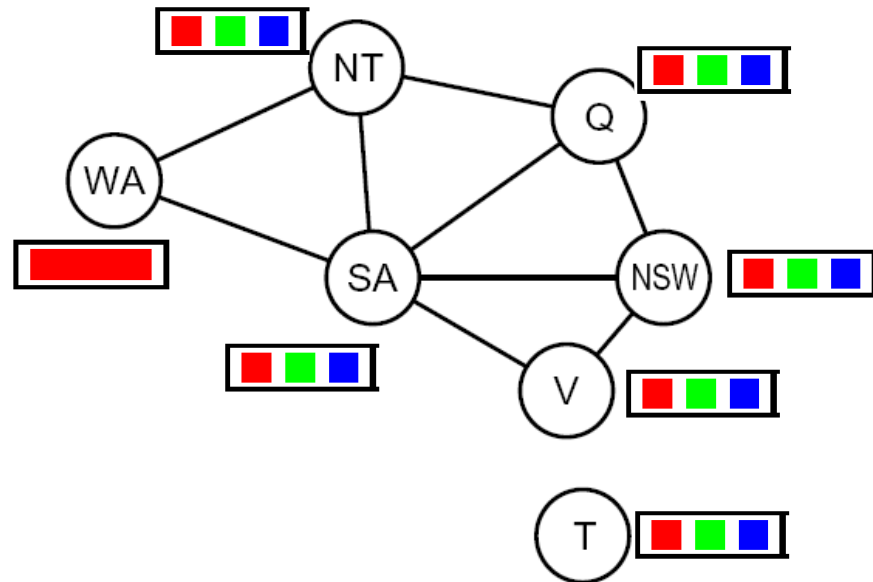
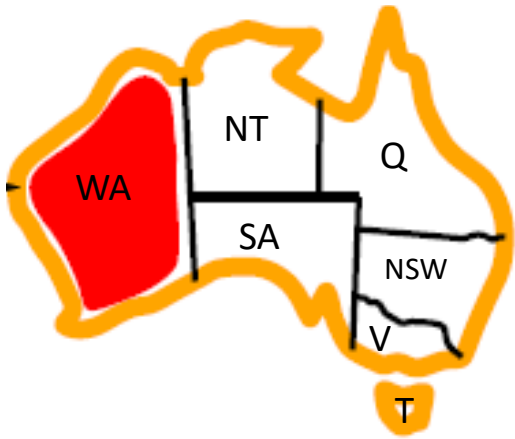
for d in D_{tail}
for d in D_{head}

- An arc $X \rightarrow Y$ is consistent iff for **every** x in the tail there is **some** y in the head which could be assigned without violating a constraint
- Enforce arc consistency: Remove values in domain of X if no corresponding legal Y exists
- Forward checking: Only enforce $X \rightarrow Y, \forall (X, Y) \in E$ and Y newly assigned



How to Enforce Arc Consistency of Entire CSP

- A simplistic algorithm: Cycle over the pairs of variables, enforcing arc-consistency, repeating the cycle until no domains change for a whole cycle
- AC-3 (short for Arc Consistency Algorithm #3): A more efficient algorithm ignoring constraints that have not been modified since they were last analyzed



AC-3: Enforce Arc Consistency of Entire CSP

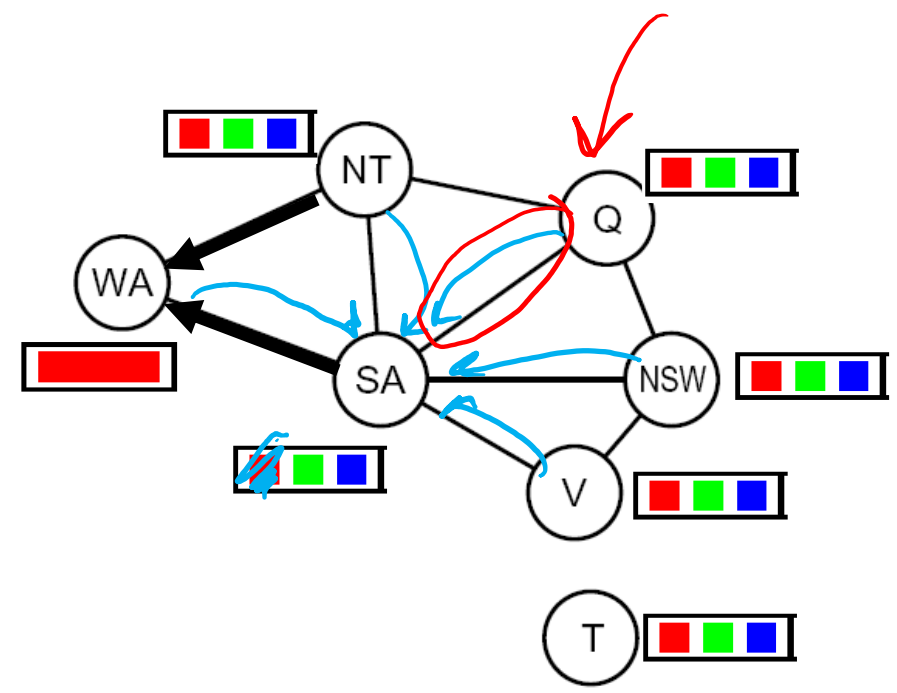
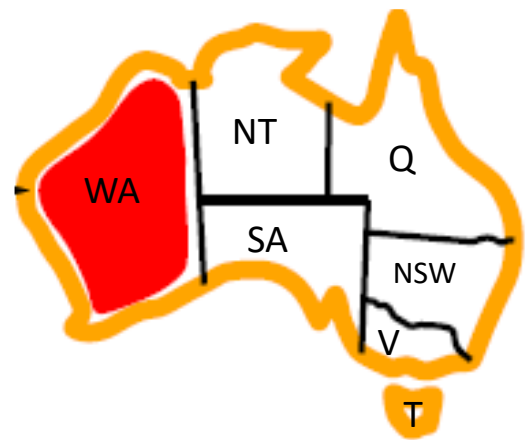
```
function AC-3(csp) returns the CSP, possibly with reduced domains
inputs: csp, a binary CSP with variables  $\{X_1, X_2, \dots, X_n\}$ 
local variables: queue, a queue of arcs, initially all the arcs in csp
while queue is not empty do
     $(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\textit{queue})$ 
    if REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) then
        for each  $X_k$  in NEIGHBORS[ $X_i$ ] do
            add  $(X_k, X_i)$  to queue


---


function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) returns true iff succeeds
    removed  $\leftarrow$  false
    for each  $x$  in DOMAIN[ $X_i$ ] do
        if no value  $y$  in DOMAIN[ $X_j$ ] allows  $(x, y)$  to satisfy the constraint  $X_i \leftrightarrow X_j$ 
            then delete  $x$  from DOMAIN[ $X_i$ ]; removed  $\leftarrow$  true
    return removed
```

Constraint Propagation!

AC-3: Enforce Arc Consistency of Entire CSP



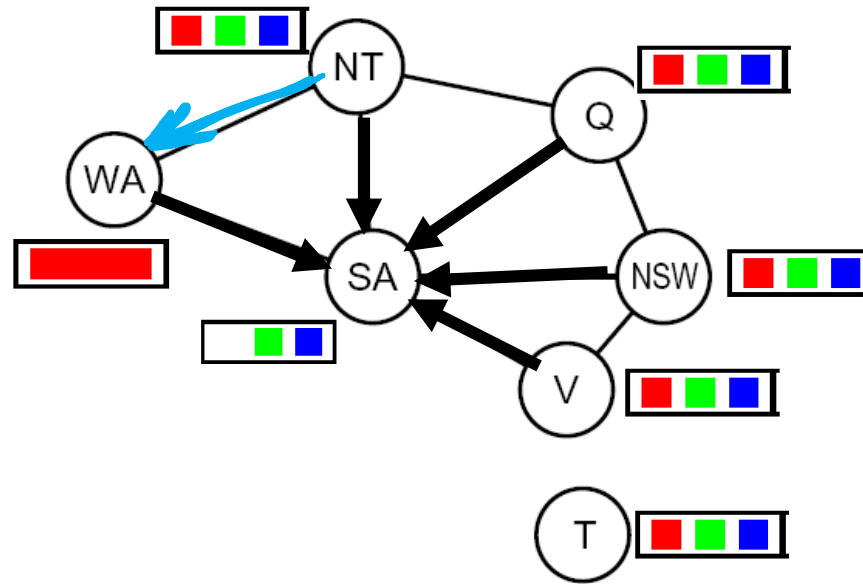
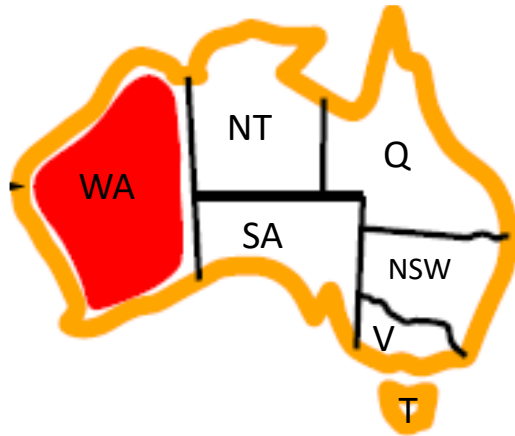
Queue:

✓ SA->WA

NT->WA

Remember: Delete from the tail!

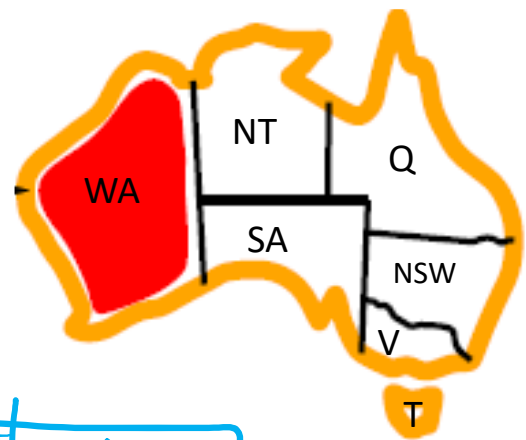
AC-3: Enforce Arc Consistency of Entire CSP



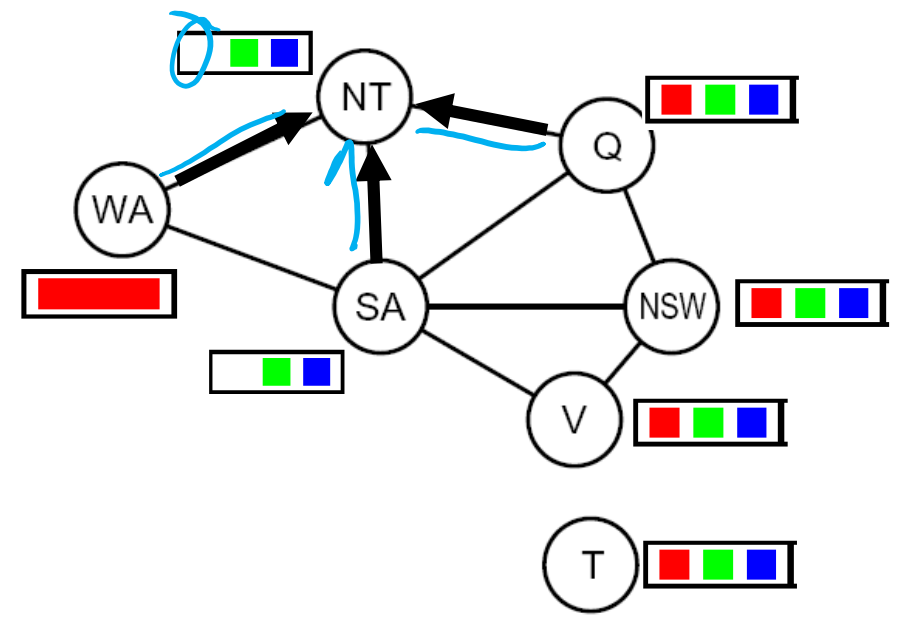
Queue:
NT->WA
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



2	34	12 4	17 34
	1		
		3	



Queue:

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

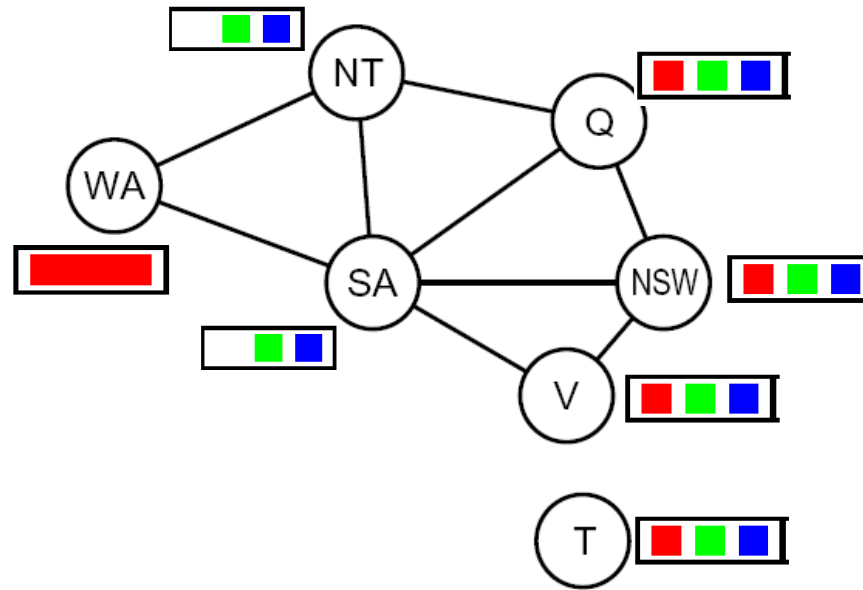
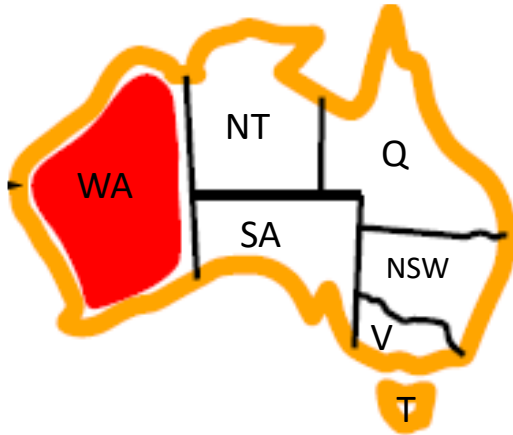
WA->NT

SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

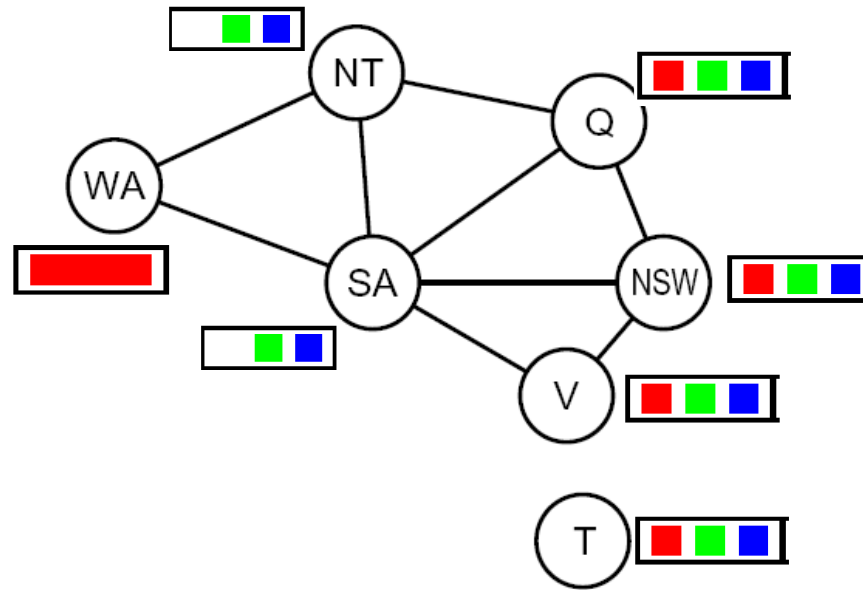
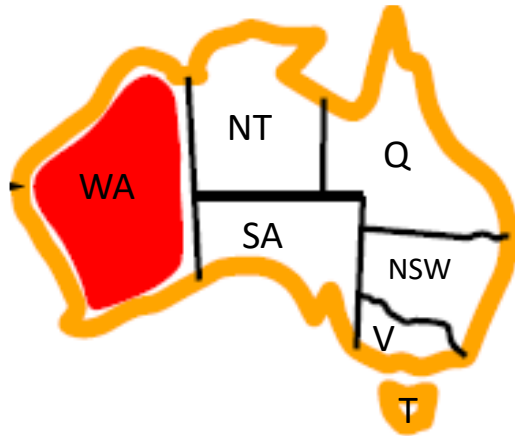
WA->NT

SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

NT->SA

Q->SA

NSW->SA

V->SA

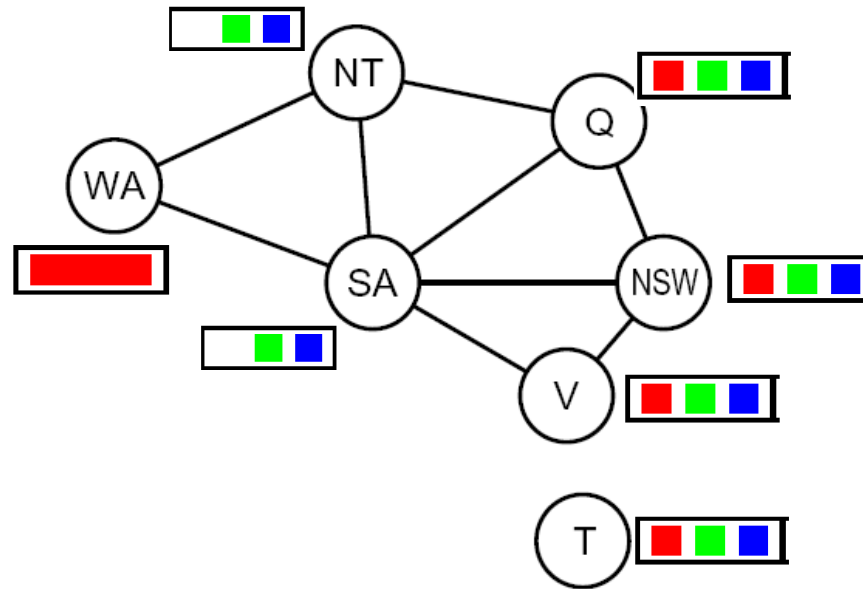
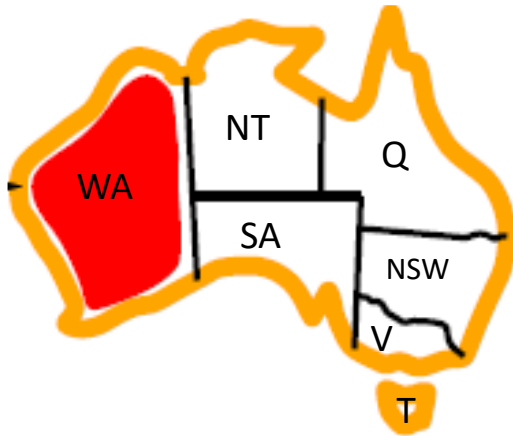
WA->NT

SA->NT

Q->NT

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

Q->SA

NSW->SA

V->SA

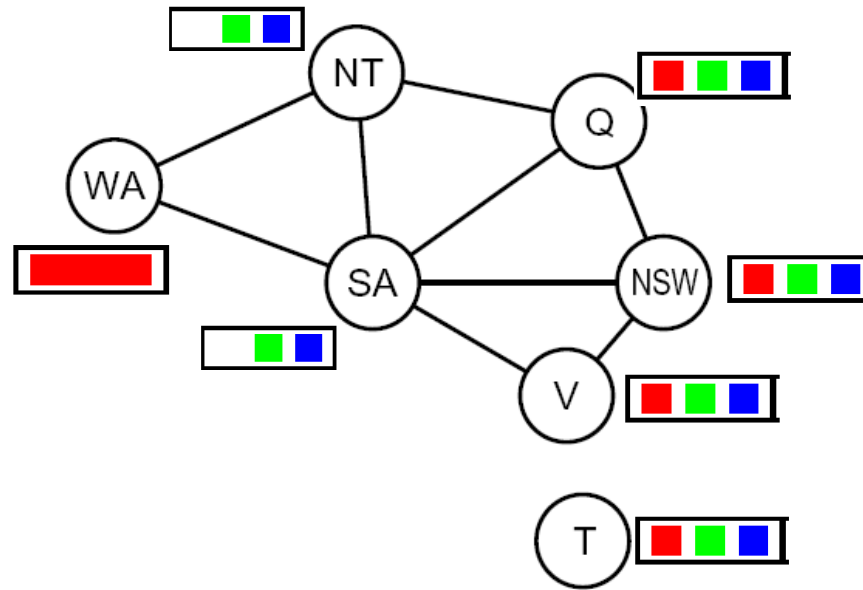
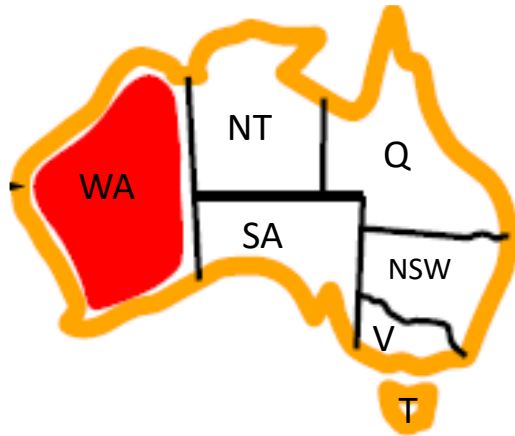
WA->NT

SA->NT

Q->NT

Remember: Delete from the tail!

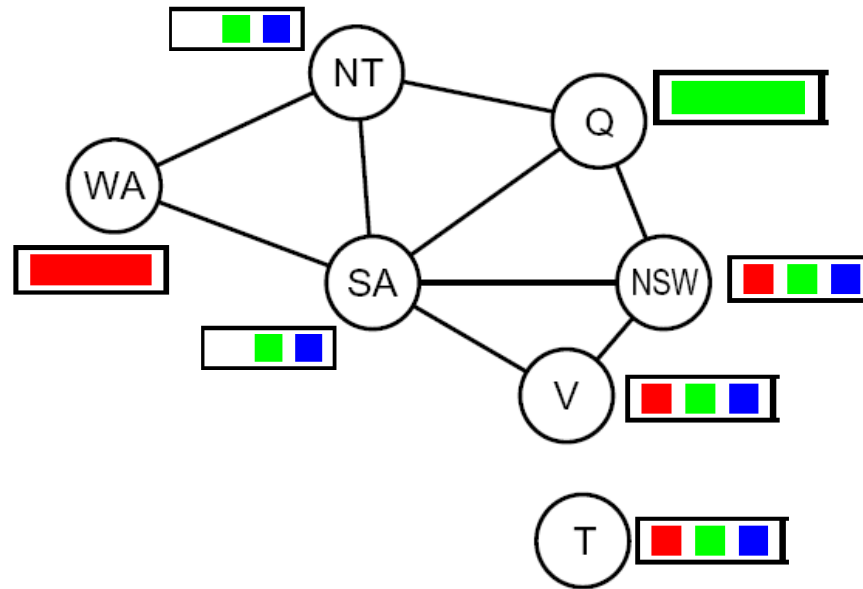
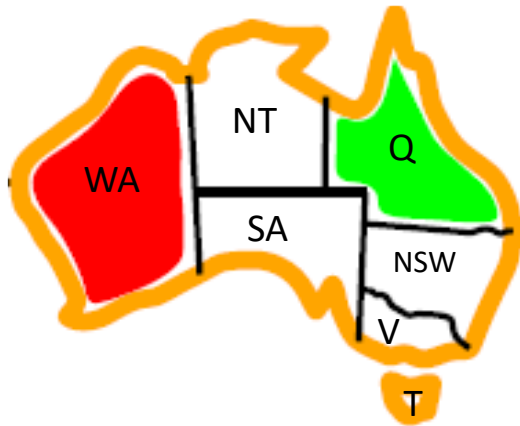
AC-3: Enforce Arc Consistency of Entire CSP



Queue:
NSW->SA
V->SA
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

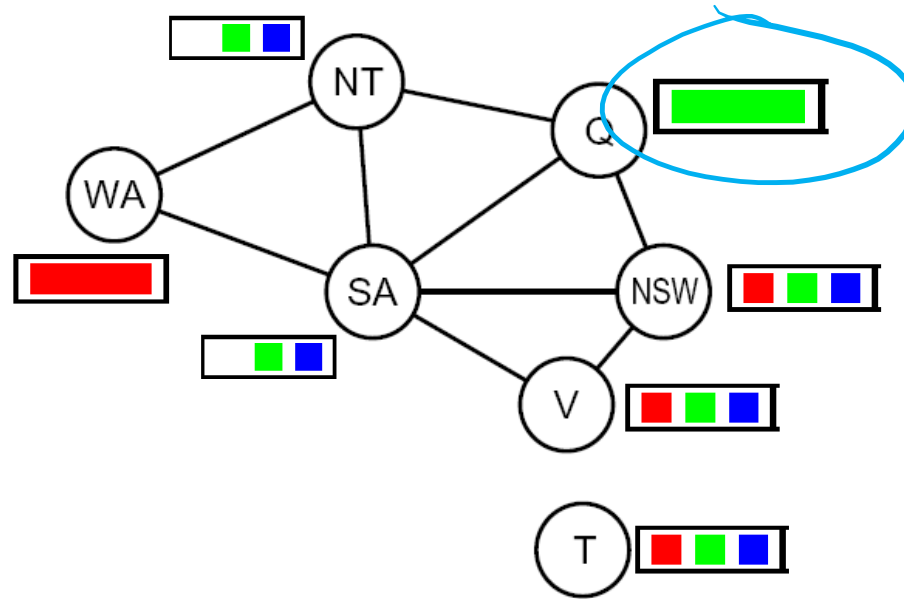
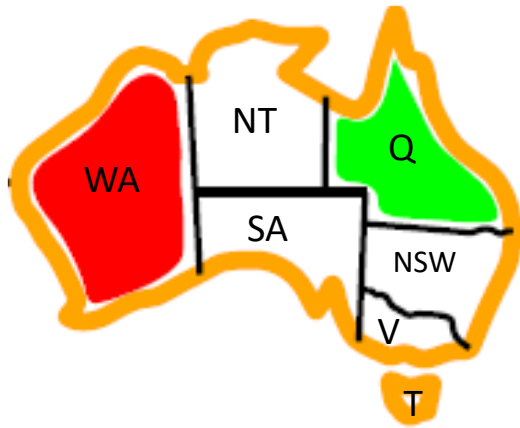
AC-3: Enforce Arc Consistency of Entire CSP



Queue:

Remember: Delete from the tail!

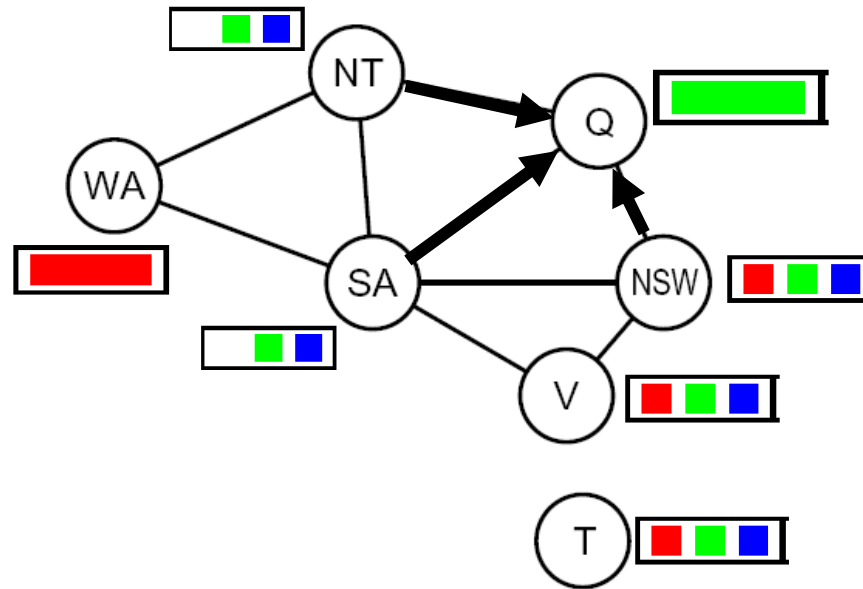
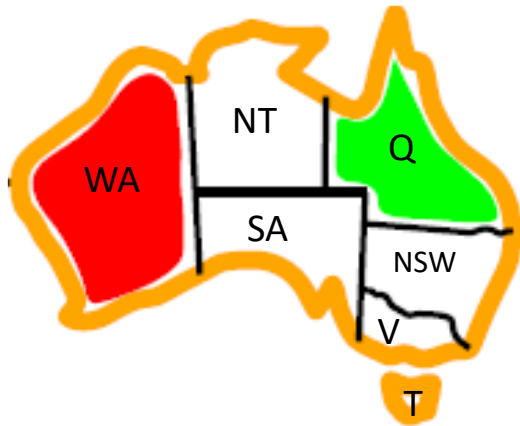
Poll 2: After assigning Q to Green, what gets added to the Queue?



Queue:

A: NSW→Q, SA→Q, NT→Q
B: Q→NSW, Q→SA, Q→NT

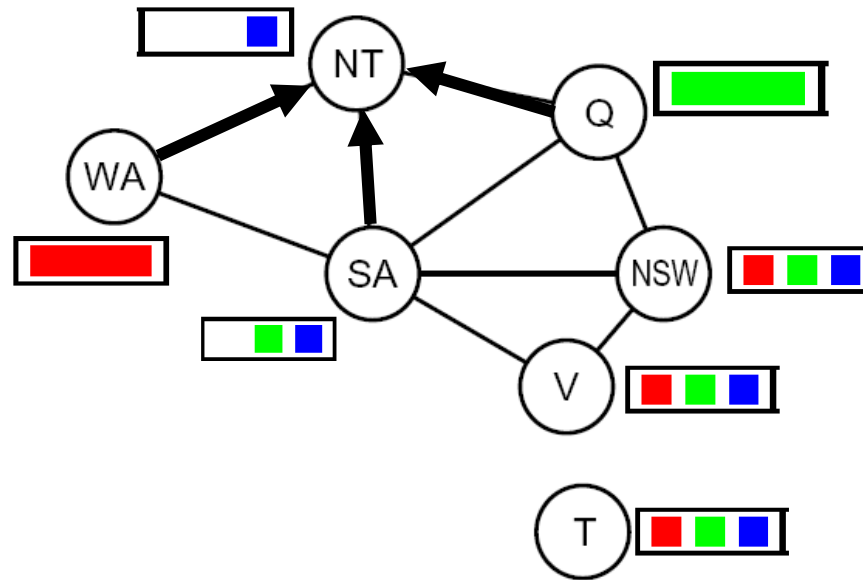
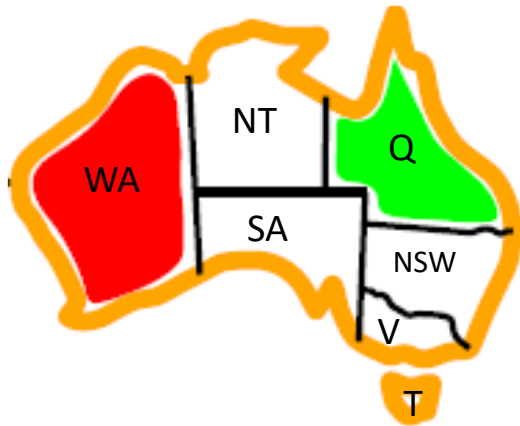
AC-3: Enforce Arc Consistency of Entire CSP



Queue:
NT->Q
SA->Q
NSW->Q

Remember: Delete from the tail!

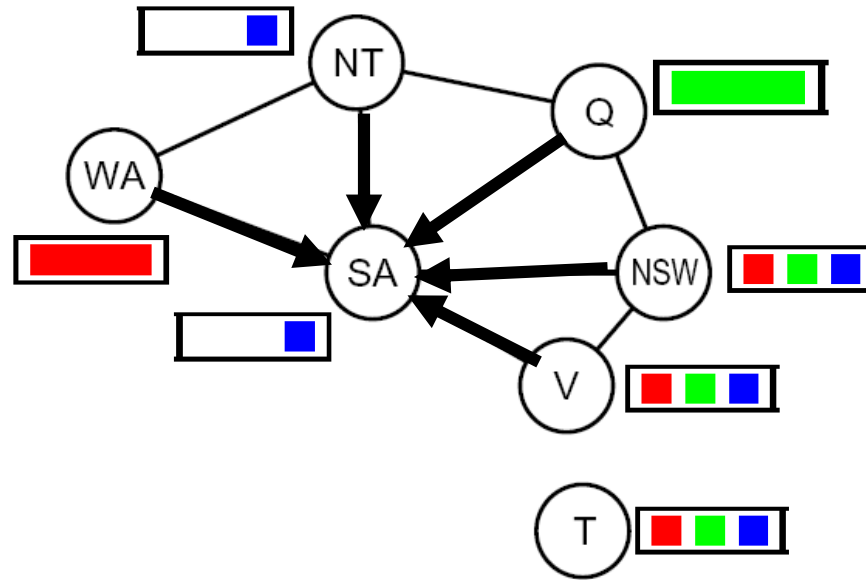
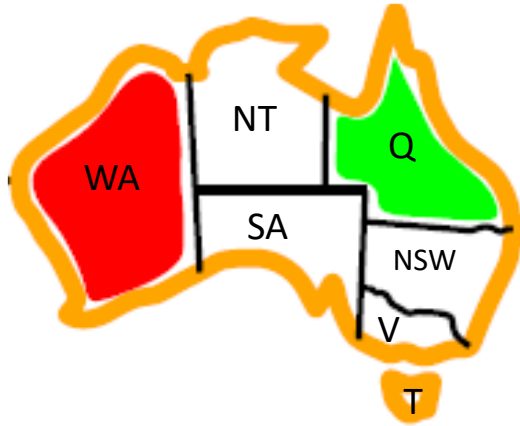
AC-3: Enforce Arc Consistency of Entire CSP



Queue:
SA->Q
NSW->Q
WA->NT
SA->NT
Q->NT

Remember: Delete from the tail!

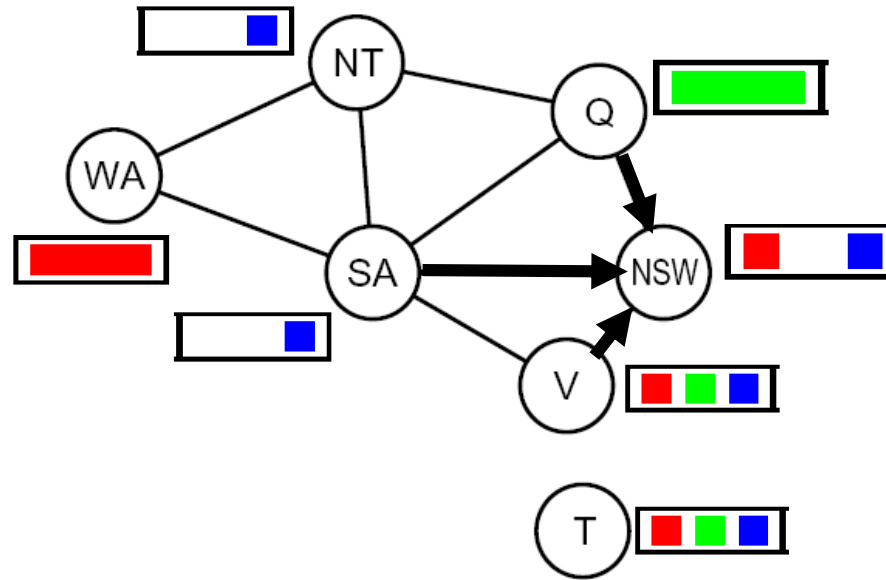
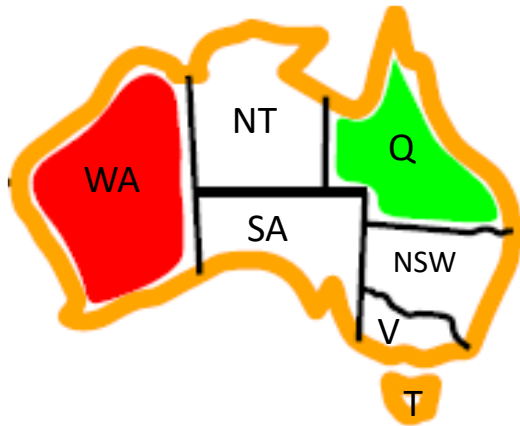
AC-3: Enforce Arc Consistency of Entire CSP



Queue:
NSW->Q
WA->NT
SA->NT
Q->NT
WA->SA
NT->SA
Q->SA
NSW->SA
V->SA

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

WA->NT

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

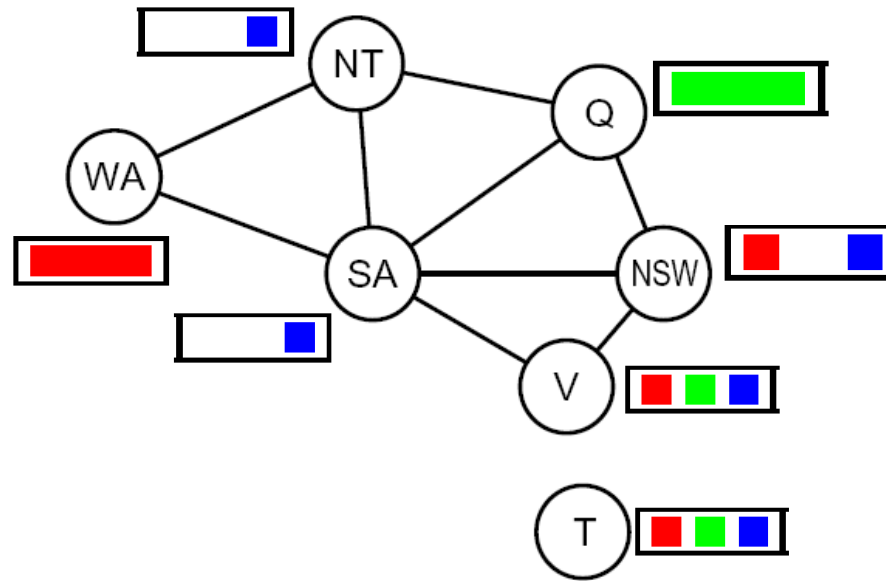
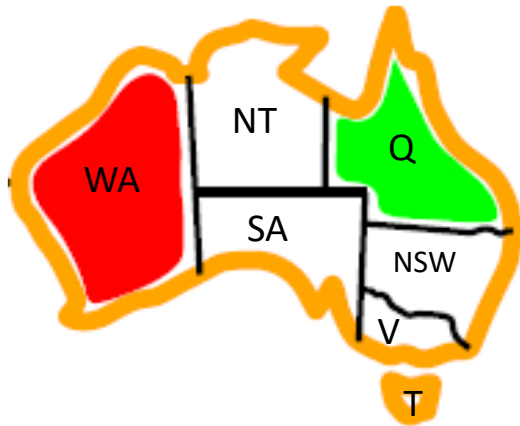
V->NSW

Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

WA->NT

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

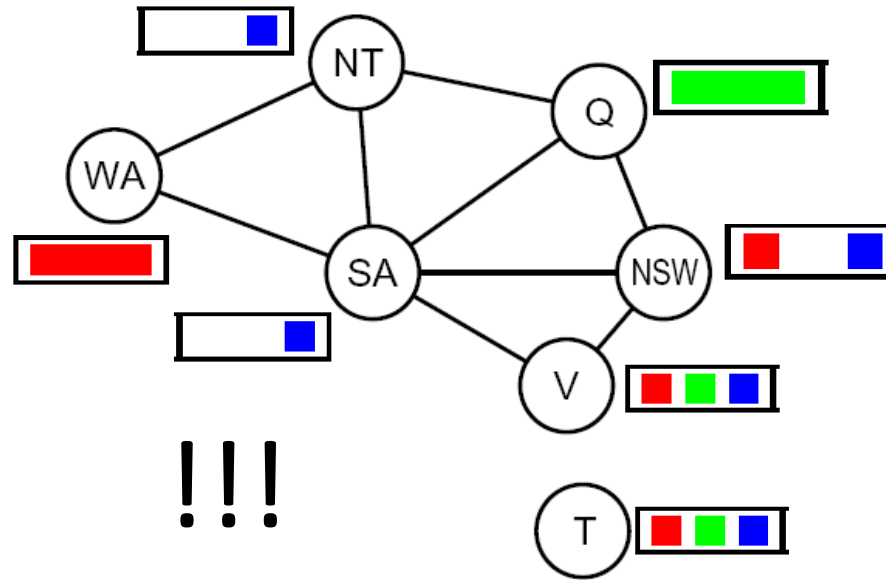
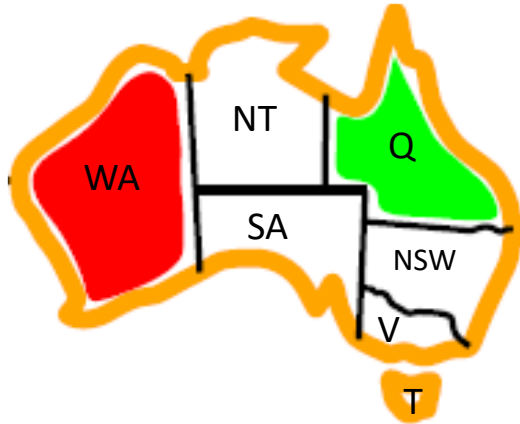
V->NSW

Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

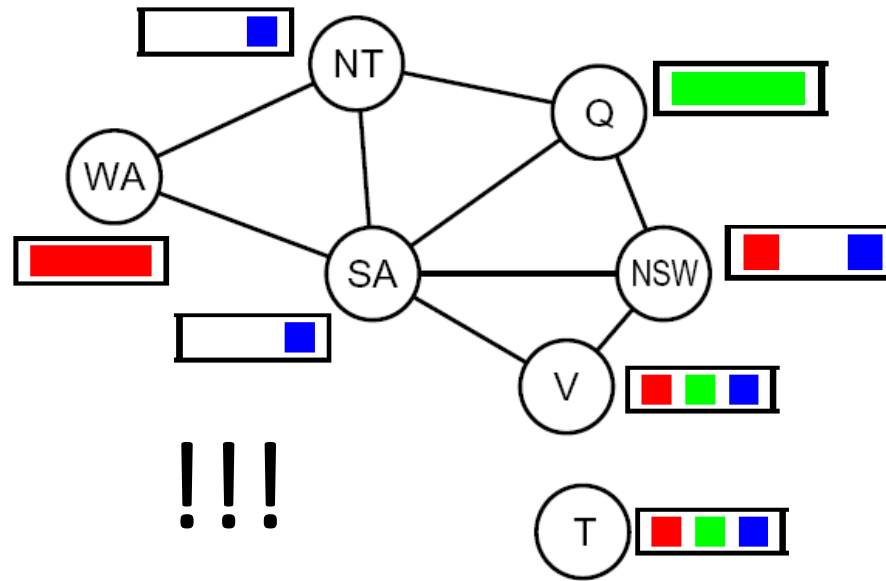
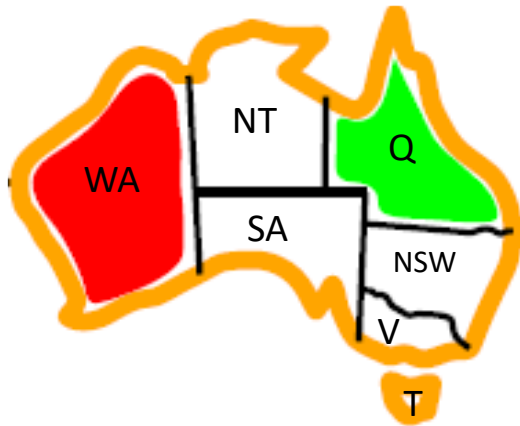
V->NSW

Q->NSW

SA->NSW

Remember: Delete from the tail!

AC-3: Enforce Arc Consistency of Entire CSP



Queue:

SA->NT

Q->NT

WA->SA

NT->SA

Q->SA

NSW->SA

V->SA

V->NSW

Q->NSW

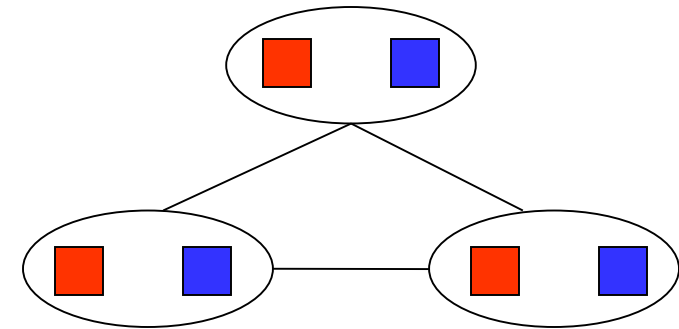
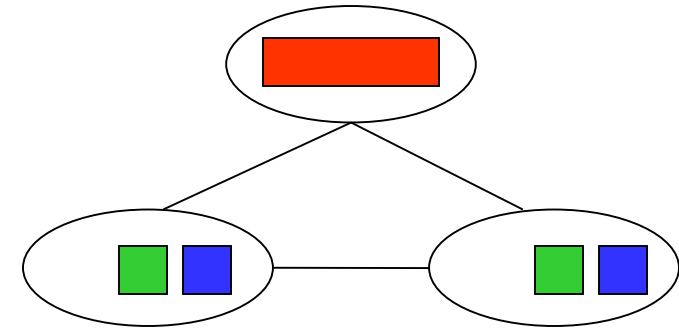
SA->NSW

- Backtrack on the assignment of Q
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Limitations of Arc Consistency

- After enforcing arc consistency:
 - Can have one solution left
 - Can have multiple solutions left
 - Can have no solutions left (and not know it)
- Arc consistency only checks local consistency conditions
- Arc consistency still runs inside a backtracking search!



What went wrong here?

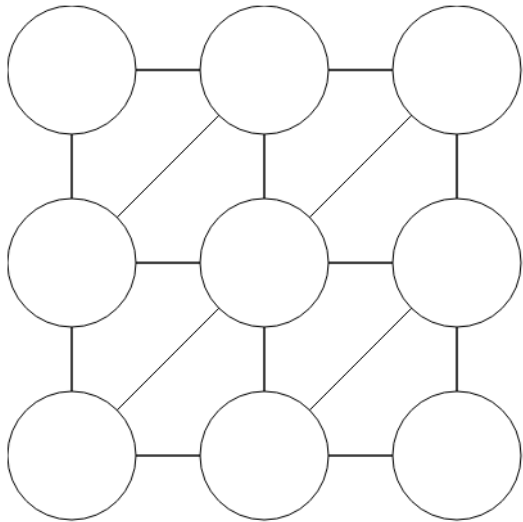
Backtracking Search with AC-3

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment given CONSTRAINTS[csp] then
      add {var = value} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove {var = value} from assignment
  return failure
```

- Where do you run AC-3?

Demo – Backtracking with AC-3



Demo – Coloring with a Complex Graph

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3

Complexity of a single run of AC-3

```
function AC-3(cs) returns the CSP, possibly with reduced domains
inputs: cs, a binary CSP with variables  $\{X_1, X_2, \dots, X_n\}$ 
local variables: queue, a queue of arcs, initially all the arcs in cs

while queue is not empty do
     $(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\textit{queue})$ 
    if REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) then
        for each  $X_k$  in NEIGHBORS[ $X_i$ ] do
            add  $(X_k, X_i)$  to queue



---



function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) returns true iff succeeds
    removed  $\leftarrow$  false
    for each  $x$  in DOMAIN[ $X_i$ ] do
        if no value  $y$  in DOMAIN[ $X_j$ ] allows  $(x, y)$  to satisfy the constraint  $X_i \leftrightarrow X_j$ 
            then delete  $x$  from DOMAIN[ $X_i$ ]; removed  $\leftarrow$  true
    return removed
```

Recall that the whole backtracking algorithm with AC-3 will call AC-3 many times

Complexity of a single run of AC-3

```
function AC-3(csp) returns the CSP, possibly with reduced domains
inputs: csp, a binary CSP with variables  $\{X_1, X_2, \dots, X_n\}$ 
local variables: queue, a queue of arcs, initially all the arcs in csp
while queue is not empty do
     $(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\textit{queue})$ 
    if REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) then
        for each  $X_k$  in NEIGHBORS[ $X_i$ ] do
            add  $(X_k, X_i)$  to queue
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) returns true iff succeeds
    removed  $\leftarrow$  false
    for each  $x$  in DOMAIN[ $X_i$ ] do
        if no value  $y$  in DOMAIN[ $X_j$ ] allows  $(x, y)$  to satisfy the constraint  $X_i \leftrightarrow X_j$ 
            then delete  $x$  from DOMAIN[ $X_i$ ]; removed  $\leftarrow$  true
    return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$

Complexity of a single run of AC-3

```
function AC-3(csp) returns the CSP, possibly with reduced domains
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    if REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) then
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```

```
function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$ ) returns true iff succeeds
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    return removed
```

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2d)$

Complexity of a single run of AC-3

$O(d^n)$

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains

inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \dots, X_n\}$

local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

while *queue* is not empty **do**

$(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\text{queue})$

if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**

for each X_k **in** NEIGHBORS[X_i] **do**

 add (X_k, X_i) to *queue*

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds

removed \leftarrow false

for each x **in** DOMAIN[X_i] **do**

if no value y in DOMAIN[X_j] allows (x, y) to satisfy the constraint $X_i \leftrightarrow X_j$

then delete x from DOMAIN[X_i]; *removed* \leftarrow true

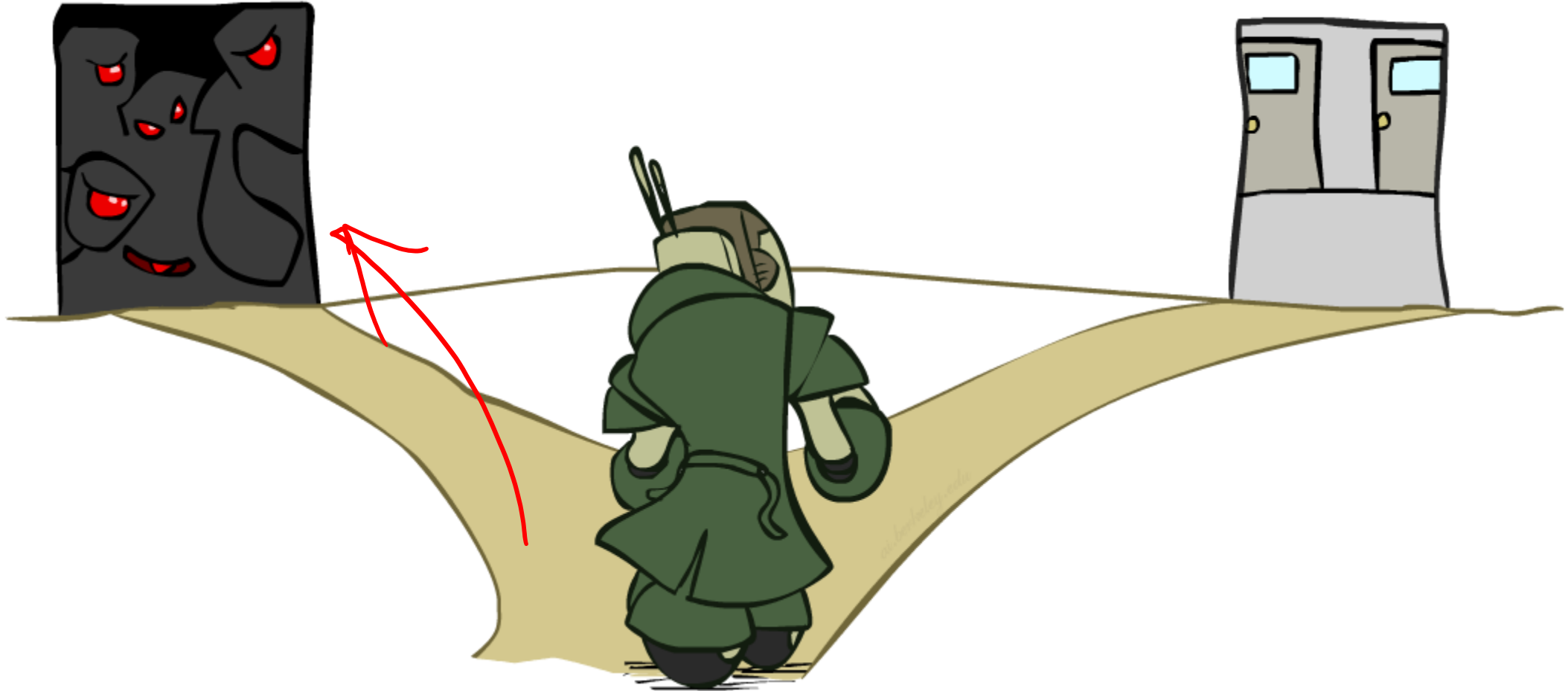
return *removed*

- An arc is added after a removal of value at a node
- n node in total, each has $\leq d$ values
- Total times of removal: $O(nd)$
- After a removal, $\leq n$ arcs added
- Total times of adding arcs: $O(n^2d)$
- Check arc consistency per arc: $O(d^2)$

Complexity of a single run of AC-3 is at most $O(n^2d^3)$

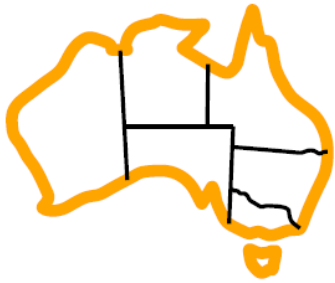
(Not required) Zhang&Yap (2001) show that its complexity is $O(n^2d^2)$

Ordering

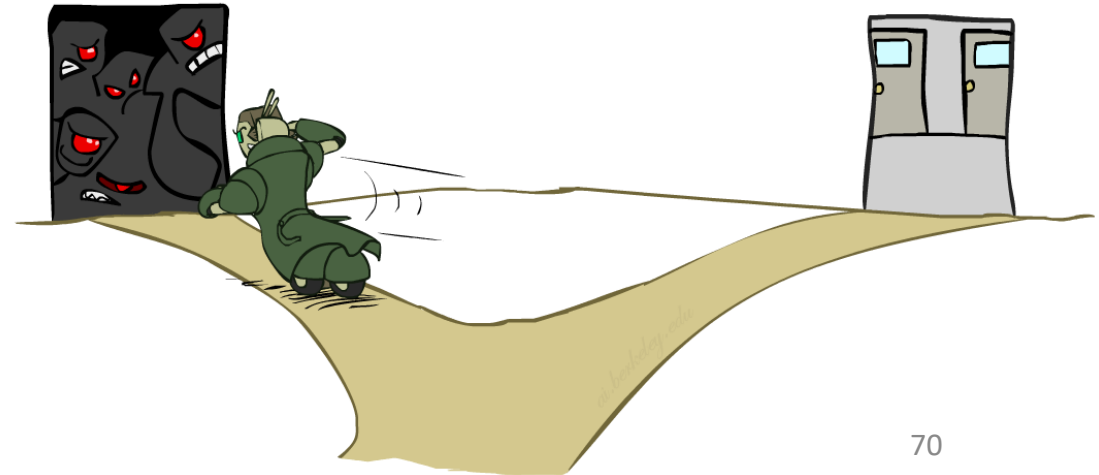


Ordering: Minimum Remaining Values

- Variable Ordering: Minimum remaining values (MRV):
 - Choose the variable with the fewest legal left values in its domain



- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering



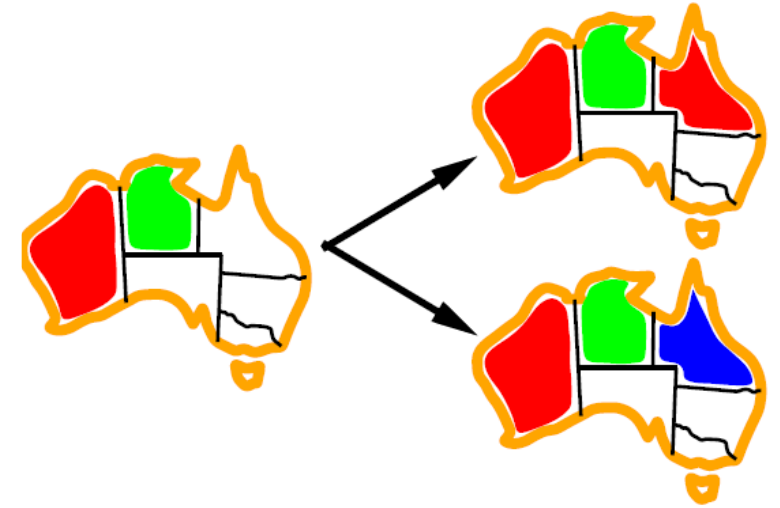
Demo – Coloring with a Complex Graph

Compare

- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)

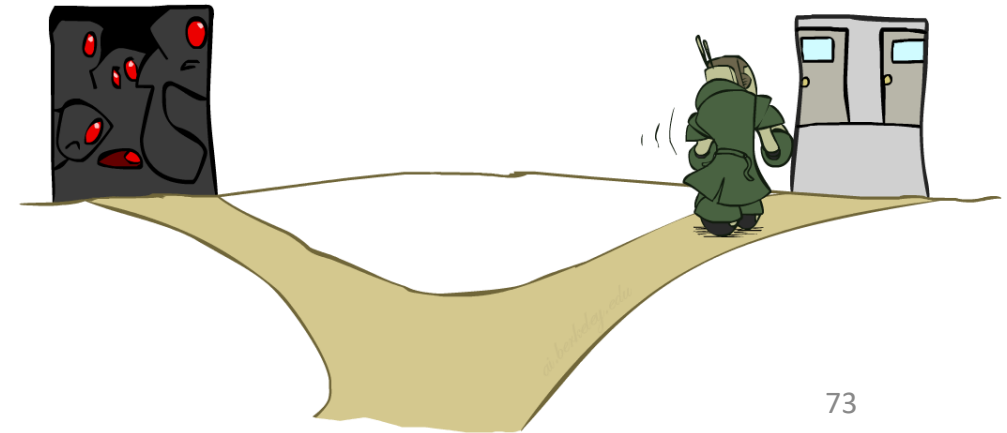
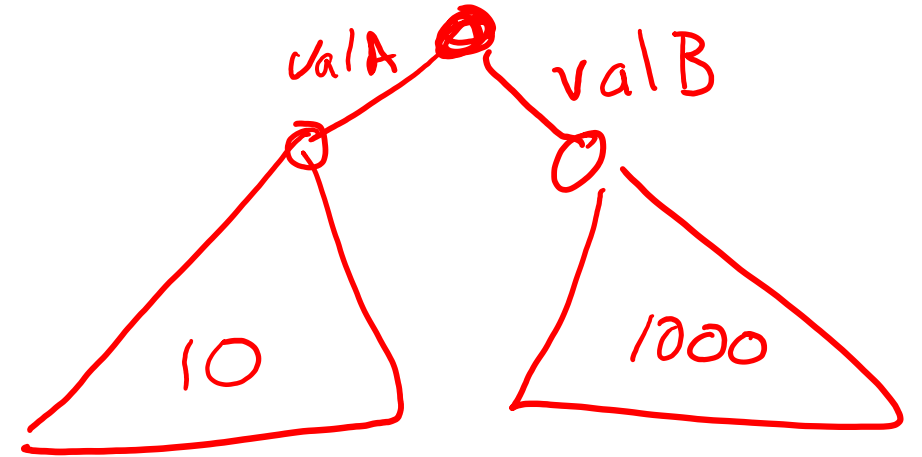
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
 - Given a choice of variable, choose the *least constraining value*
 - i.e., the one that rules out the fewest values in the remaining variables
 - Note that it may take some computation to determine this! (E.g., rerunning filtering)



Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
 - Given a choice of variable, choose the *least constraining value*
 - i.e., the one that rules out the fewest values in the remaining variables
 - Note that it may take some computation to determine this! (E.g., rerunning filtering)
- Why least rather than most?
- Combining these ordering ideas makes 1000 queens feasible

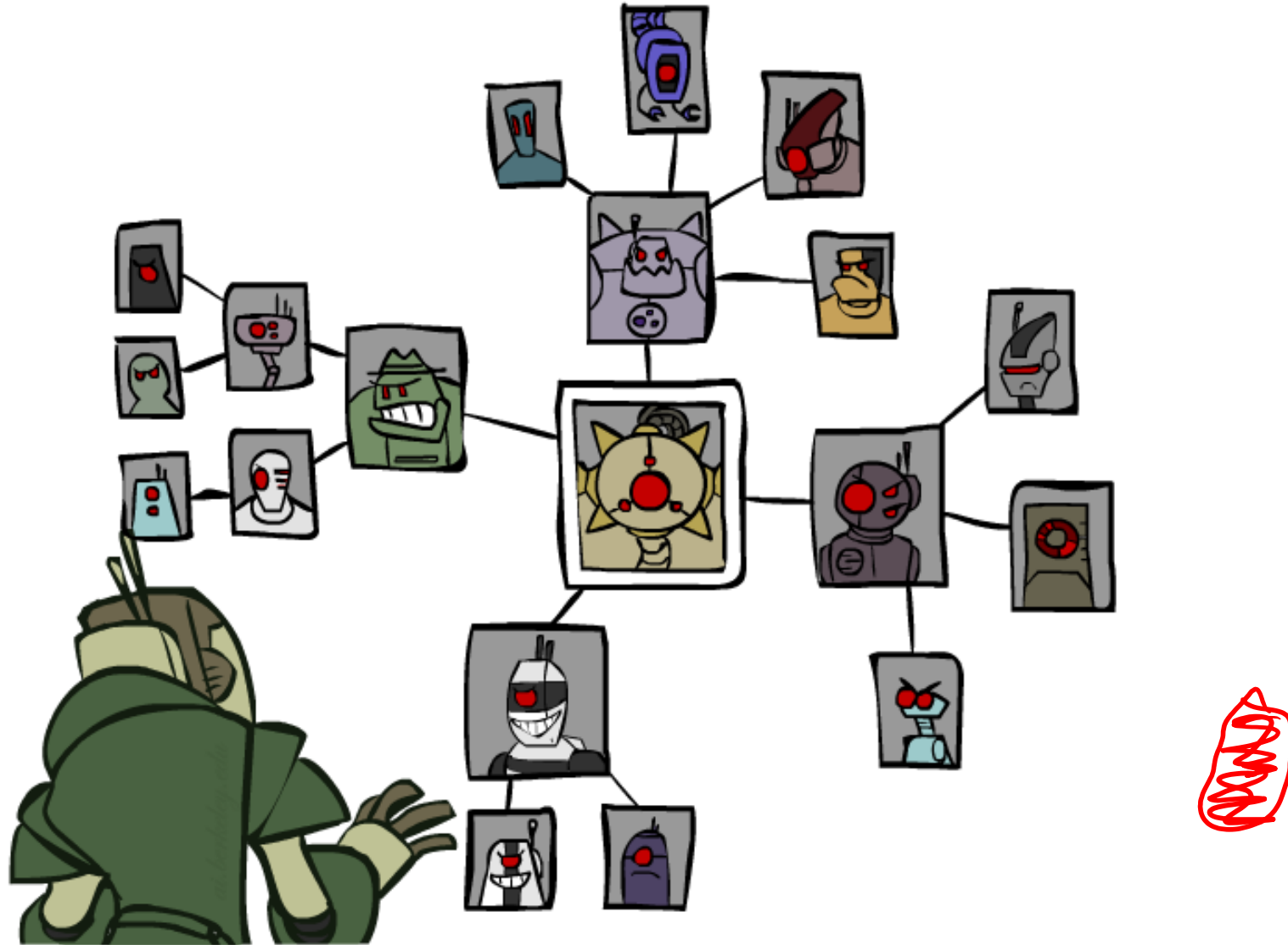


Demo – Coloring with a Complex Graph

Compare

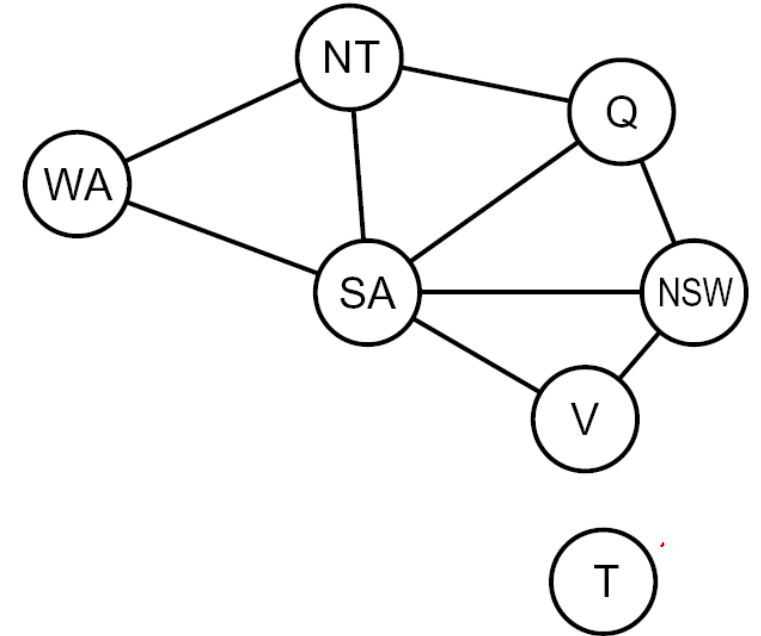
- Backtracking with Forward Checking
- Backtracking with AC-3
- Backtracking + Forward Checking + Minimum Remaining Values (MRV)
- Backtracking + AC-3 + MRV + LCV

Structure

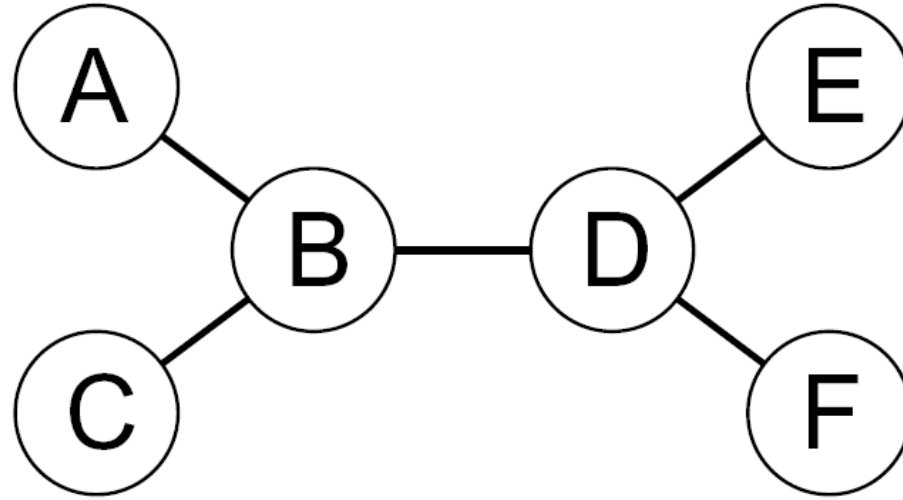


Problem Structure

- For general CSPs, worst-case complexity with backtracking algorithm is $O(d^n)$
- When the problem has special structure, we can often solve the problem more efficiently
- Special Structure 1: Independent subproblems
 - Example: Tasmania and mainland do not interact
 - Connected components of constraint graph
 - Suppose a graph of n variables can be broken into subproblems, each of only c variables:
 - Worst-case complexity is $O((n/c)(d^c))$, linear in n
 - E.g., $n = 80$, $d = 2$, $c = 20$
 - $2^{80} = 4$ billion years at 10 million nodes/sec
 - $(4)(2^{20}) = 0.4$ seconds at 10 million nodes/sec



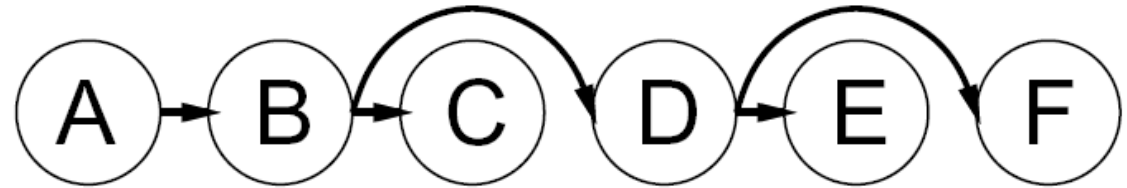
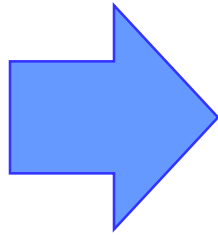
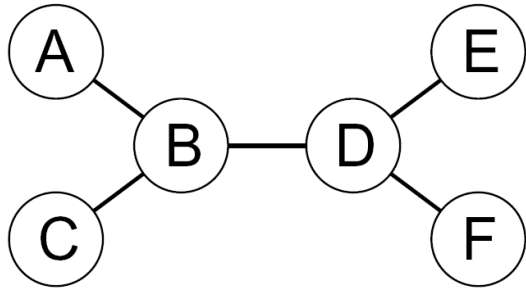
Tree-Structured CSPs



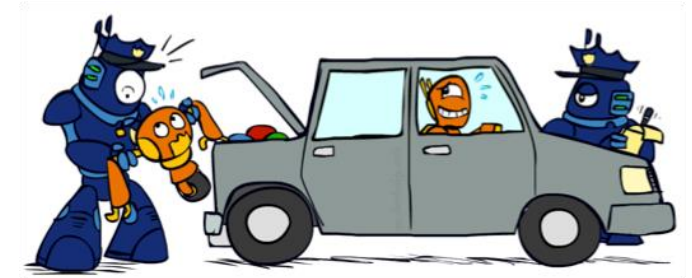
- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(nd^2)$ time
 - Much smaller compare to general CSPs, where worst-case time is $O(d^n)$
 - How?

Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
 - Order: Choose a root variable, order variables so that parents precede children

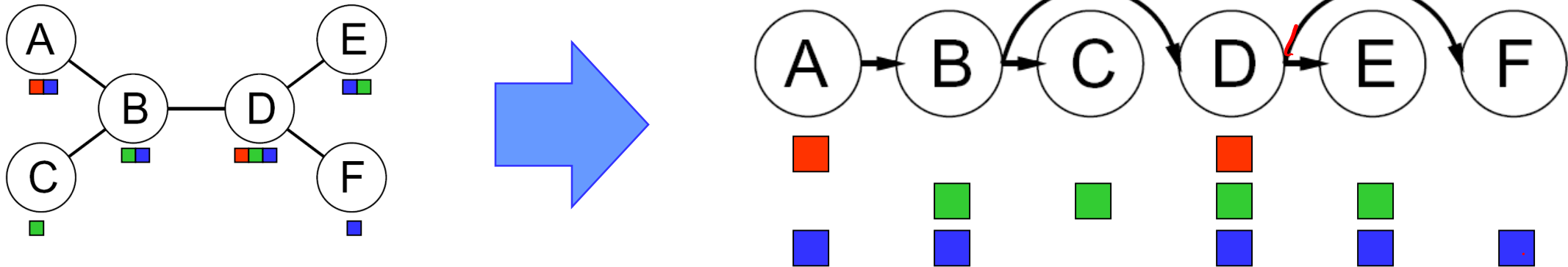


Tree-Structured CSPs



- Algorithm for tree-structured CSPs:

- Order: Choose a root variable, order variables so that parents precede children



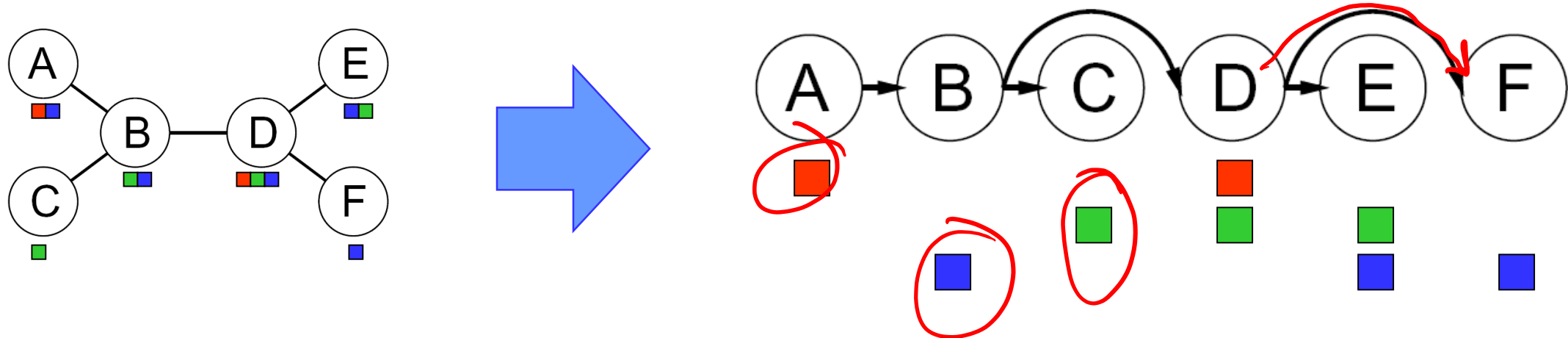
- Remove backward: For $i = n : 2$, apply $\text{RemoveInconsistent}(\text{Parent}(X_i), X_i)$

Tree-Structured CSPs



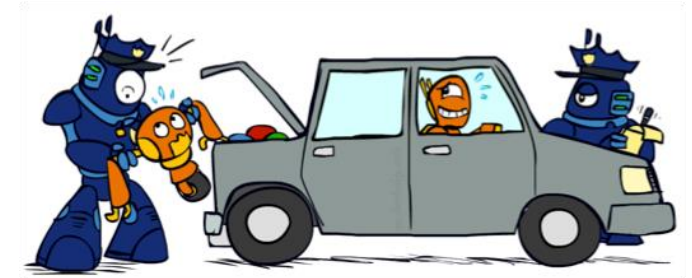
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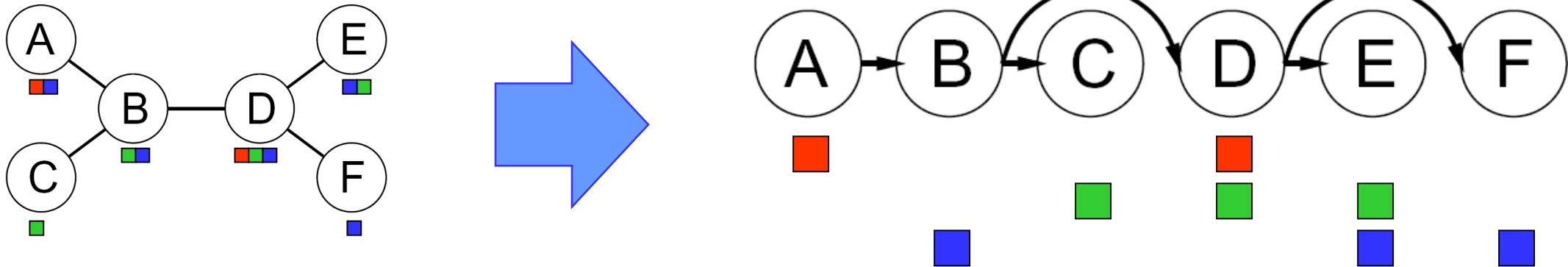


- Remove backward: For $i = n : 2$, apply $\text{RemoveInconsistent}(\text{Parent}(X_i), X_i)$
 - Assign forward: For $i = 1 : n$, assign X_i consistently with $\text{Parent}(X_i)$
- Runtime: $O(nd^2)$ (why?)
 - Can always find a solution when there is one (why?)

Tree-Structured CSPs



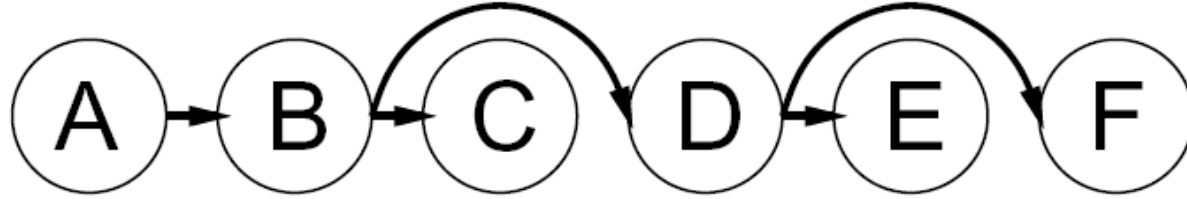
- Algorithm for tree-structured CSPs:
 - Order: Choose a root variable, order variables so that parents precede children



- Remove backward: For $i = n : 2$, apply $\text{RemoveInconsistent}(\text{Parent}(X_i), X_i)$
- Assign forward: For $i = 1 : n$, assign X_i consistently with $\text{Parent}(X_i)$
- Runtime: $O(nd^2)$ (why?)
 - Remove backward $O(nd^2) : O(d^2)$ per arc and $O(n)$ arcs
 - Assign forward $O(nd) : O(d)$ per node and $O(n)$ nodes
- Can always find a solution when there is one (why?)

Tree-Structured CSPs

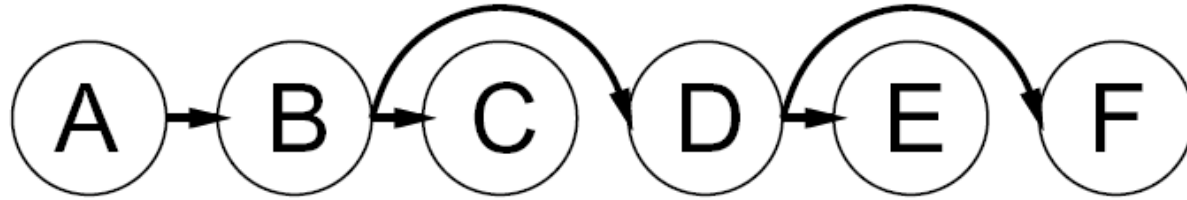
Remove backward: For $i = n : 2$, apply $\text{RemoveInconsistent}(\text{Parent}(X_i), X_i)$



- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: During backward pass, every node except the root node was “visited” once.
 - a. $\text{Parent}(X_i) \rightarrow X_i$ was made consistent when X_i was visited
 - b. After that, $\text{Parent}(X_i) \rightarrow X_i$ kept consistent until the end of the backward pass.

Tree-Structured CSPs

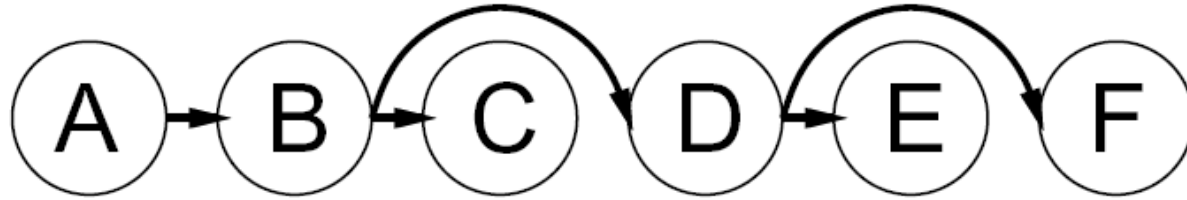
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- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: During backward pass, every node except the root node was “visited” once.
 - a. $\text{Parent}(X_i) \rightarrow X_i$ was made consistent when X_i was visited
 - When X_i was visited, we enforced arc consistency of $\text{Parent}(X_i) \rightarrow X_i$ by reducing the domain of $\text{Parent}(X_i)$. By definition, for every value in the reduced domain of $\text{Parent}(X_i)$, there was some x in the domain of X_i which could be assigned without violating the constraint involving $\text{Parent}(X_i)$ and X_i
 - b. After that, $\text{Parent}(X_i) \rightarrow X_i$ kept consistent until the end of the backward pass.
 - Domain of X_i would not have been reduced after X_i is visited because X_i ’s children were visited before X_i . Domain of $\text{Parent}(X_i)$ could have been reduced further. Arc consistency would still hold by definition.

Tree-Structured CSPs

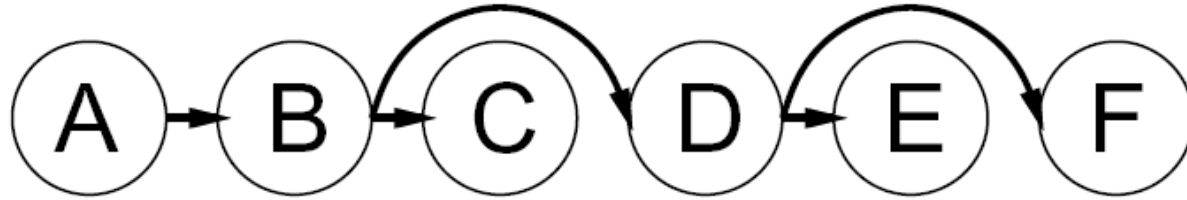
Assign forward: For $i = 1 : n$, assign X_i consistently with $\text{Parent}(X_i)$



- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack.
- Proof: Follow the backtracking algorithm (on the reduced domains and with the same ordering). Induction on position.

Tree-Structured CSPs

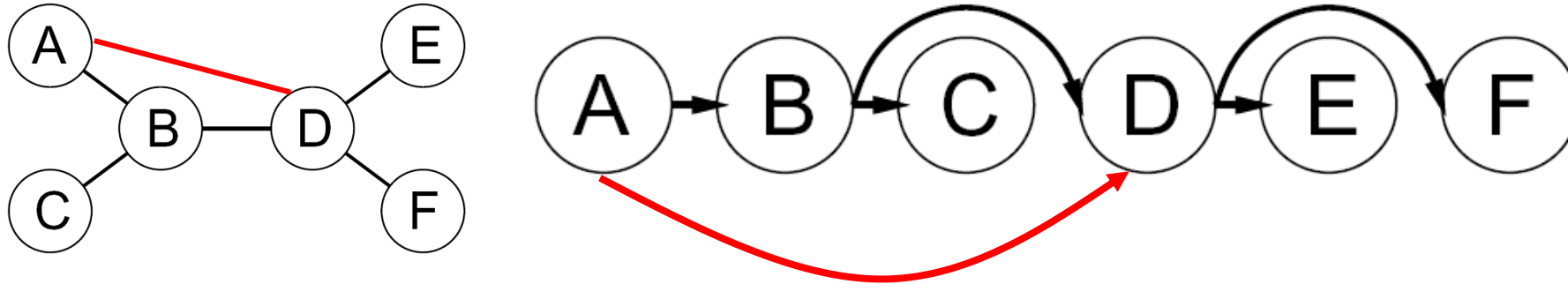
Assign forward: For $i = 1 : n$, assign X_i consistently with $\text{Parent}(X_i)$



- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack.
- Proof: Follow the backtracking algorithm (on the reduced domains and with the same ordering). Induction on position. Suppose we have successfully reached node X_i . In the current step, the potential failure can only be caused by the constraint between X_i and $\text{Parent}(X_i)$, since all other variables that are in a same constraint of X_i have not assigned a value yet. Due to the arc consistency of $\text{Parent}(X_i) \rightarrow X_i$, there exists a value x in the domain of X_i that does not violate the constraint. So we can successfully assign value to X_i and go to the next node. By induction, we can successfully assign a value to a variable in each step of the algorithm. A solution is found in the end.

Tree-Structured CSPs

- Why doesn't this algorithm work with cycles in the constraint graph?

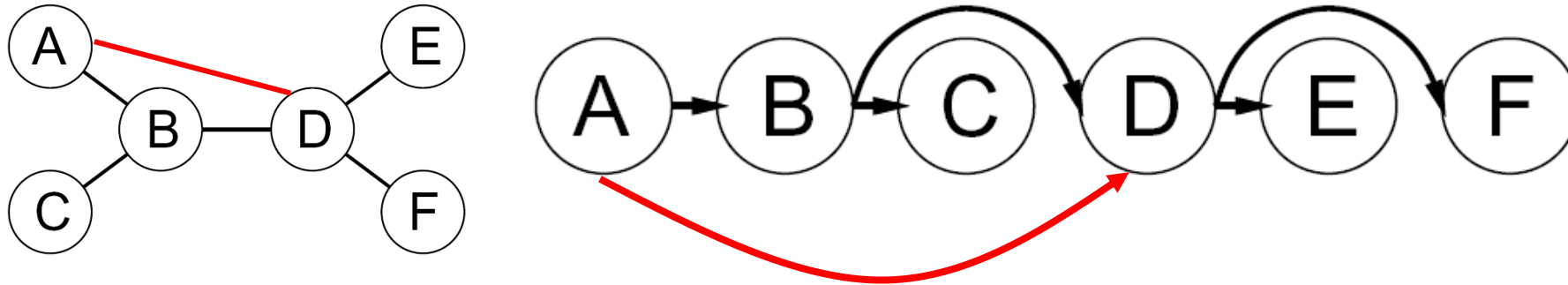


We can still apply the algorithm (choose an arbitrary order and draw “forward” arcs). For remove backward, what would happen?

For assign forward, what would happen?

Tree-Structured CSPs

- Why doesn't this algorithm work with cycles in the constraint graph?



We can still apply the algorithm (choose an arbitrary order and draw “forward” arcs).
For remove backward, what would happen?

We can enforce all arcs pointing to X_i when X_i is visited. The complexity is $O(n^2 d^2)$.

After backward pass, the reduced domains do not exclude any solution and all the forward arcs are consistent

For assign forward, what would happen?

In a step of assigning values, we may encounter failure because we need to make sure the constraints involving the current node and any parent node is satisfied, which could be impossible. Therefore, we may need to backtrack.

How to deal with non-binary CSPs?

- Variables:

$F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$

- Domains:

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Constraints:

$\text{alldiff}(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$

\dots

$$\begin{array}{r} T \ W \ O \\ + \ T \ W \ O \\ \hline F \ O \ U \ R \end{array}$$

Constraint graph for non-binary CSPs

- Variable nodes: nodes to represent the variables
- Constraint nodes: auxiliary nodes to represent the constraints
- Edges: connects a constraint node and its corresponding variables

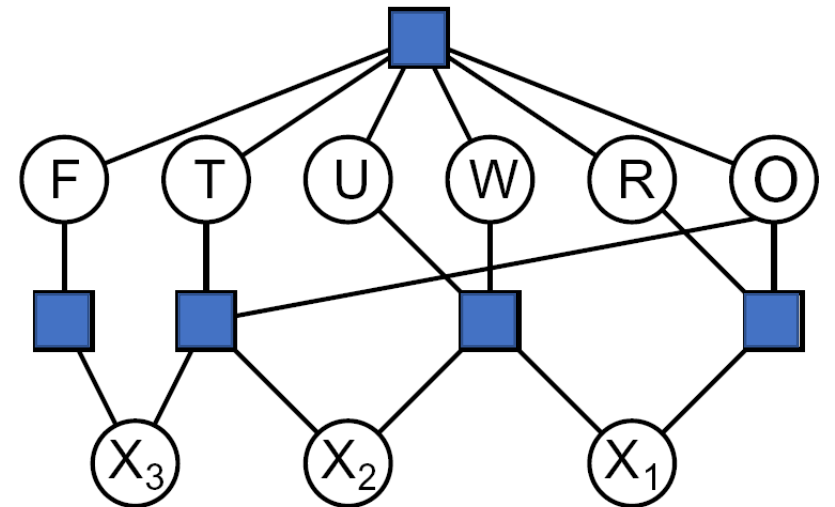
$$\begin{array}{r} \text{T} \text{ W} \text{ O} \\ + \text{T} \text{ W} \text{ O} \\ \hline \text{F} \text{ O} \text{ U} \text{ R} \end{array}$$

Constraints:

$$\text{alldiff}(F, T, U, W, R, O)$$

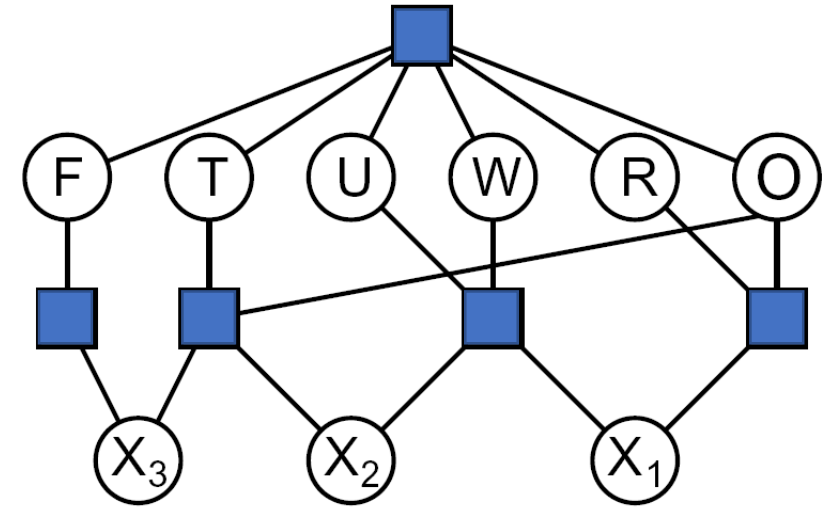
$$O + O = R + 10 \cdot X_1$$

...



Solve non-binary CSPs

- Naïve search?
 - Yes!
- Backtracking?
 - Yes!
- Forward Checking?
 - Need to generalize the original FC operation
 - (nFC0) After a variable is assigned a value, find all constraints with only one unassigned variable and cross off values of that unassigned variable which violate the constraint
 - There exist other ways to do generalized forward checking

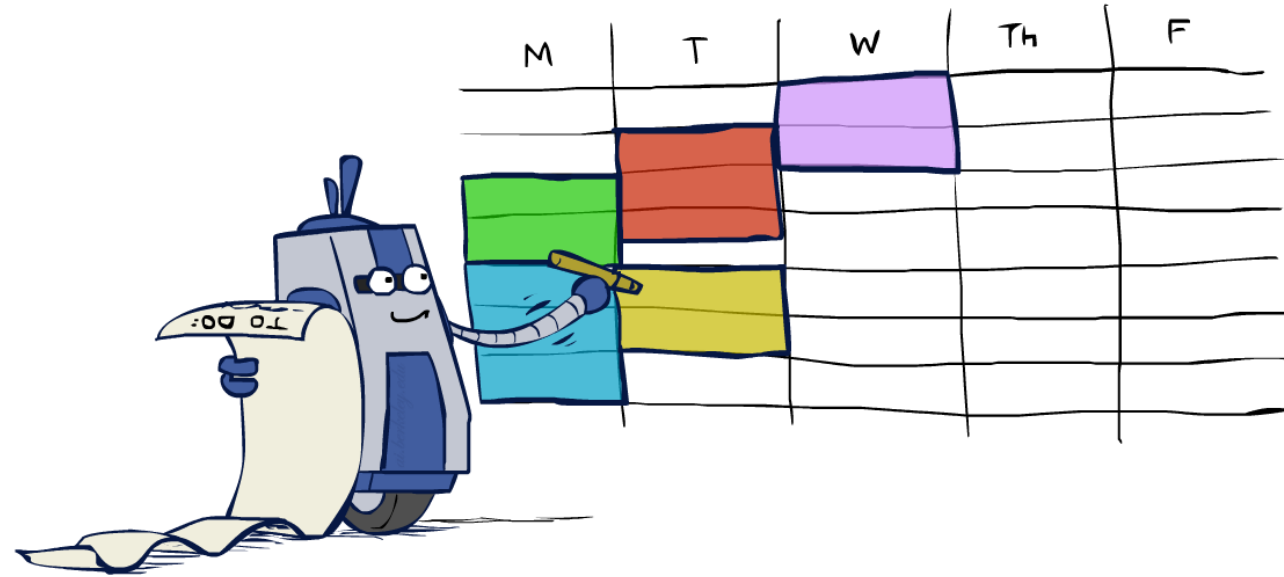


Solve non-binary CSPs

- (Bonus material, not required)
- AC-3? Need to generalize the definition of AC and enforcement of AC
- Generalized arc-consistency (GAC)
 - A non-binary constraint is GAC iff for **every** value for a variable there **exist** consistent value combinations for **all other variables** in the constraint
 - Reduced to AC for binary constraints
- Enforcing GAC
 - Simple schema: enumerate value combination for all other variables
 - $O(d^k)$ on k -ary constraint on variables with domains of size d
- There are other algorithms for non-binary constraint propagation, e.g., (i,j)-consistency [Freuder, JACM 85]

Summary: CSPs

- CSPs are a special kind of search problem:
 - States are partial assignments
 - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
 - Ordering
 - Filtering
 - Structure



Learning Objectives

- Describe definition of **CSP problems** and its connection with general search problems
- Formulate a real-world problem as a CSP
- Describe and implement **backtracking algorithm**
- Define **arc consistency**
- Describe and implement **forward checking** and **AC-3**
- Explain the differences between **MRV** and **LCV heuristics**
- Understand the complexity of general binary CSP and **tree-structured binary CSP**

Additional Resources (Not required)

- References

- Zhang, Yuanlin, and Roland HC Yap. "Making AC-3 an optimal algorithm." In *IJCAI*, vol. 1, pp. 316-321. 2001.
- Freuder, Eugene C. "A sufficient condition for backtrack-bounded search." *Journal of the ACM (JACM)* 32, no. 4 (1985): 755-761.