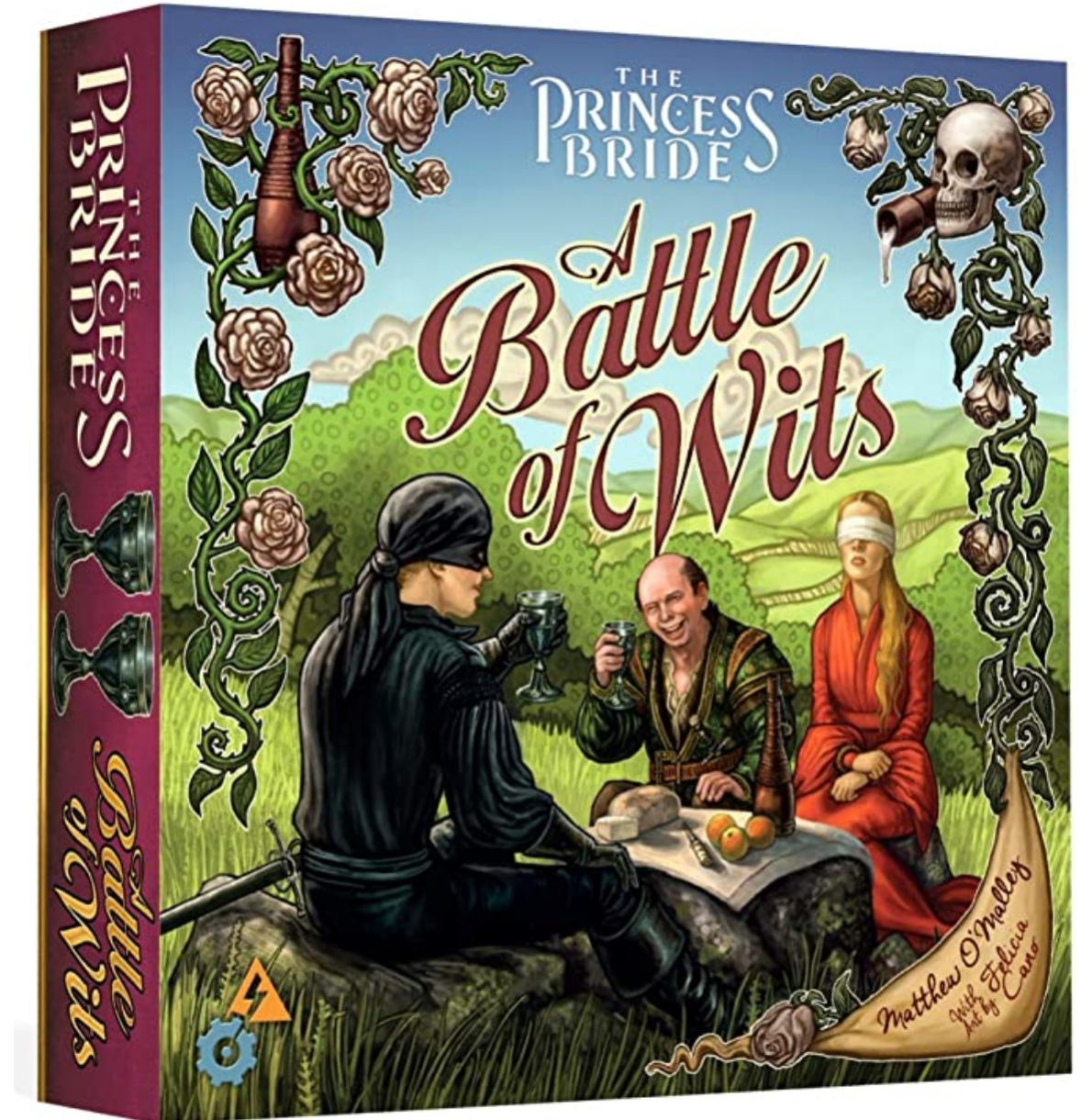


A Battle of Wits

The Princess Bride

<https://www.youtube.com/watch?v=EZSx3zNZOaU>



<https://www.amazon.com/Princess-Bride-Battle-Wits-3rd>

Rock, Paper, Scissors, Lizard, Spock

CBS, Big Bang Theory

<https://www.youtube.com/watch?v=iSHPVCBsnLw>

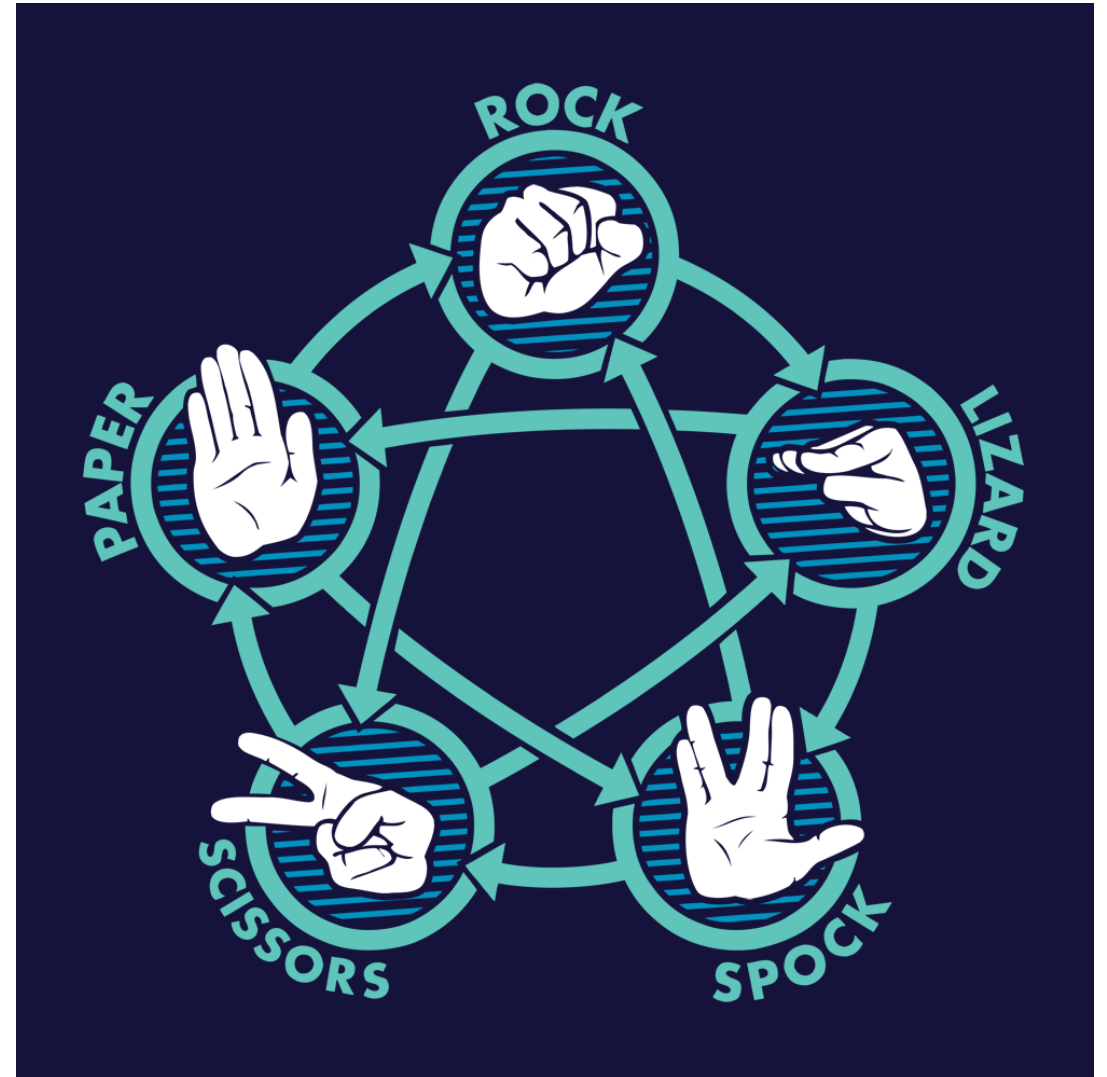
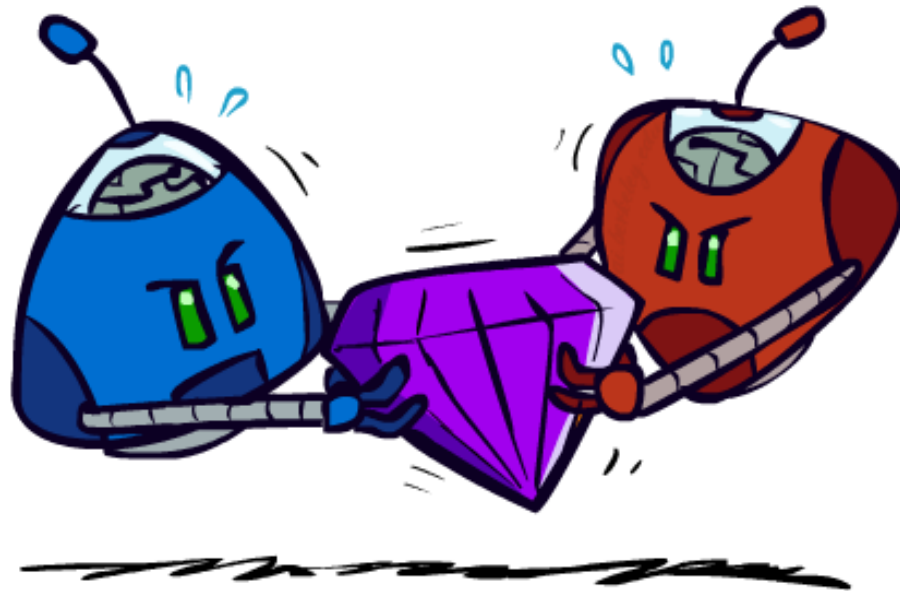


Image credit: <https://www.snorgtees.com/rock-paper-scissors-lizard-spock>

AI: Representation and Problem Solving

Game Theory: Equilibrium



Instructor: Pat Virtue

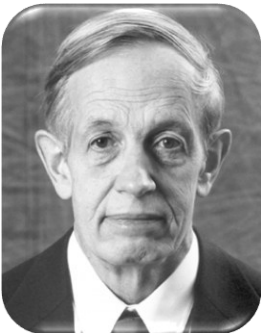
Slide credits: CMU AI, Fei Fang

From Games to Game Theory



The study of mathematical models of conflict and cooperation between intelligent decision makers

Used in economics, political science, etc

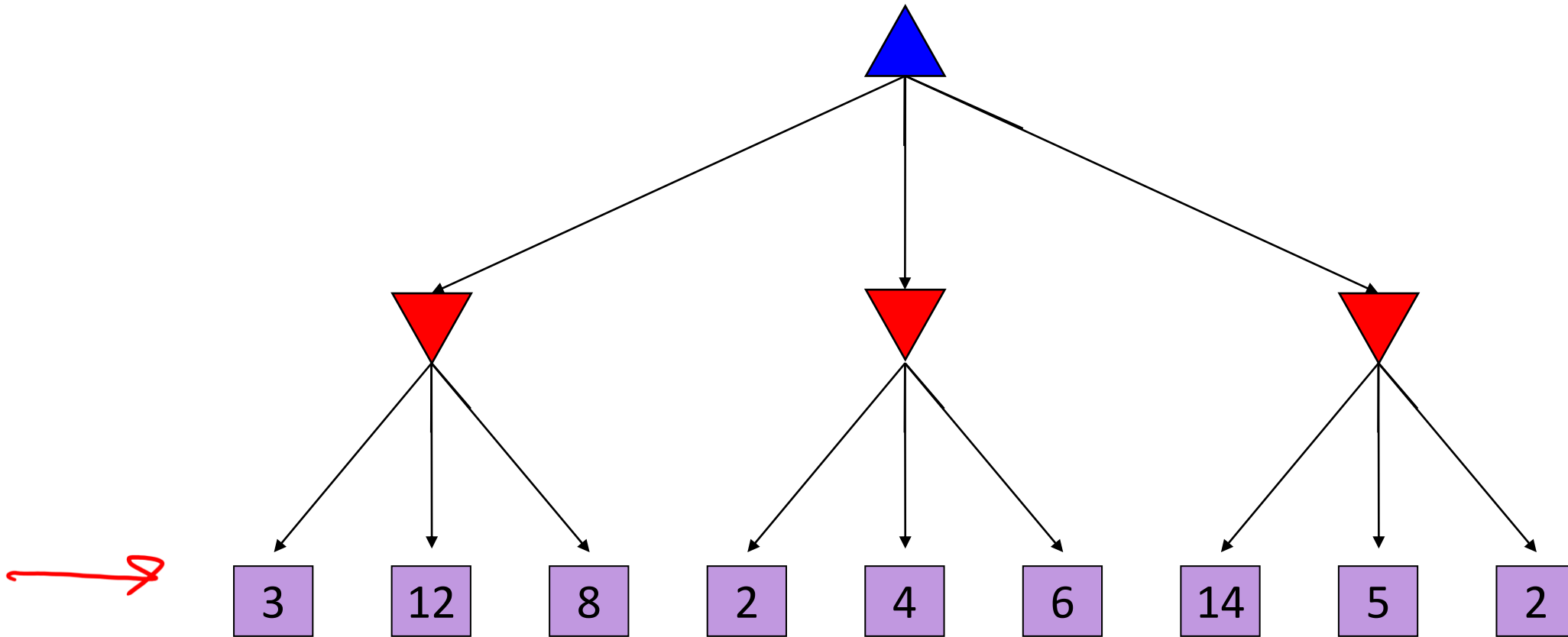


John Nash

Winner of Nobel Memorial Prize in Economic Sciences

Recall: Adversarial Search

Zero-sum, perfect information, two player games with turn-taking moves



Simultaneous Play and Payoff Matrices

Rock-Paper-Scissors (RPS)

- Rock beats Scissors
- Scissors beats Paper
- Paper beats Rock

Player 1	Player 2		
	Rock	Paper	Scissors
	Rock	0, 0	-1, 1
	Paper	1, -1	0, 0
	Scissors	-1, 1	1, -1

2-player normal-form game with finite set of actions taken simultaneously

represented in a (bi)matrix

Player 1 is row player (typically first number)

Player 2 is column player (typically second number)

$u_1(R, P)$

Classical Games and Payoff Matrices

Football vs Concert (FvsC)

- ~~Historically known as Battle of Sexes~~
- If football together: Alex 😊😊, Berry 😊
- If concert together: Alex 😊, Berry 😊😊
- If not together: Alex 😞, Berry 😞

Fill in the payoff matrix

		Berry	
		Football	Concert
Alex	Football	2 1	-1 -1
	Concert	-1 -1	1 2

Classical Games and Payoff Matrices

Prisoner's Dilemma (PD)

- If both Cooperate: 1 year in jail each
- If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
- If both Defect: 2 years in jail each
- Let's play!

Player 2

Player 1		Cooperate	Defect
	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2



Variation: Split or Steal



Normal-Form Games

A game in normal form consists of the following elements

- Set of players
- Set of actions for each player
- Payoffs / Utility functions
 - Determines the utility for each player given the actions chosen by all players (referred to as action profile)
- Bimatrix game is special case: two players, finite action sets

Players move simultaneously and the game ends immediately afterwards

What are the players, set of actions and utility functions of Football vs Concert (FvsC) game?

Strategy: Rock, Paper, Scissors

Design an AI to play Rock-Paper-Scissors for T rounds

If $T=1$, $\text{roundID}=1$, what action does your AI choose?

```
function playRPS(roundID,T)
```

Input: T , $\text{roundID} \in \{1..T\}$

Output: action $a \in \{\text{Rock, Paper, Scissors}\}$

pure

return Rock

mixed

return rand_choice([Rock, Pap, Sci])

```
Return a
```

Strategy

Pure strategy: choose an action deterministically

Mixed strategy: choose actions according to a probability distribution

- Notation: $s = (0.3, 0.7, 0)$
- Support: set of actions chosen with non-zero probability

Notation Alert! We use s to represent strategy here (not states)

Does your AI play a deterministic strategy or a mixed strategy?

What is the support size of your AI's strategy?

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Zero-sum vs General-sum

Zero-sum Game

- No matter what actions are chosen by the players, the utilities for all the players sum up to zero or a constant

General-sum Game

- The sum of utilities of all the players is not a constant

Which ones are general-sum games?

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0,0	-1,1	1,-1
	Paper	1,-1	0,0	-1,1
	Scissor	-1,1	1,-1	0,0

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Expected Utility

$$s = s_1, s_2$$

Notation Alert!

Use a, s, u to represent action, strategy, utility of a player

Use **a, s, u** to represent action, strategy, utility profile

Given the strategies of all players,

Expected Utility for player i $u_i = u_i(s_1, s_2)$

$$\sum_{\mathbf{a}} \text{Prob}(\text{action profile } \mathbf{a}) \times \text{Utility for player } i \text{ in } \mathbf{a} = (\text{Football}, \text{Football})$$

Can skip action profiles with probability 0 or utility 0

If Alex's strategy $s_A = (\frac{1}{2}, \frac{1}{2})$, Berry's strategy $s_B = (1, 0)$

What is the probability of action profile $\mathbf{a} = (\text{Concert}, \text{Football})$?

$$\frac{1}{2} \cdot 1.0$$

What is Alex's utility in this action profile?

Alex $\frac{1}{2}$
 $\frac{1}{2}$

Berry

1 0

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Poll 1

$$\sum_a P(a) u_1(a)$$

$$P(R,R)u_1(R,R) +$$

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $s_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, how many non-zero terms need to be added up when computing the expected utility for player 1?

- ~~A. 9~~ — Take 2
- B. 6
- C. 4
- D. 7
- E. 3

↓ Player 2

		<u>0</u>	$\frac{1}{2}$	$\frac{1}{2}$
		Rock	Paper	Scissors
Player 1 $\frac{1}{3}$ $\frac{2}{3}$ <u>0</u>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Poll 2

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $s_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$,
what is the utility of player 1?

- A. -1
- B. -1/3
- C. 0
- D. 2/3
- E. 1

		Player 2		
		0	1/2	1/2
Player 1		Rock	Paper	Scissors
	1/3	0, 0	-1, 1	1, -1
	2/3	1, -1	0, 0	-1, 1
	0	-1, 1	1, -1	0, 0

Poll 2

In Rock-Paper-Scissors, if $s_1 = (\frac{1}{3}, \frac{2}{3}, 0)$, $s_2 = (0, \frac{1}{2}, \frac{1}{2})$, how many non-zero terms need to be added up when computing the expected utility for player 1?

- A. 9
- B. 6
- C. 4
- D. 7
- E. 3

$$u_1 = \begin{array}{l} 0 \times \frac{1}{3} \times 0 + (-1) \times \frac{1}{3} \times \frac{1}{2} + 1 \times \frac{1}{3} \times \frac{1}{2} \\ + 1 \times \frac{2}{3} \times 0 + 0 \times \frac{2}{3} \times \frac{1}{2} + (-1) \times \frac{2}{3} \times \frac{1}{2} \\ + (-1) \times 0 \times 0 + 1 \times 0 \times \frac{1}{2} + 0 \times 0 \times \frac{1}{2} = -\frac{1}{3} \end{array}$$

Player 2

Player 1		Rock	Paper	Scissors
	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Best Response

Best Response (BR): Given the strategies or actions of all players but player i (denoted as s_{-i} or a_{-i}), Player i 's best response to s_{-i} or a_{-i} is the set of actions or strategies of player i that can lead to the highest expected utility for player i

In RPS, what is Player 1's best response to Rock (i.e., assuming Player 2 plays Rock)?

$a = \text{Paper}$

$s = (0, 1, 0, 0)$

In Prisoner's Dilemma, what is Player 1's best response to Cooperate?

What is Player 1's best response to Defect?

Defect →

		Player 2	
Player 1		Cooperate	Defect
	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

← Defect

Best Response

Best Response (BR): Given the strategies or actions of all players but player i (denoted as s_{-i} or a_{-i}), Player i 's best response to s_{-i} or a_{-i} is the set of actions or strategies of player i that can lead to the highest expected utility for player i

What is Alex's best response to Berry's mixed strategy $s_B = \left(\frac{1}{2}, \frac{1}{2}\right)$?

$$u_{\text{Alex}}(F, s_B) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0$$

$$u_{\text{Alex}}(C, s_B) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1$$

		Berry	
		$\frac{1}{2}$ Football	$\frac{1}{2}$ Concert
Alex	Football	<u>2,1</u>	<u>0,0</u>
	Concert	<u>0,0</u>	<u>1,2</u>

Poll 3

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which actions or strategies are player 2's best responses to s_1 ?

Select all that apply

- A. Rock
- B. Paper
- C. Scissors
- D. Lizard

E. $s_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

F. $s_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$u_2(s_1, s_2)$

$u_2(s, R)$

Player 2

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Player 1

$\frac{1}{3}$
 $\frac{1}{3}$
 $\frac{1}{3}$

Best Response

Theorem 1 (Nash 1951): A mixed strategy is a BR iff all actions in the support are a BR

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Dominance

s_i and s_i' are two strategies for player i

s_i **strictly** dominates s_i' if s_i is **always better** than s_i' , no matter what strategies are chosen by other players

s_i **strictly** dominates s_i' if

$$u_i(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i} \quad \text{always better}$$

s_i **very weakly** dominates s_i' if

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i} \quad \text{never worse}$$

s_i **weakly** dominates s_i' if

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i} \quad \text{never worse and}$$

and $\exists \mathbf{s}_{-i}, u_i(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$ **sometimes better**

Dominance

Can you find any dominance relationships between the pure strategies in these games?

$D > C$

Player 2

	Rock	Paper	Scissors
Player 1 → Rock	<u>0,0</u>	<u>-1,1</u>	<u>1,-1</u>
Paper	<u>1,-1</u>	<u>0,0</u>	<u>-1,1</u>
Scissor	-1,1	1,-1	0,0

Player 2

	Cooperate	Defect
Player 1 → Cooperate	<u>-1,-1</u>	<u>-3,0</u>
Defect	0,-3	-2,-2

Berry

	Football	Concert
Alex → Football	<u>2,1</u>	<u>0,0</u>
Concert	0,0	1,2

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium



- Nash Equilibrium
- (Minimax strategy)
- (Maximin strategy)
- (Stackelberg Equilibrium)

Dominant Strategy

→ A strategy could be (always better / never worse / never worse and sometimes better) than any other strategy

s_i is a (strictly/very weakly/weakly) dominant strategy if it (strictly/very weakly/weakly) dominates s'_i , $\forall s'_i \in S_i \setminus \{s_i\}$

Focus on single player's strategy

Doesn't always exist

Is there a strictly dominant strategy for player 1 in PD?



		Player 2	
		Cooperate	Defect
Player 1	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Dominant Strategy Equilibrium

Sometimes called dominant strategy solution

Every player plays a dominant strategy

Focus on strategy profile for all players

Doesn't always exist

What is the dominant strategy equilibrium for PD?

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- (Minimax strategy)
- (Maximin strategy)
- (Stackelberg Equilibrium)

Nash Equilibrium

Nash Equilibrium (NE)

- Every player's strategy is a best response to others' strategy profile
- In other words, one cannot gain by unilateral deviation
- Pure Strategy Nash Equilibrium (PSNE)
 - $a_i \in BR(\mathbf{a}_{-i}), \forall i$
- Mixed Strategy Nash Equilibrium
 - At least one player use a randomized strategy
 - $s_i \in BR(\mathbf{s}_{-i}), \forall i$

Nash Equilibrium

What are the PSNEs in these games?

What is the mixed strategy NE in RPS?

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0,0	-1,1	1,-1
	Paper	1,-1	0,0	-1,1
	Scissor	-1,1	1,-1	0,0

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Handwritten annotations: A blue circle around 'Player 1' with a red arrow pointing to it. A red arrow points down to the 'Defect' column. A red box highlights the bottom-right cell (-2,-2). A blue arrow points from -3,0 to -2,-2 with the word 'No' written next to it. A green arrow points from 0,-3 to -2,-2 with the word 'No' written below it.

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Handwritten annotations: A red box highlights the top-left cell (2,1). A blue arrow points from 2,1 to 0,0 with the word 'No' written next to it. A green arrow points from 0,0 to 1,2 with the word 'No' written below it. A red box highlights the bottom-right cell (1,2). A blue arrow points from 0,0 to 1,2 with the word 'No' written next to it. A green arrow points from 2,1 to 1,2 with the word 'No' written above it.

Nash Equilibrium

Theorem 2 (Nash 1951): NE always exists in finite games

- Finite number of players, finite number of actions
- NE: can be pure or mixed
- Proof: Through Brouwer's fixed point theorem

Find PSNE

Find pure strategy Nash Equilibrium (PSNE)

- Enumerate all action profile
- For each action profile, check if it is NE
 - For each player, check other available actions to see if he should deviate
- Other approaches?

		Player 2		
		L	C	R
Player 1	U	10, 3	1, 5	5, 4
	M	3, 1	2, 4	5, 2
	D	0, 10	1, 8	7, 0

Find PSNE

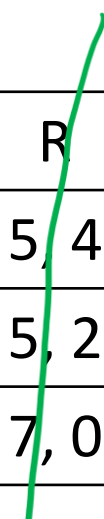
A strictly dominated strategy is one that is always worse than **some other strategy**

Strictly dominated strategies cannot be part of an NE **Why?**

Which are the strictly dominated strategies for player 1?

How about player 2?

		Player 2		
		L	C	R
Player 1	U	10, 3	1, 5	5, 4
	M	3, 1	2, 4	5, 2
	D	0, 10	1, 8	7, 0



Find PSNE through Iterative Removal

Remove strictly dominated actions (pure strategies) and then find PSNE in the remaining game

Can have new strictly dominated actions in the remaining game!

Repeat the process until no actions can be removed

This is the Iterative Removal algorithm (also known as Iterative Elimination of Strictly Dominated Strategies)

Find PSNE in this game using iterative removal

		Player 2		
Player 1		L	C	R
	U	10, 3	1, 5	5, 4
	M	3, 1	2, 4	5, 2
	D	0, 10	1, 8	7, 0

Find Mixed Strategy Nash Equilibrium

Can we still apply iterative removal?

- Yes! The removed strategies cannot be part of any NE
- You can always apply iterative removal first



Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

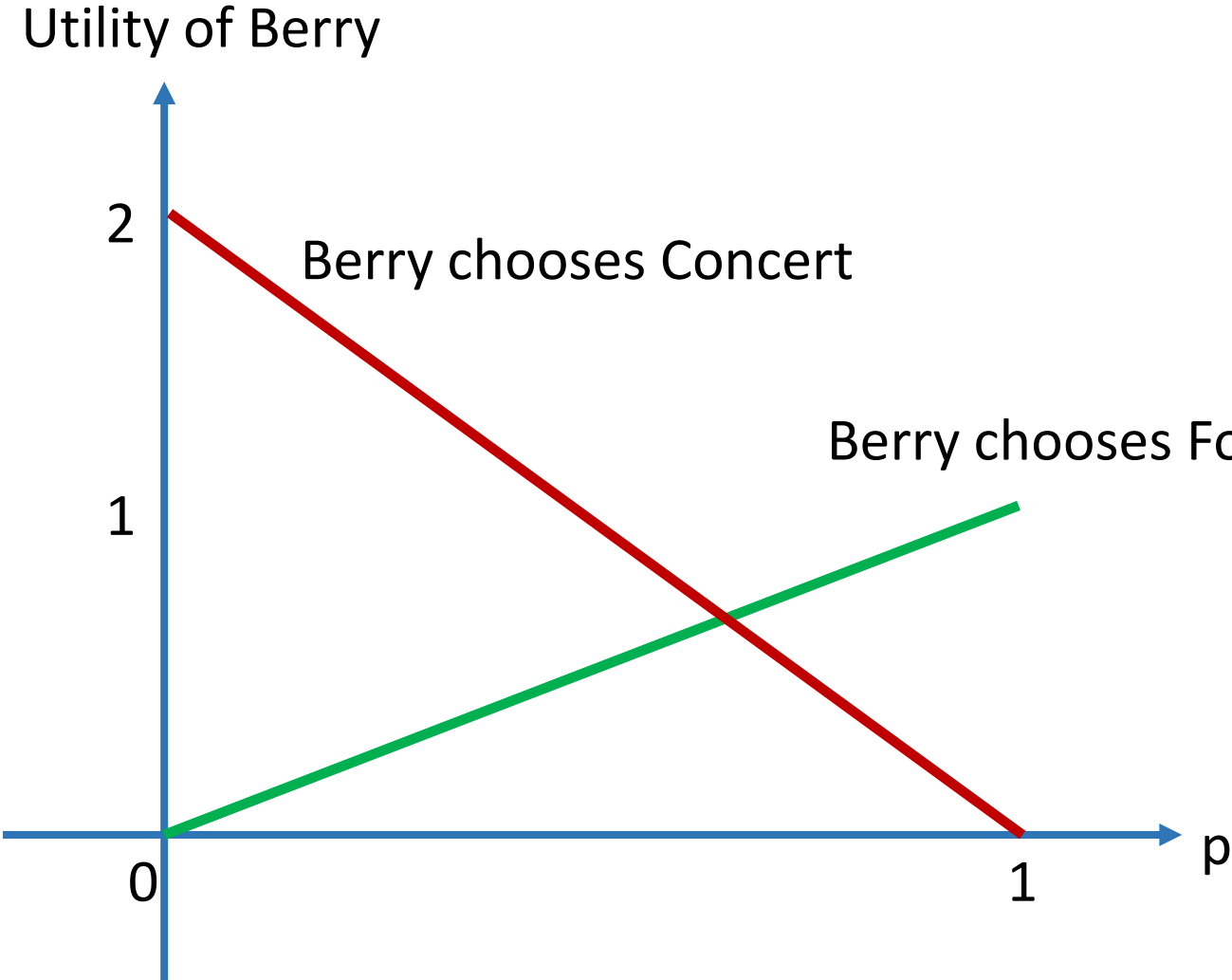
If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q ?

Visualizing Equilibrium

Alex wants to choose p such that Berry doesn't want to deviate from their strategy

Berry wants the most reward they can get, so they will deviate if one strategy has more utility than another

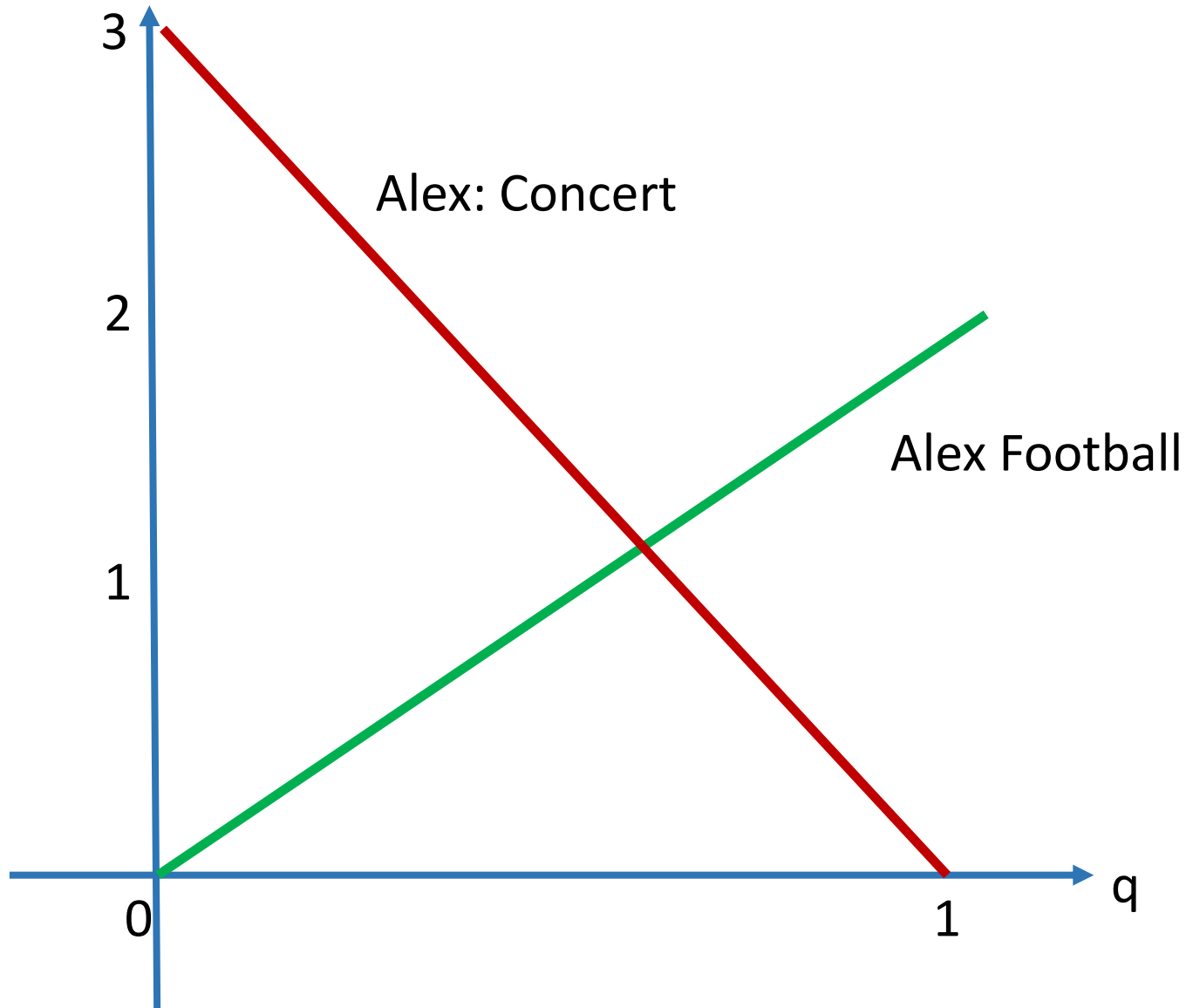
The only way these two conditions is met is if we choose the p such that any strategy Berry picks will yield equal utility for Berry



		Berry	
		Football	Concert
p 1-p Alex	Football	2,1	0,0
	Concert	0,0	3,2

Visualizing Equilibrium

Utility of Alex



Berry wants to choose q such that Alex doesn't want to deviate from their strategy

Alex wants the most reward they can get, so they will deviate if one strategy has more utility than another

The only way these two conditions is met is if we choose the q such that any strategy Alex picks will yield equal utility for Alex

		Berry	
		q	$1-q$
Alex	Football	2,1	0,0
	Concert	0,0	3,2

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q ?

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q ?

$$u_A(F, s_B) = u_A(C, s_B)$$

$$u_B(s_A, F) = u_B(s_A, C)$$

Why? Remember Theorem 1: A mixed strategy is BR iff all actions in the support are BR.

So...if $s_A \in BR(s_B)$, then $F \in BR(s_B)$ and $C \in BR(s_B)$

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q ?

$$u_A(F, s_B) = u_A(C, s_B)$$

$$u_B(s_A, C) = u_B(s_A, F)$$

$$u_A(F, s_B) = 2 * q + 0 * (1 - q) = u_A(C, s_B) = 0 * q + 1 * (1 - q)$$

$$\text{So } 2q = 1 - q, \text{ we get } q = \frac{1}{3}$$

$$\text{Similarly, } u_B(s_A, F) = 1 * p + 0 * (1 - p) = u_B(s_A, C) = 0 * p + 2 * (1 - p)$$

$$\text{So } p = 2(1 - p), \text{ we get } p = \frac{2}{3}$$

Poll 4

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE of the game, which equations should p and q satisfy?

- A. $2q = 3(1 - q)$
- B. $2p = 3(1 - p)$
- C. $q = 2(1 - q)$
- D. $p = 2(1 - p)$
- E. $p = q$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	3,2

Poll 4

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE of the game, which equations should p and q satisfy?

A. $2q = 3(1 - q)$

B. $2p = 3(1 - p)$

C. $q = 2(1 - q)$

D. $p = 2(1 - p)$

E. $p = q$

$$u_A(F, s_B) = u_A(C, s_B)$$

$$u_B(s_A, C) = u_B(s_A, F)$$

$$u_A(F, s_B) = 2 * q + 0 * (1 - q)$$

$$u_A(C, s_B) = 0 * q + 3 * (1 - q)$$

$$u_B(s_A, F) = 1 * p + 0 * (1 - p)$$

$$u_B(s_A, C) = 0 * p + 2 * (1 - p)$$

Berry

	Football	Concert
Football	2,1	0,0
Concert	0,0	3,2

Alex

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Minimax strategy
- Maximin strategy
- Stackelberg Equilibrium

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- (Minimax strategy)
- (Maximin strategy)
- Stackelberg Equilibrium

Power of Commitment

What's the PSNEs in this game and the players' utilities?

What action should player 2 choose if player 1 commits to playing b ?

What is player 1's utility?

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1's expected utility?

		Player 2	
		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Power of Commitment

What's the PSNEs in this game and the players' utilities? $(a, c), 2$

What action should player 2 choose if player 1 commits to playing b ?

What is player 1's utility? $d, 3$

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1's utility? $d, 3.5$

		Player 2	
		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Stackelberg Equilibrium

Stackelberg Game

- Leader commits to a strategy first
- Follower responds after observing the leader's strategy

Stackelberg Equilibrium

- Follower best responds to leader's strategy
- Leader commits to a strategy that maximize her utility assuming follower best responds

		Player 2	
		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Stackelberg Equilibrium

If the leader can only commit to a pure strategy, or you know that the leader’s strategy in equilibrium is a pure strategy, the equilibrium can be found by enumerating leader’s pure strategy

If ties for the follower are broken by the follower such that the leader benefits, the leader can exploit this. This is the **strong Stackelberg equilibrium (SSE)**

In general, the leader can commit to a mixed strategy and

$$u^{SSE} \geq u^{NE} \text{ (first-mover advantage)!}$$

[Conitzer & Sandholm 2006, von Stengel and Zamir 2010; see also Prof. Fei Fang’s work]

		Berry				Player 2	
		Football	Concert			c	d
Alex	Football	2,1	0,0	Player 1	a	2,1	4,0
	Concert	0,0	1,2		b	1,0	3,2