AI: Representation and Problem Solving

Bayes Nets: Independence

Instructor: Pat Virtue

Slide credits: CMU AI and http://ai.berkeley.edu

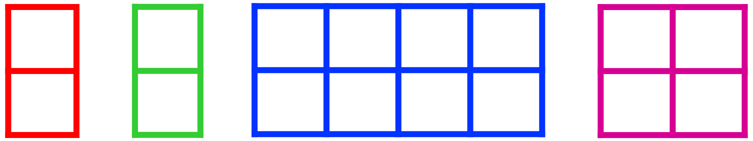
Bayesian Networks

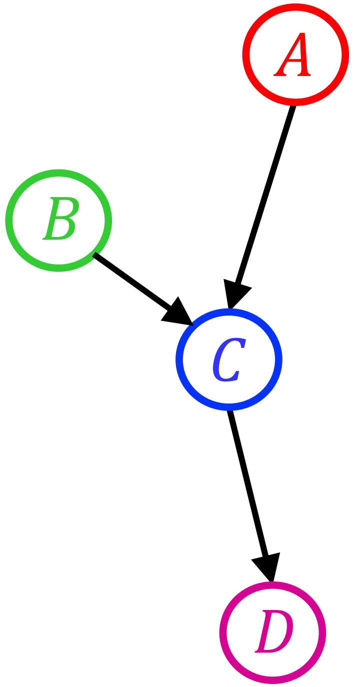
Bayes net

One node per random variable

Directed-Acyclic-Graph

One CPT per node: P(node | *Parents*(node))





$$P(A,B,C,D) = P(A)\,P(B)\,P(C|A,B)\,P(D|C)$$

Encode joint distributions as product of conditional distributions on each variable

$$P(X_1,\dots,X_N) = \prod_i P(X_i|\,Parents(X_i))$$

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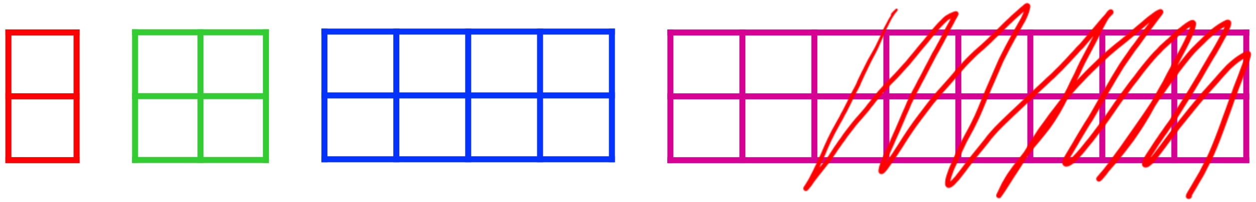

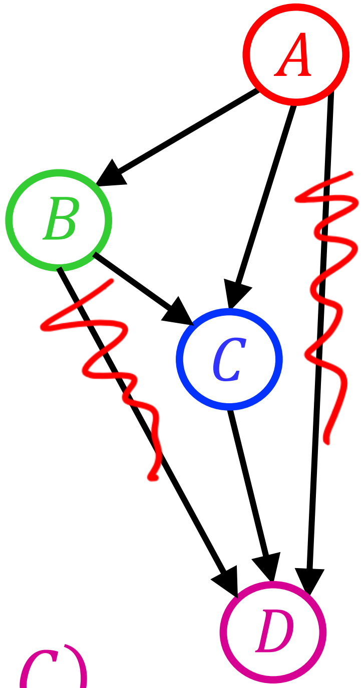

$$P(A,B,C,D) = P(A)\,P(B|A)\,P(C|A,B)\,P(D|\cancel{A},\cancel{B},C)$$

Encode joint distributions as product of conditional distributions on each variable

$$P(X_1,\ldots,X_N) = \prod_i P(X_i|\,Parents(X_i))$$

Bayesian Networks

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One node per random variable

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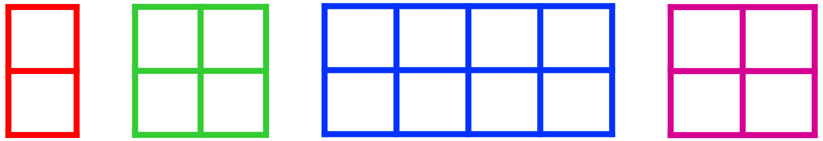
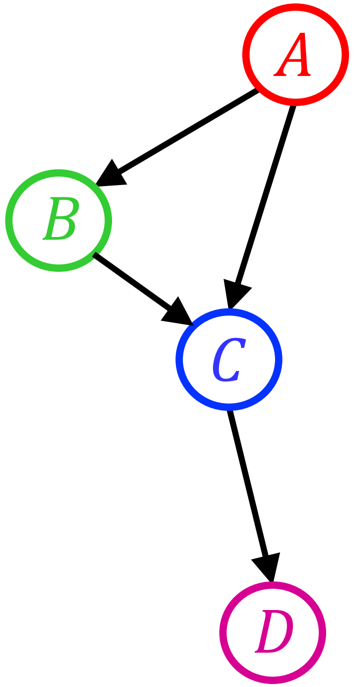




$$P(A,B,C,D) = P(A)\,P(B|A)\,P(C|A,B)\,P(D|C)$$

Encode joint distributions as product of conditional distributions on each variable

$$P(X_1, \dots, X_N) = \prod_i P(X_i|\,Parents(X_i))$$

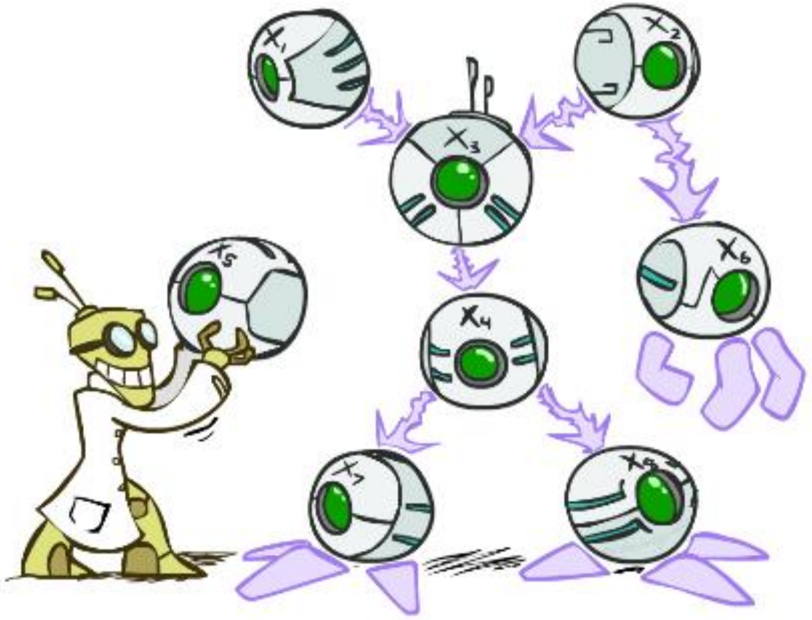
Bayes' Nets: Big Picture

Two problems with using full joint distribution tables as our probabilistic models:

- Usually, the joint is WAY too big to represent explicitly
- Hard to learn (estimate) anything empirically about more than a few variables at a time

Bayes nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)

- A type of probabilistic graphical models
- We describe how variables locally interact
- Local interactions chain together to give global, indirect interactions



Graphical Model Notation

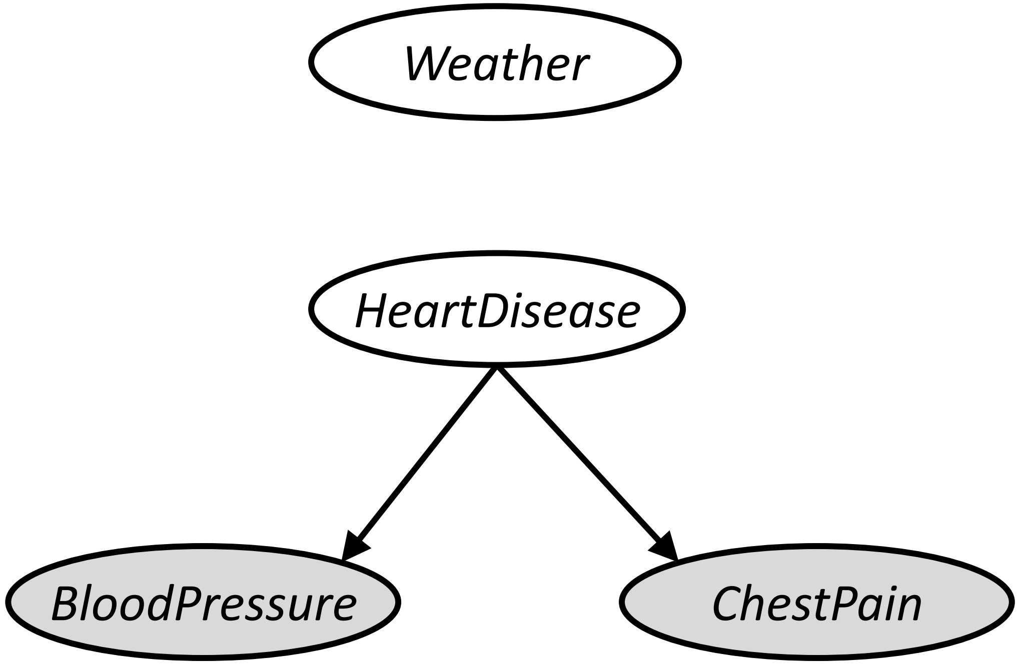
Nodes: variables (with domains)

- Can be assigned (observed) or unassigned (unobserved)
- We'll shade node to indicate observed variables
- Observed does not mean $Variable = true$
  Observed just means that we will have the value for that variable

Edges

- Indicate "direct influence" between variables
- Absence of edges: encode conditional independence

For now: imagine that arrows mean direct causation (in general, they don't!)



Query: $P(HD \mid \underline{BP}, \underline{CP})$

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https://mymakar.github.io/

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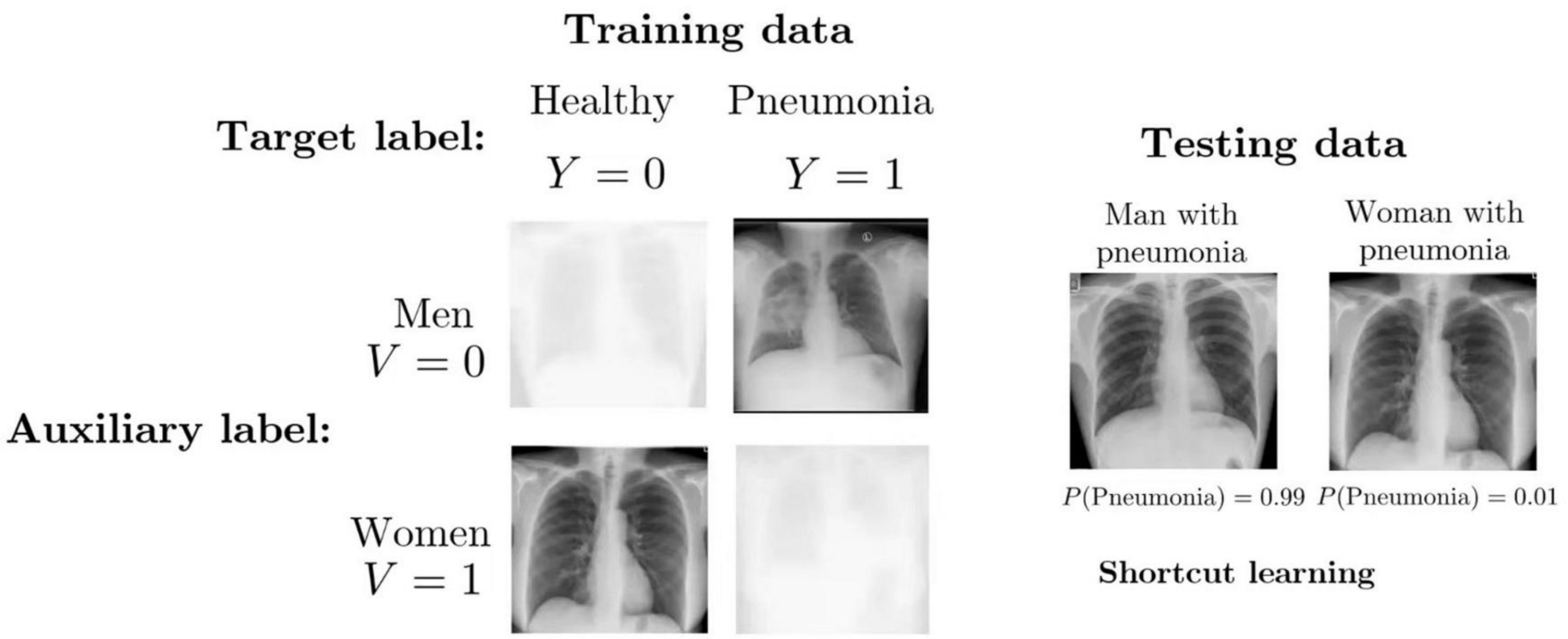
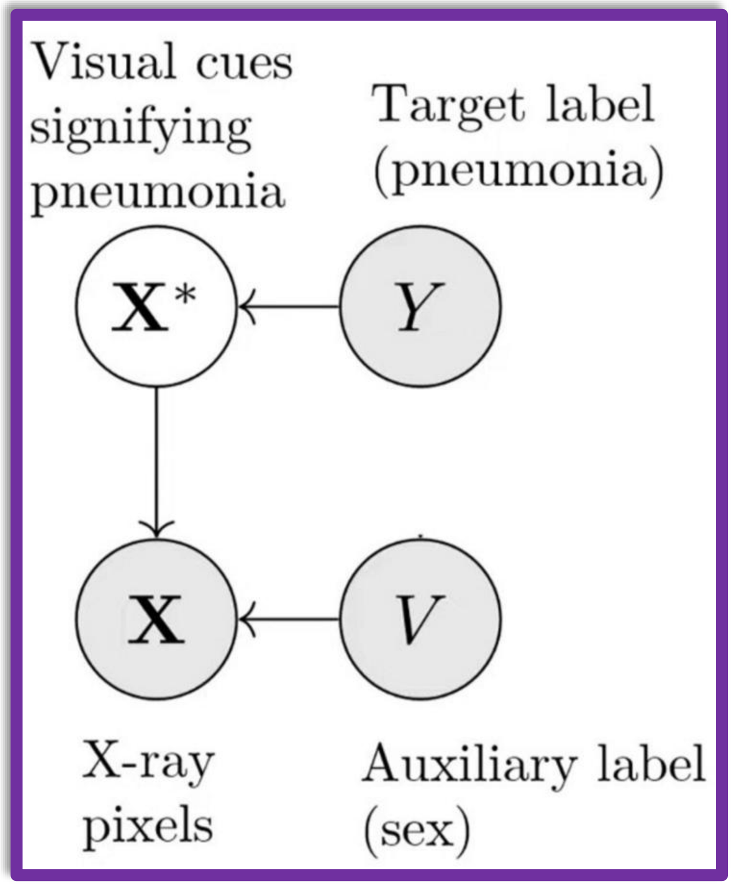
Causally-motivated shortcut removal using auxiliary labels
M. Makar, B. Packer, D. Moldovan, D. Blalock, Y. Halpern, A. D'Amour
AIStats 2022 [paper]

Pneumonia detection under biased sampling

$Y \to \cancel{X}$



Cases courtesy of Dr. Andrew Dixon, Dr Henry Knip, Dr. Usman Bashir, and Dr. Ian Bickle , Radiopaedia.org, rID: 48366, 31388, 18394 and rID: 50318



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Danielle Belgrave, Microsoft Research

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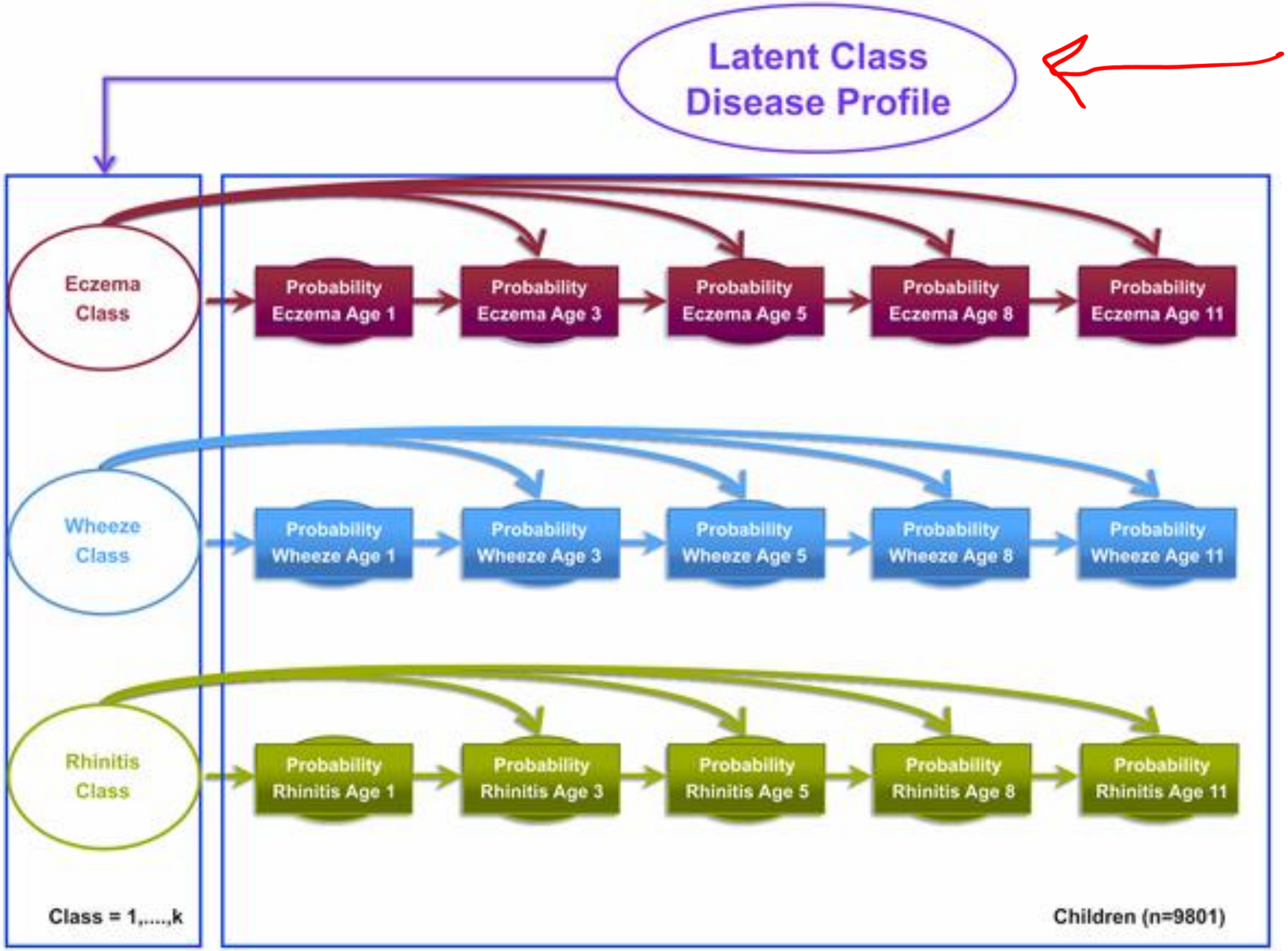
Developmental Profiles of Eczema, Wheeze, and Rhinitis:
Two Population-Based Birth Cohort Studies
Danielle Belgrave, et al. *PLOS Medicine*, 2014
https://journals.plos.org/plosmedicine/article?id=10.1371/journal.pmed.1001748



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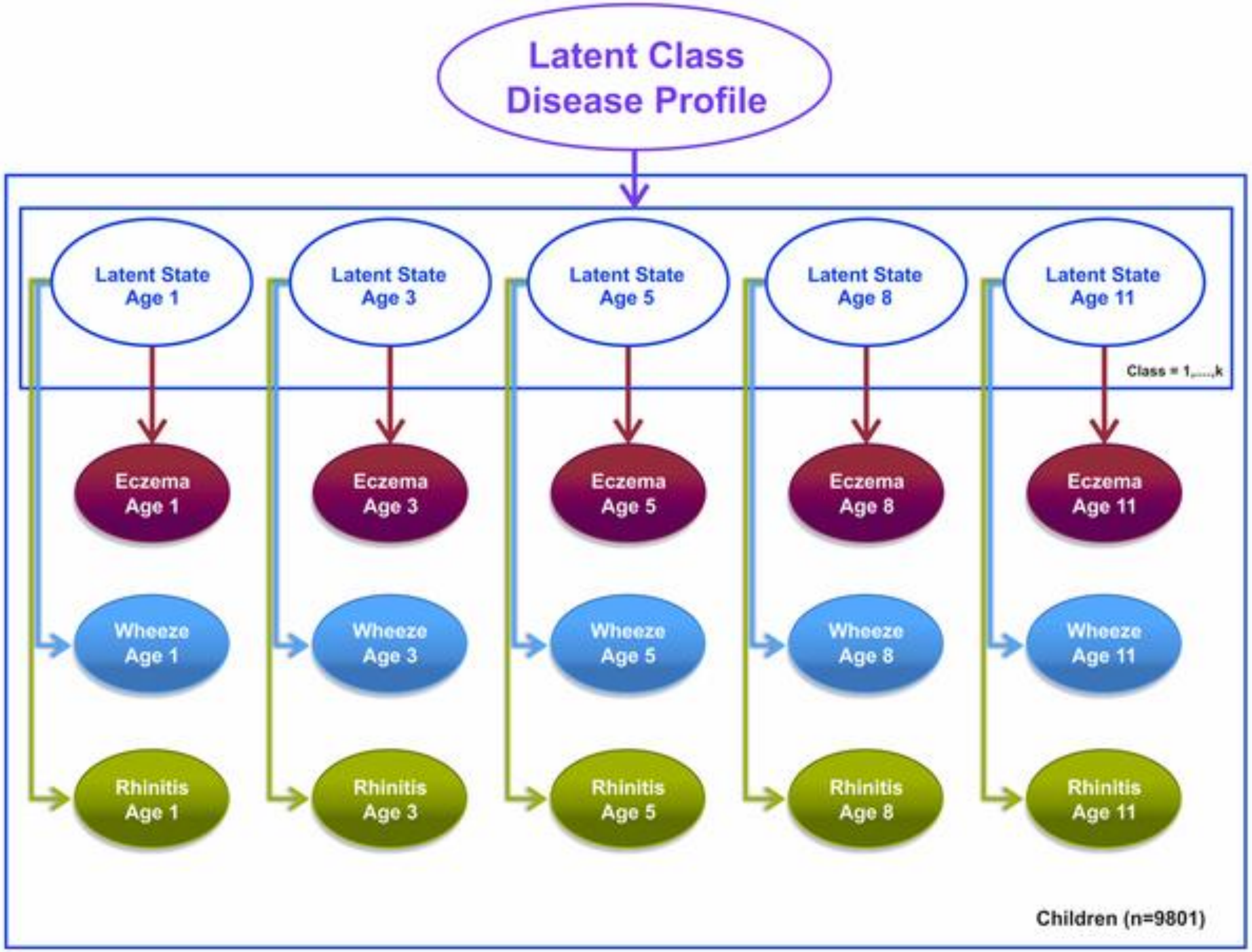


Danielle Belgrave, et al. *PLOS Medicine*, 2014

https://journals.plos.org/plosmedicine/article?id=10.1371/journal.pmed.1001748

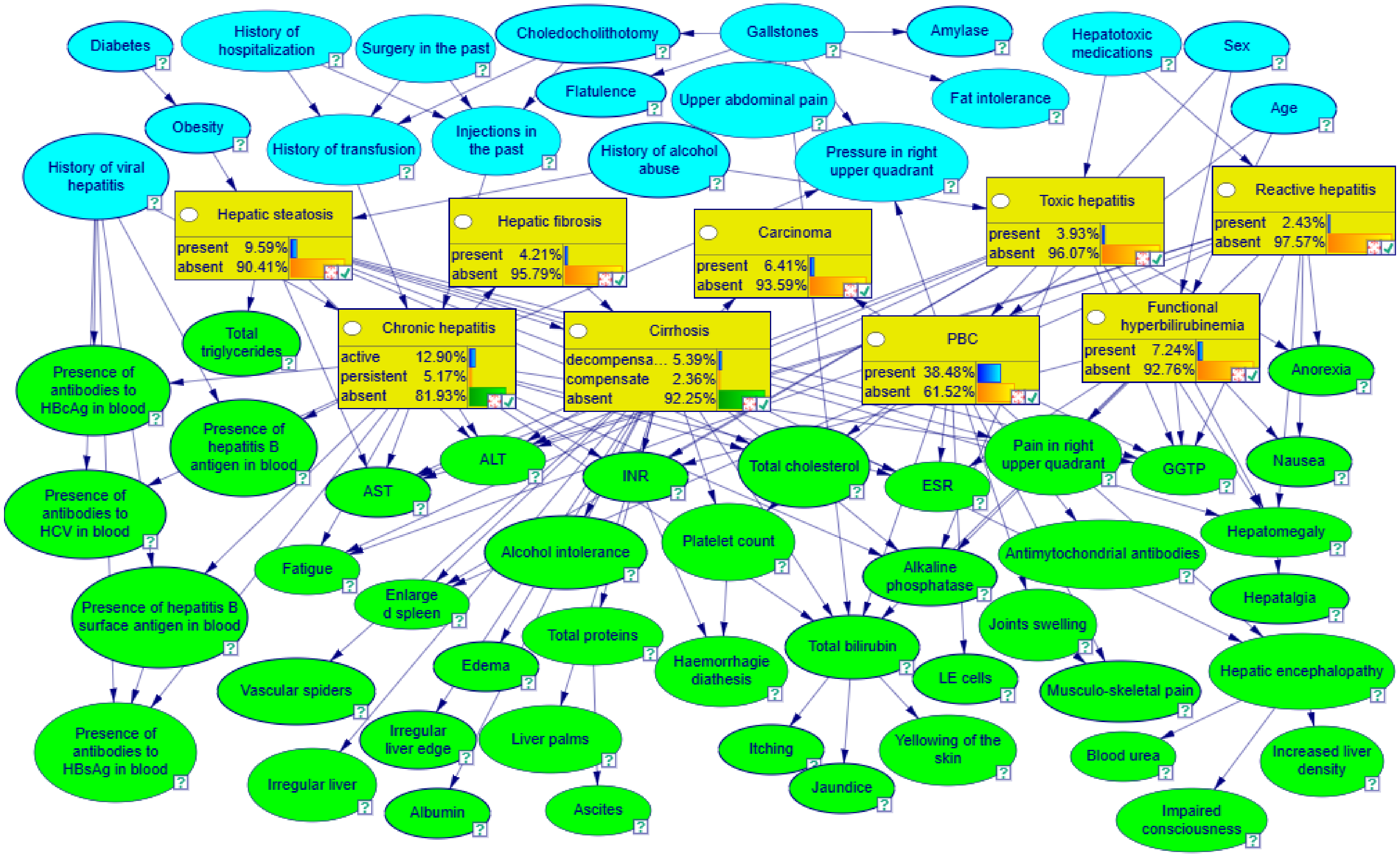
Characteristic	MAAS Cohort		ALSPAC Cohort		Joint MAAS and ALSPAC Cohort	
	n/Total	Percent	*n*/Total	Percent	*n*/Total	Percent
Gender (Female)	617/1,136	54.3	4,212/8,665	48.61	4,829/9,801	49.3
Eczema						
Age 1 y	383/1,077					
Age 3 y	355/1,061					
Age 5 y	340/1,050					
Age 8 y	285/1,027					
Age 11 y	216/924					
Wheeze						
Age 1 y	300/1,087					
Age 3 y	257/1,095					
Age 5 y	238/1,056					
Age 8 y	185/1,024					
Age 11 y	173/916					
Rhinitis						
Age 1 y	8/943					
Age 3 y	49/1,075					
Age 5 y	292/1,039					
Age 8 y	297/1,027					
Age 11 y	321/927					

	n/Total	Percent
Gender (Female)	617/1,136	54.3
Eczema		
Age 1 y	383/1,077	35.6
Age 3 y	355/1,061	33.5
Age 5 y	340/1,050	32.4
Age 8 y	285/1,027	27.8
Age 11 y	216/924	23.4

Danielle Belgrave, et al. *PLOS Medicine*, 2014

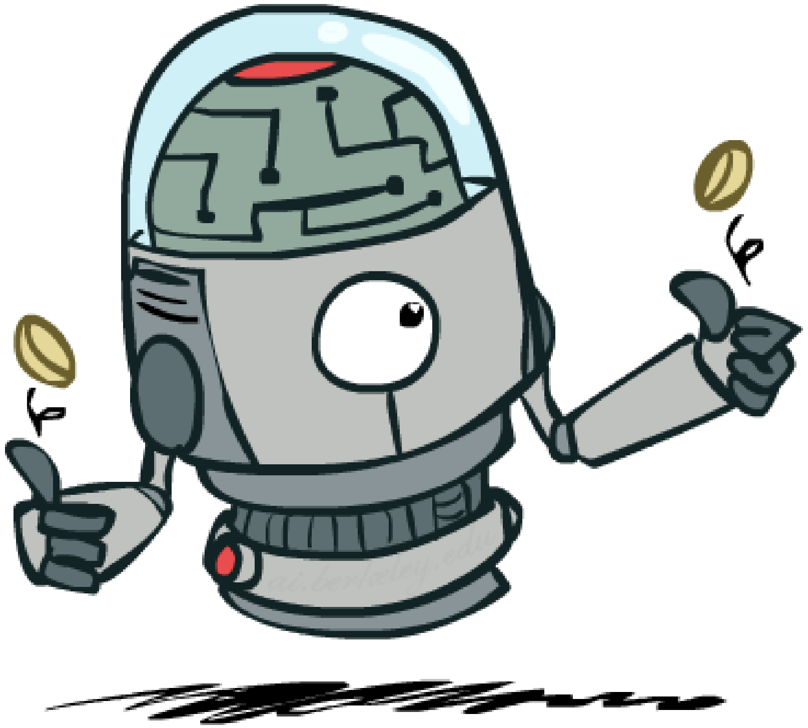
https://journals.plos.org/plosmedicine/article?id=10.1371/journal.pmed.1001748

Example: Liver Disorders



https://demo.bayesfusion.com/bayesbox.html

Independence



Independence

$X \perp\!\!\!\perp Y$

Two variables X and Y are *independent* if

$$\forall x,y \qquad P(x, y) = P(x)\, P(y)$$

- This says that their joint distribution ***factors*** into a product of two simpler distributions

$\propto$

- Combine with product rule $P(x,y) = P(x|y)P(y)$ we obtain another form:

$$\forall x,y\ P(x \mid y) = P(x) \quad \text{or} \quad \forall x,y\ P(y \mid x) = P(y)$$

Example: two dice rolls R_1 and R_2

$P(R_1{=}5, R_2{=}5) = P(R_1{=}5)\, P(R_2{=}5) = 1/6 \times 1/6 = 1/36$

$P(R_2{=}5 \mid R_1{=}5) = P(R_2{=}5) \approx 1/6$

Example: Independence

n fair, independent coin flips:

$P(X_1)$

H	0.5
T	0.5

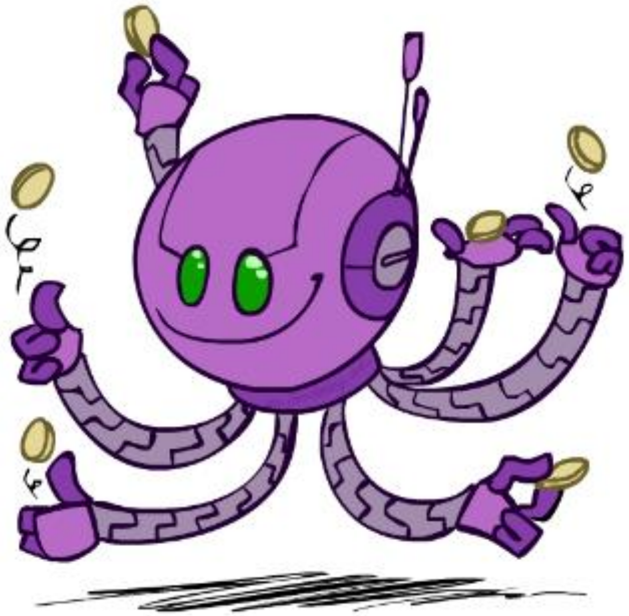
$P(X_2)$

H	0.5
T	0.5

. . .

$P(X_n)$

H	0.5
T	0.5



$P(X_1,X_2,\ldots,X_n)$

2^n

Poll 1

Are T and W independent?

$P(T)$

T	P
hot	0.5
cold	0.5

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(W)$

W	P
sun	0.6
rain	0.4

$P(T)P(W)$

T W

Poll 1

Are T and W independent?

No

$P(T)$

T	P
hot	0.5
cold	0.5

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)P(W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

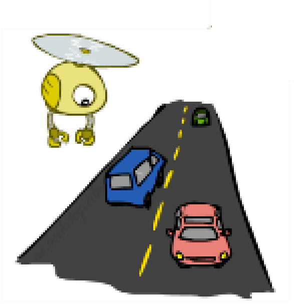
$P(W)$

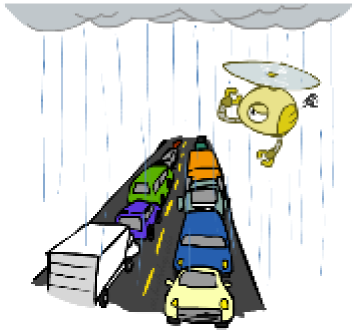
W	P
sun	0.6
rain	0.4

Example: Traffic

Variables:

- R: rain or not
- T: traffic or not



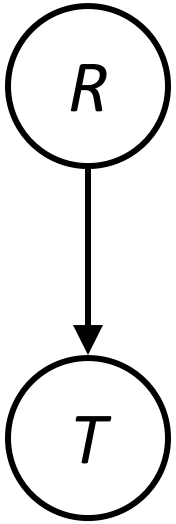


Model 1: independence

$P(R)P(T)$

$R \perp\!\!\!\perp T$

Model 2: rain affects traffic



$P(R)P(T|R)$

no assump.

Why is an agent using model 2 better?

Conditional Independence

Absolute (unconditional) independence very rare (why?)

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

X is conditionally independent of Y given Z

$X \perp\!\!\!\perp Y \mid Z$

if and only if:

$$\forall x,y,z \qquad P(x \mid y, z) = P(x \mid z)$$

or, equivalently, if and only if

$$\forall x,y,z \qquad P(x, y \mid z) = P(x \mid z)\, P(y \mid z)$$

Independence Rules

Independence

If A and B are independent, then:

$A \perp\!\!\!\perp B$

$$P(A, B) = P(A)P(B)$$
$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$

Conditional independence

If A and B are conditionally

independent given C, then:

$A \perp\!\!\!\perp B \mid C$

$$P(A, B \mid \underline{C}) = P(A \mid \underline{C})P(B \mid \underline{C})$$
$$P(A \mid B, \underline{C}) = P(A \mid \underline{C})$$
$$P(B \mid A, \underline{C}) = P(B \mid \underline{C})$$

Conditional Independence

P(Traffic, Rain, Umbrella)

If it's rainining, the probability that there is traffic doesn't depend on whether see an umbrella:

- P(+traffic | +umbrella, +rain) = P(+traffic | +rain)

$\boxed{T \perp\!\!\!\perp U \mid R}$

The same independence holds if it's not raining:

- P(+traffic | +umbrella, -rain) = P(+traffic| -rain)

Traffic is *conditionally independent* of Umbrella given Rain:

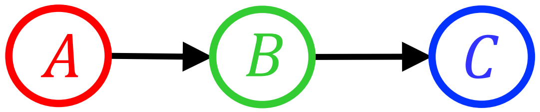
- P(Traffic | Umbrella, Rain) = P(Traffic | Rain)

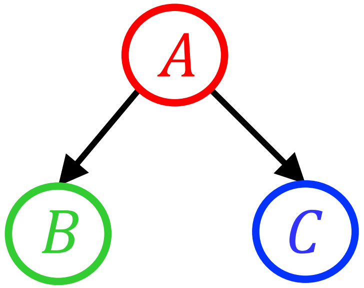
Equivalent statements:

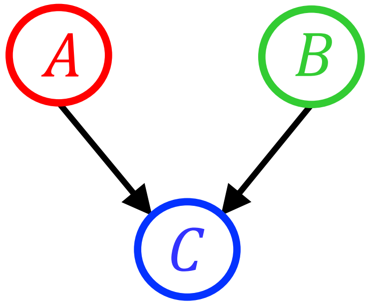
- P(Umbrella | Traffic , Rain) = P(Umbrella | Rain)
- P(Umbrella, Traffic | Rain) = P(Umbrella | Rain) P(Traffic | Rain)
- One can be derived from the other easily

Poll 2

Match the product of CPTs to the Bayes net.







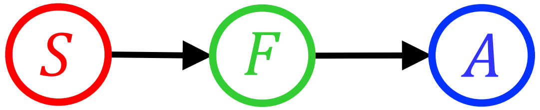
q2

I.	$P(A)\,P(B\vert A)\,P(C\vert B)$	$P(A)\,P(B\vert A)\,P(C\vert A)$	$P(A)\,P(B)\,P(C\vert A,B)$
II.	$P(A)\,P(B)\,P(C\vert A,B)$	$P(A)\,P(B\vert A)\,P(C\vert B)$	$P(A)\,P(B\vert A)\,P(C\vert A)$
III.	$P(A)\,P(B\vert A)\,P(C\vert A)$	$P(A)\,P(B)\,P(C\vert A,B)$	$P(A)\,P(B\vert A)\,P(C\vert B)$

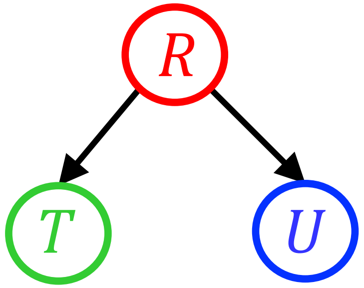
Conditional Independence Semantics

Common local releationships within a Bayes net

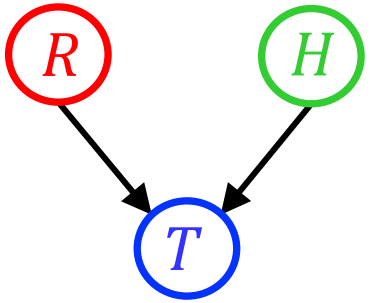
Causal Chain

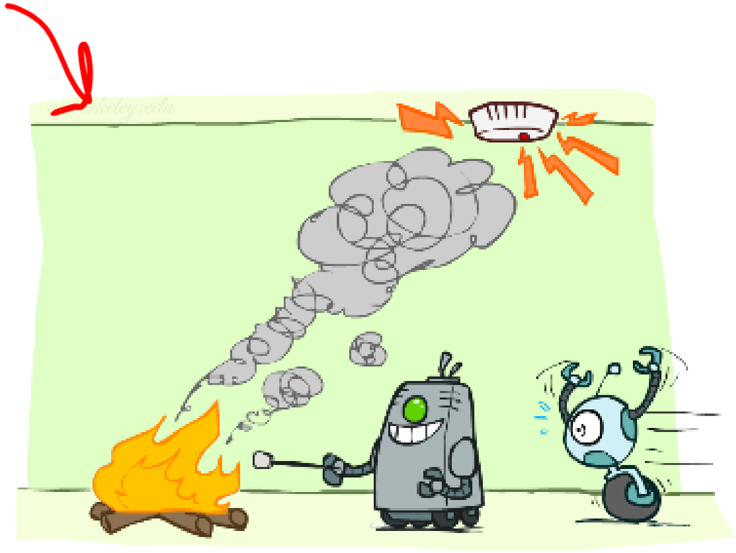


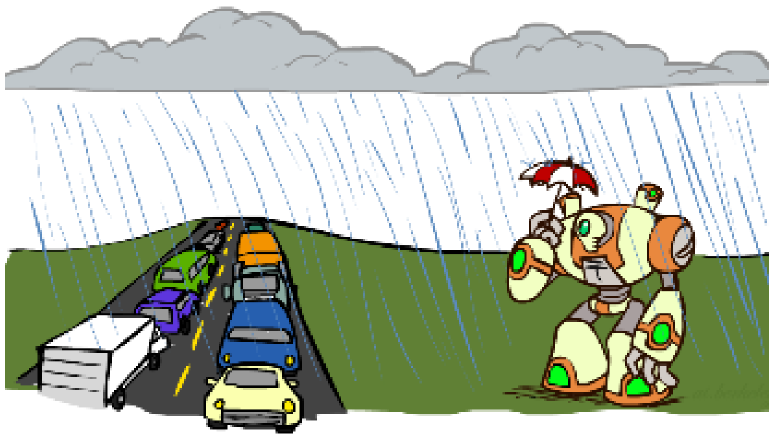
Common Cause



Common Effect







Causal Chain

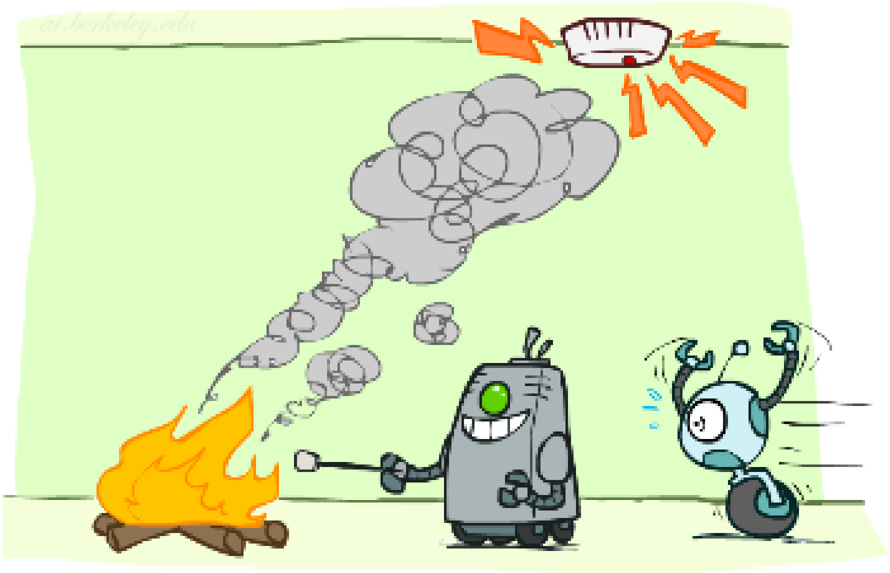
No: $F \perp\!\!\!\perp A$

Fire, Smoke, Alarm

- Causal story to create Bayes net

$$F \rightarrow S \rightarrow A$$

$$F \perp\!\!\!\perp A \mid S$$



- Assumptions
- Joint distribution

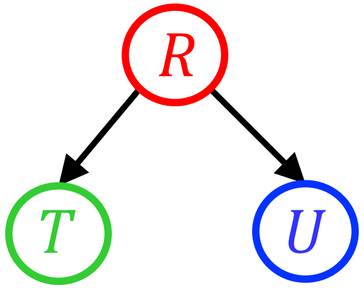
$$P(F,S,A) = P(F)\,P(S \mid F)\,P(A \mid F\,S)$$

$$P(F,S,A) = P(F)\,P(S \mid F)\,P(A \mid S)$$

Common Cause

Chain rule:

$P(x_1, x_2,\ldots, x_n) = \prod_i P(x_i \mid x_1,\ldots, x_{i-1})$



Trivial decomposition:

$P(Rain, Traffic, Umbrella) =$

With assumption of conditional independence:

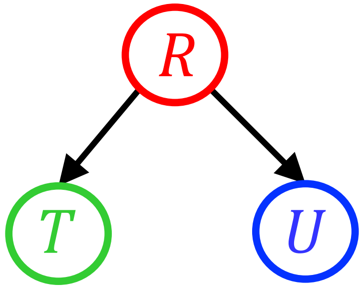
$P(Raint, Traffic, Umbrella) =$

Bayes nets / graphical models help us express conditional independence assumptions

Common Cause

$T \perp\!\!\!\perp U \mid R$

$T \perp\!\!\!\perp U$?



Chain rule:

$P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i \mid x_1, \ldots, x_{i-1})$

Trivial decomposition:

$P(Rain, Traffic, Umbrella) = P(Rain)\, P(Traffic \mid Rain)\, P(Umbrella \mid Rain, Traffic)$

With assumption of conditional independence:

$P(Rain, Traffic, Umbrella) = P(Rain)\, P(Traffic \mid Rain)\, P(Umbrella \mid Rain)$

Bayes nets / graphical models help us express conditional independence assumptions

Common Effect

$$\begin{tikzcd} R \arrow[dr] & & H \arrow[dl] \\ & T & \end{tikzcd}$$

R

H

T

Chain rule:

$P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i \mid x_1, \ldots, x_{i-1})$

Trivial decomposition:

$P(Rain, Hockey, Traffic) = P(Rain)\, P(Hockey \mid Rain)\, P(Traffic \mid Rain, Hockey)$

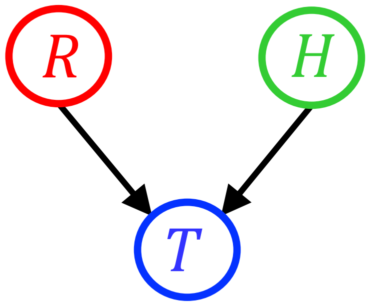
With assumption of conditional independence:

$P(Rain, Hockey, Traffic) =$ $P(Rain)\, P(Hockey)\;\; P(Traf \mid Rain, Hockey)$

Bayes nets / graphical models help us express conditional independence assumptions

$H \perp\!\!\!\perp R$

Common Effect



Chain rule:

$P(x_1, x_2,\ldots, x_n) = \prod_i P(x_i \mid x_1,\ldots, x_{i-1})$

Trivial decomposition:

$P(Rain, Hockey, Traffic) = P(Rain)\ P(Hockey \mid Rain)\ P(Traffic \mid Rain, Hockey)$

With assumption of conditional independence:

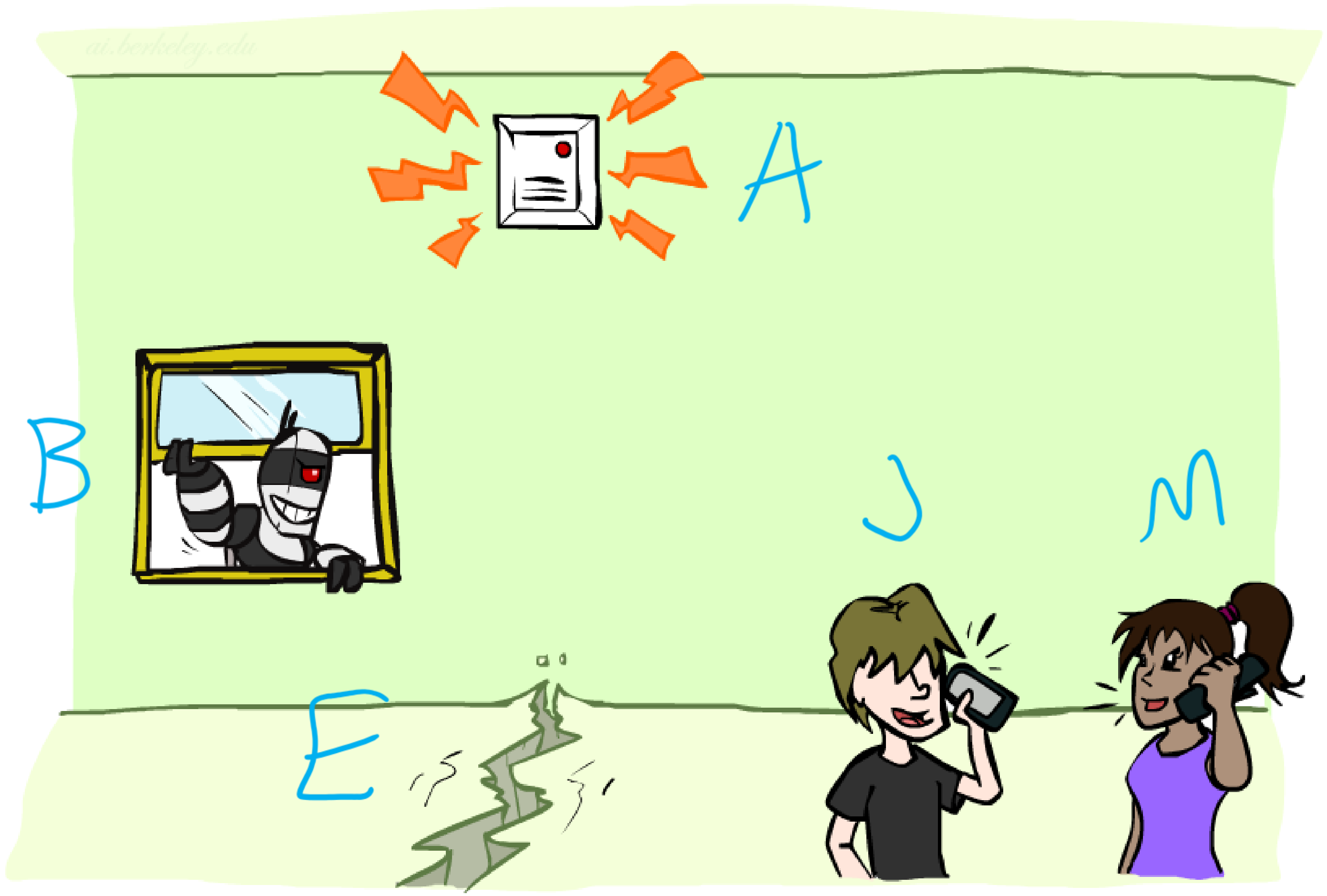
$P(Rain, Hockey, Traffic) = P(Rain)\ P(Hockey)\quad P(Traffic \mid Rain, Hockey)$

Bayes nets / graphical models help us express conditional independence assumptions

Example: Alarm Network

Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



Example: Alarm Network

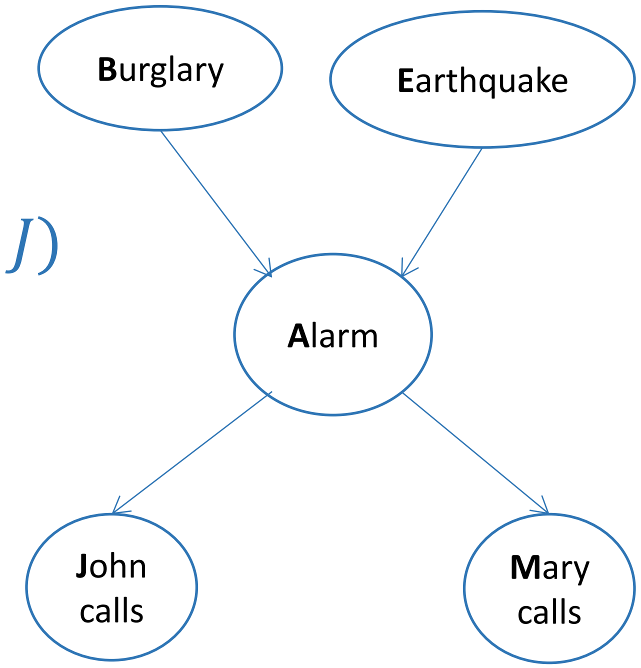
Joint distribution factorization example

Generic chain rule

- $P(X_1 \ldots X_2) = \prod_i P(X_i |\, X_1 \ldots X_{i-1})$

$$P(B,E,A,J,M) = P(B)\,P(E|B)\,P(A|B,E)\,P(J|B,E,A)\,P(M|B,E,A,J)$$

$$P(B,E,A,J,M) = P(B)\,P(E)\quad P(A|B,E)\,P(J|A)\qquad P(M|A)$$

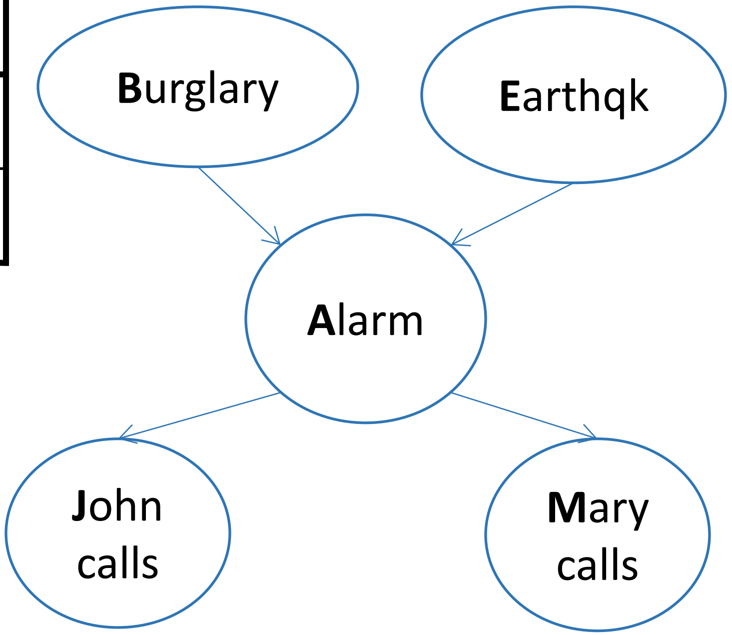


Bayes nets

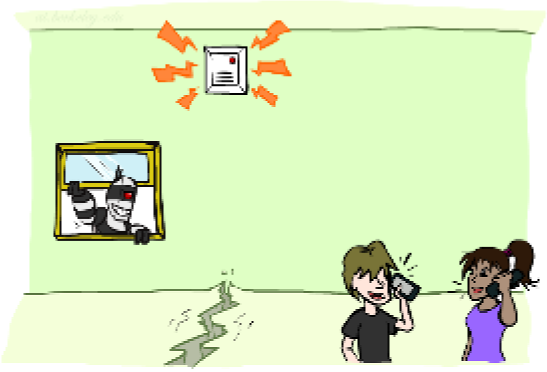
- $P(X_1 \ldots X_2) = \prod_i P(X_i |\, Parents(X_i))$

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

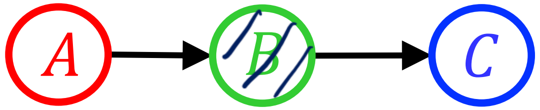
A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

Conditional Independence Semantics

For the following Bayes nets, write the joint $P(A, B, C)$

1. Using the chain rule (with top-down order A,B,C)
2. Using Bayes net semantics (product of CPTs)



Chain

$P(A)\, P(B|A)\, P(C|A, B)$

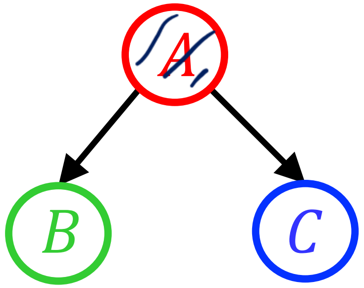
Bayes Net

$P(A)\, P(B|A)\, P(C|B)$

Assumption:

$P(C|A, B) = P(C|B)$

C is independent from A given B



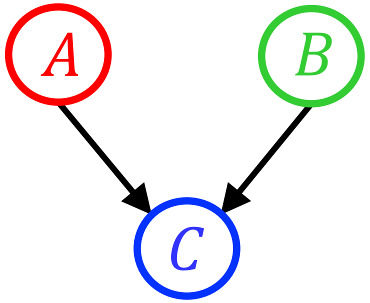
$P(A)\, P(B|A)\, P(C|A, B)$

$P(A)\, P(B|A)\, P(C|A)$

Assumption:

$P(C|A, B) = P(C|A)$

C is independent from B given A



$P(A)\, P(B|A)\, P(C|A, B)$

$P(A)\, P(B)\, P(C|A, B)$

Assumption:

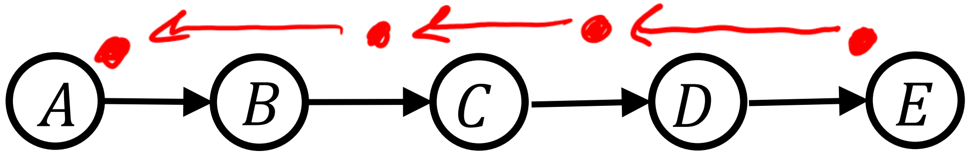
$P(B|A) = P(B)$

A is independent from B given { }

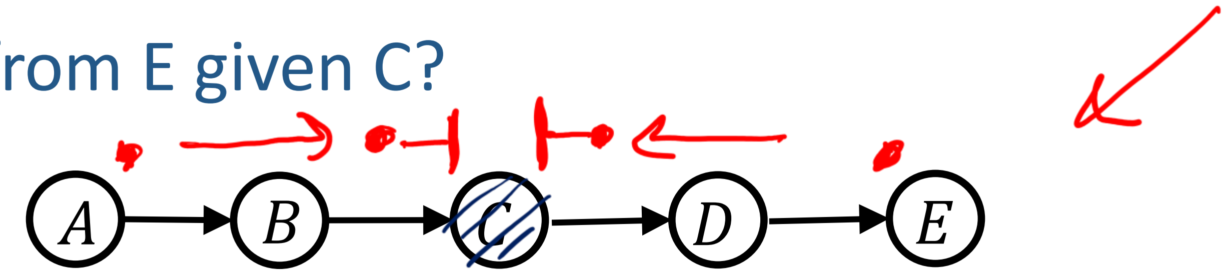
Bayes Net Independence

Answering Independence Questions

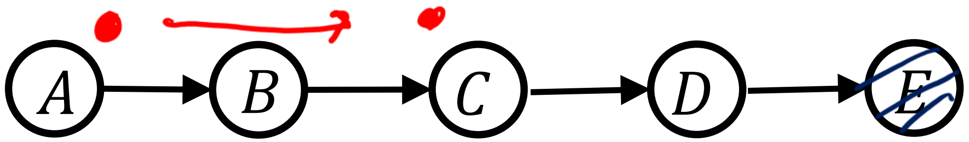
- Is A independent from E?



- Is A independent from E given C?



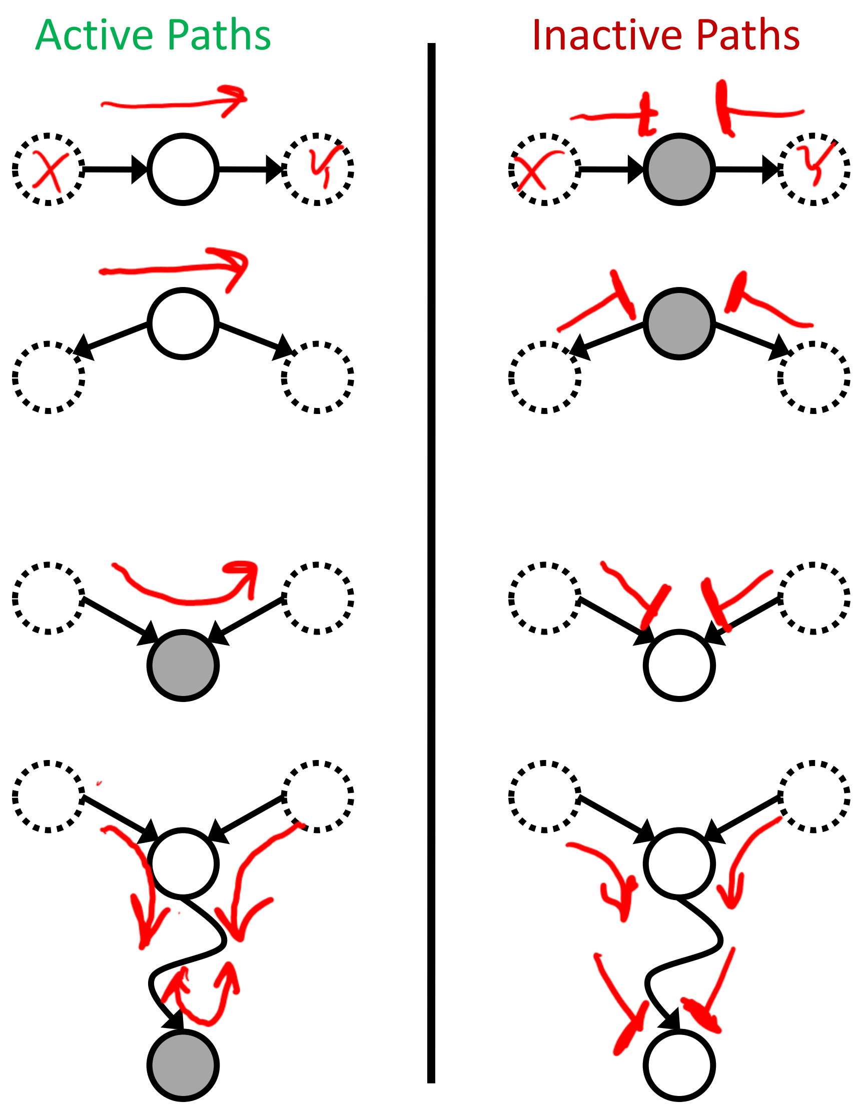
- Is A independent from C given E?



Active / Inactive Paths

Question: Are X and Y conditionally independent given evidence variables {Z}?

- Yes, if X and Y "d-separated" by Z
- Consider all (undirected) paths from X to Y
- No active paths = independence!



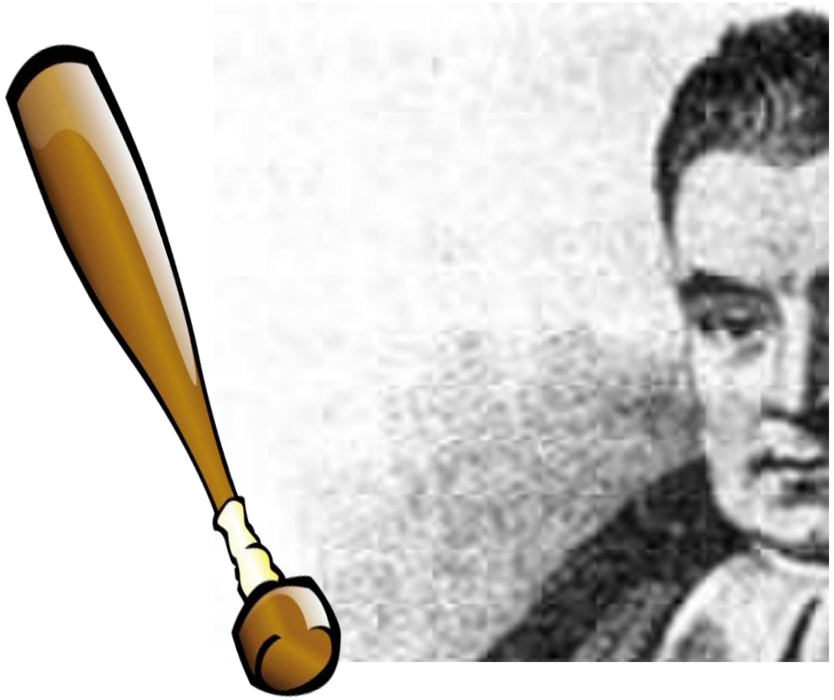
A path is active if each triple is active:

- Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
- Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
- Common effect (aka v-structure)
  $A \rightarrow B \leftarrow C$ where B *or one of its descendents* is observed

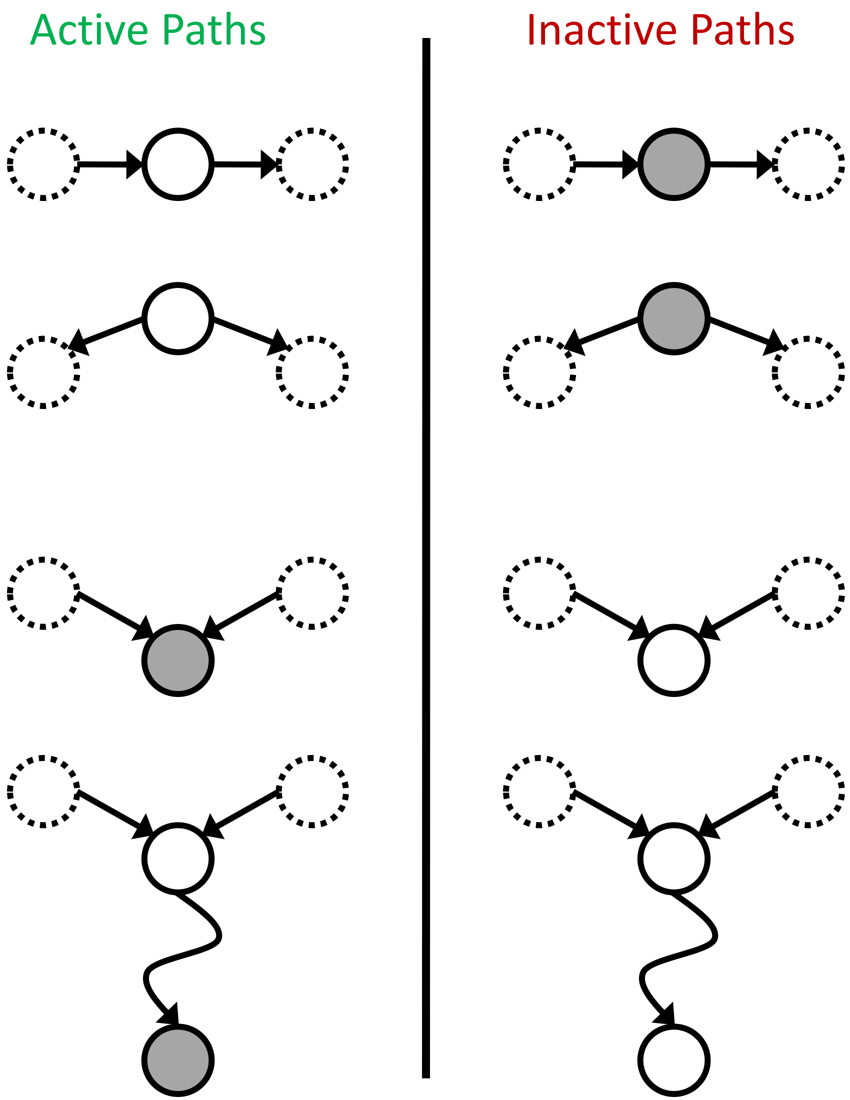
All it takes to block a path is a single inactive segment

Bayes Ball

Question: Are X and Y conditionally independent given evidence variables {Z}?



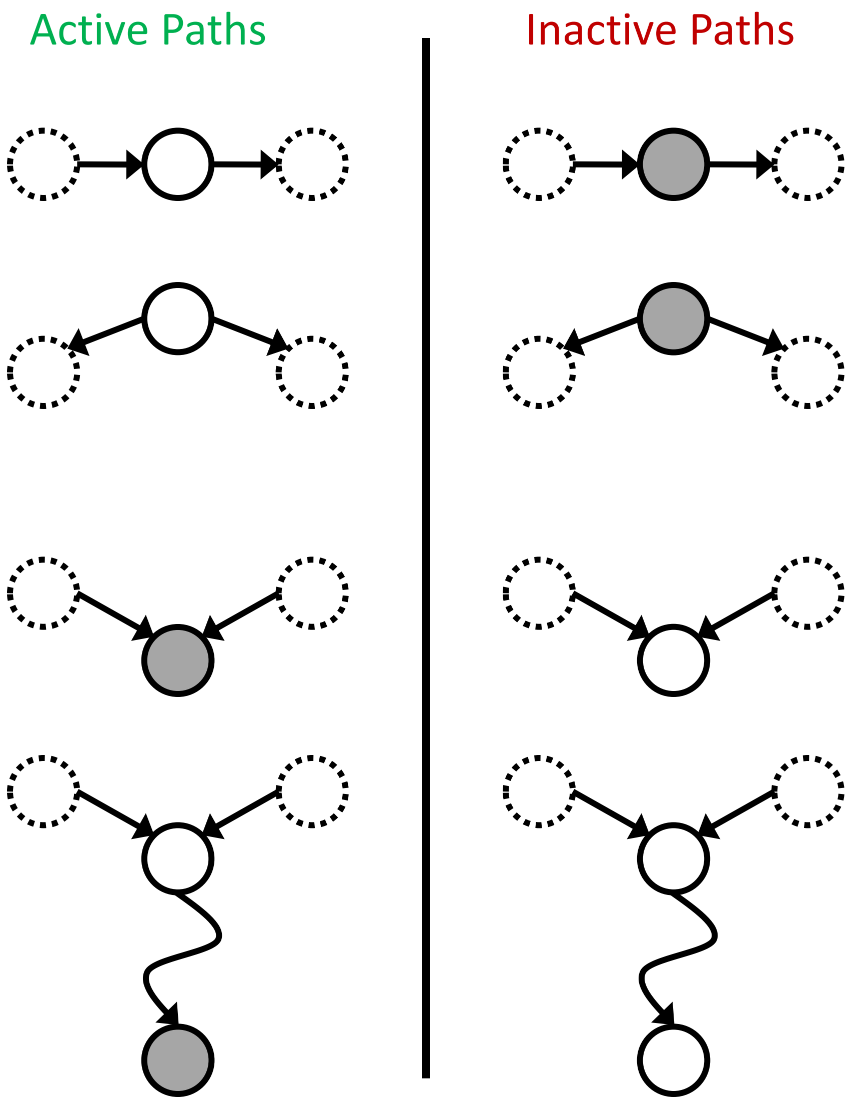
Shachter, Ross D. "Bayes-Ball: Rational Pastime (for Determining Irrelevance and Requisite Information in Belief Networks and Influence Diagrams)." *Proceedings of the Fourteenth conference on Uncertainty in Artificial Intelligence.* 1998.



Bayes Ball

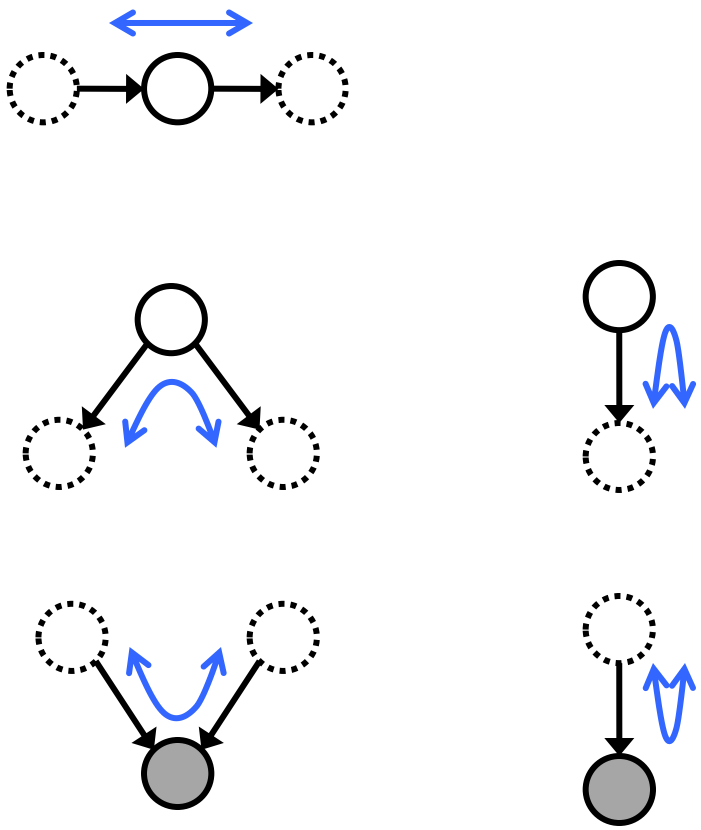
Question: Are X and Y conditionally independent given evidence variables {Z}?

1. Shade in Z
2. Drop a ball at X
3. The ball can pass through any *active* path and is blocked by any *inactive* path (ball can move either direction on an edge)
4. If the ball reaches Y, then X and Y are NOT conditionally independent given Z

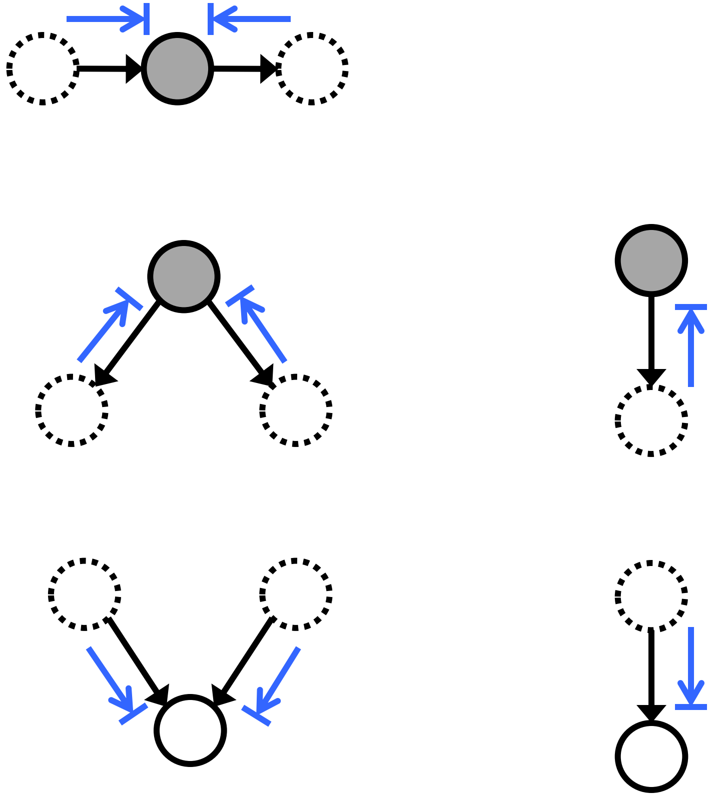


Bayes Ball

Active Paths

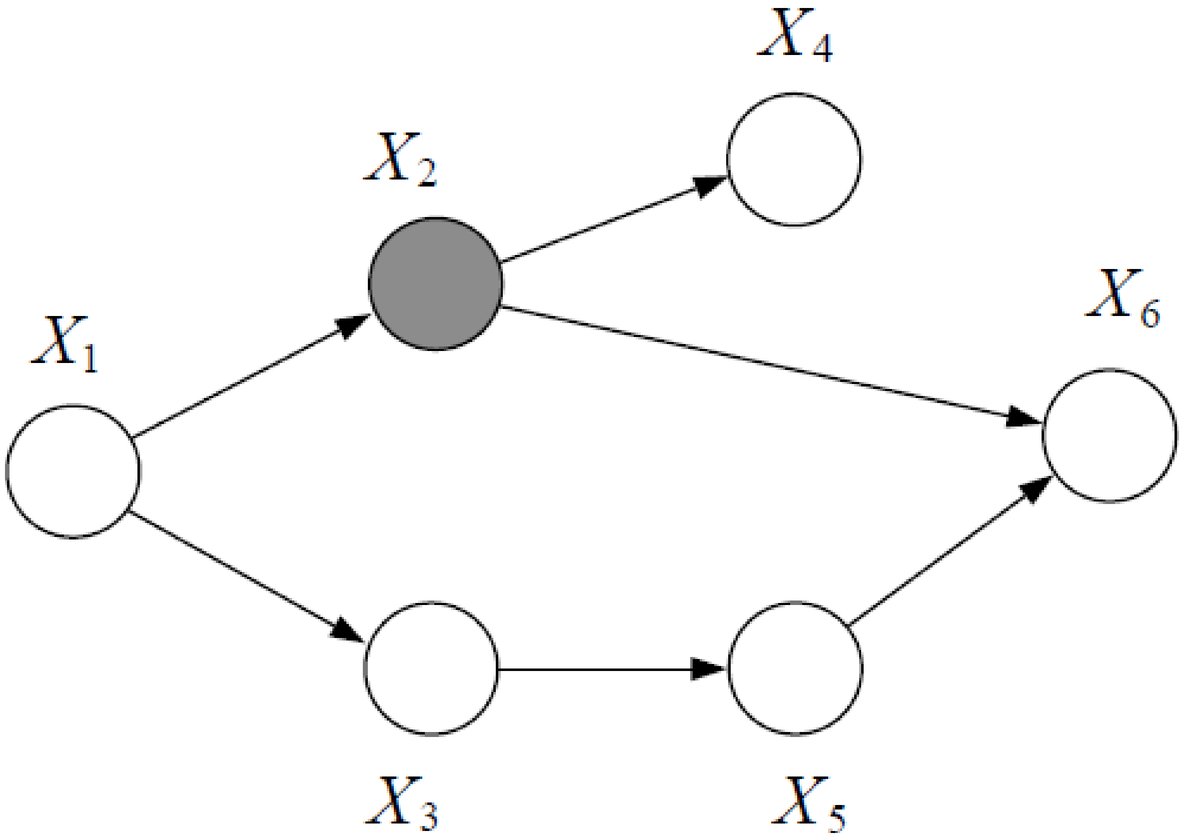


Inactive Paths



Poll 3

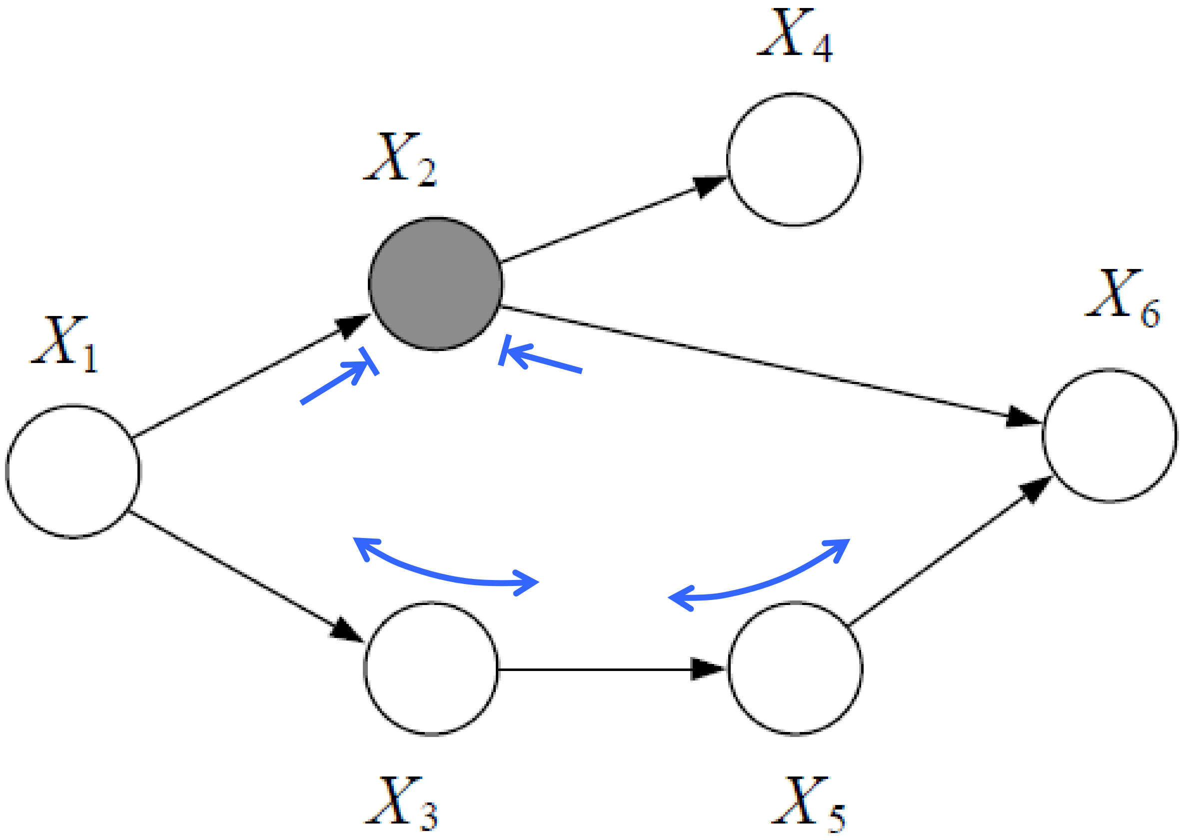
Is X_1 independent from X_6 given X_2?



Poll 3

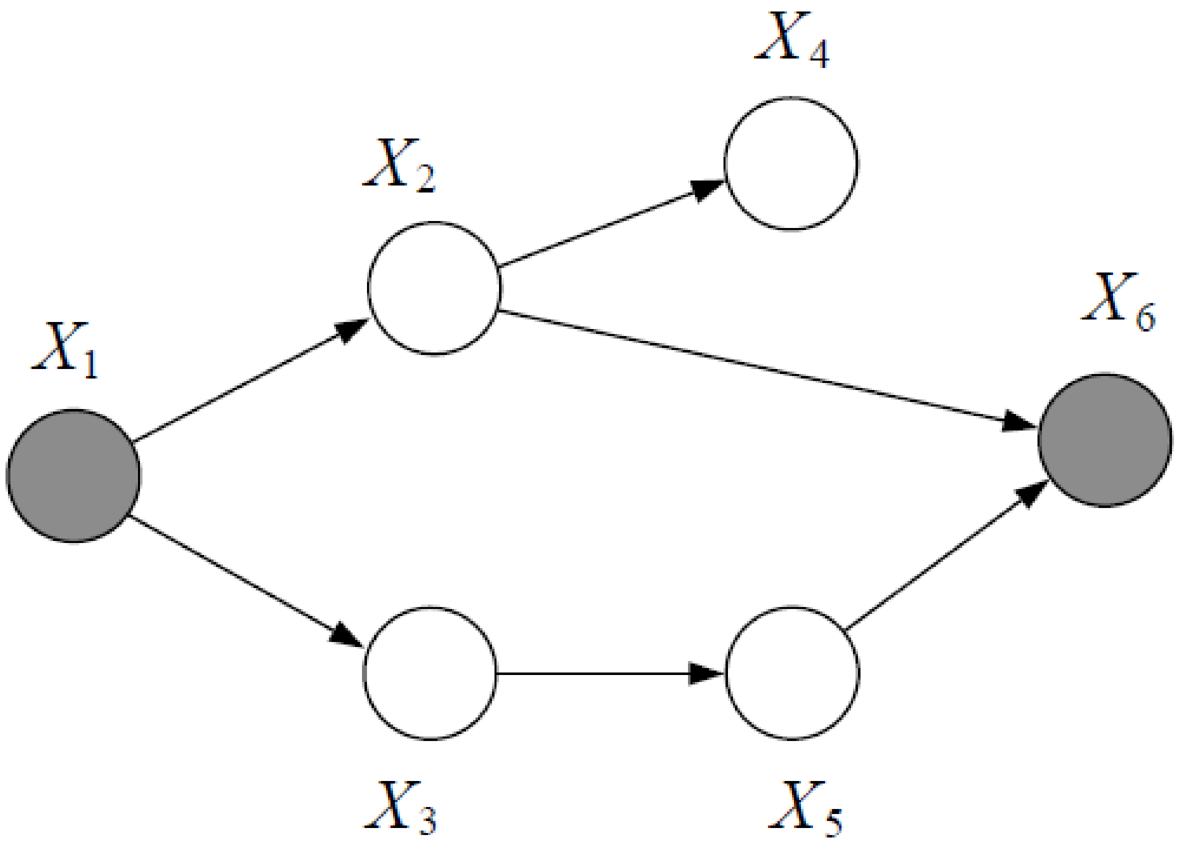
Is X_1 independent from X_6 given X_2?

No, the Bayes ball can travel through X_3 and X_5.



Poll 4

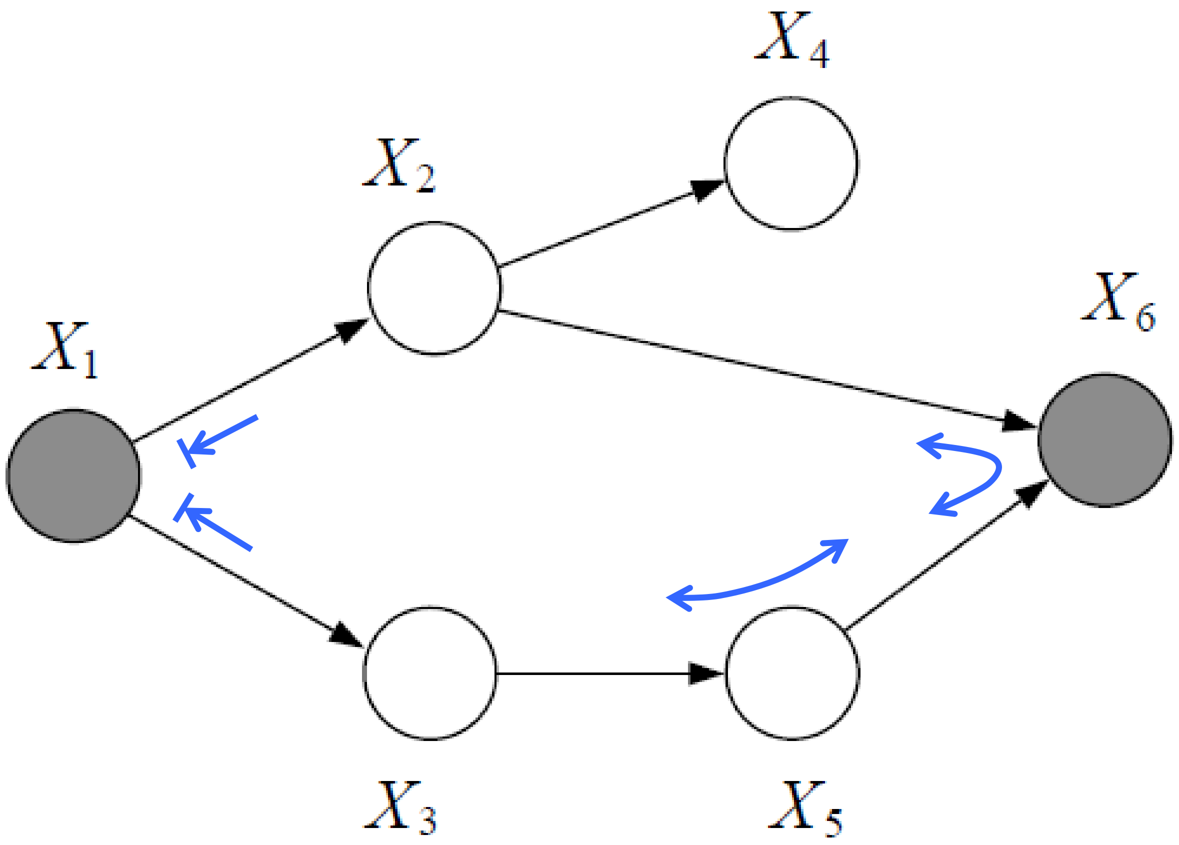
Is X_2 independent from X_3 given X_1 and X_6?



Poll 4

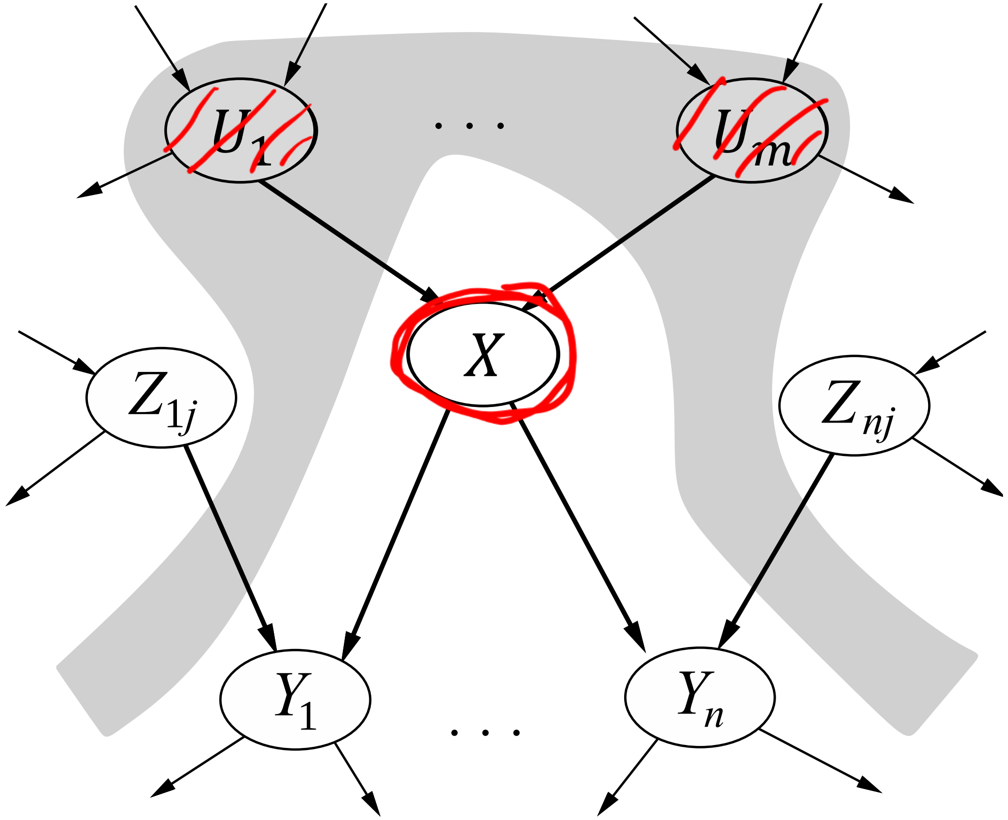
Is X_2 independent from X_3 given X_1 and X_6?

No, the Bayes ball can travel through X_5 and X_6.



Conditional Independence Semantics

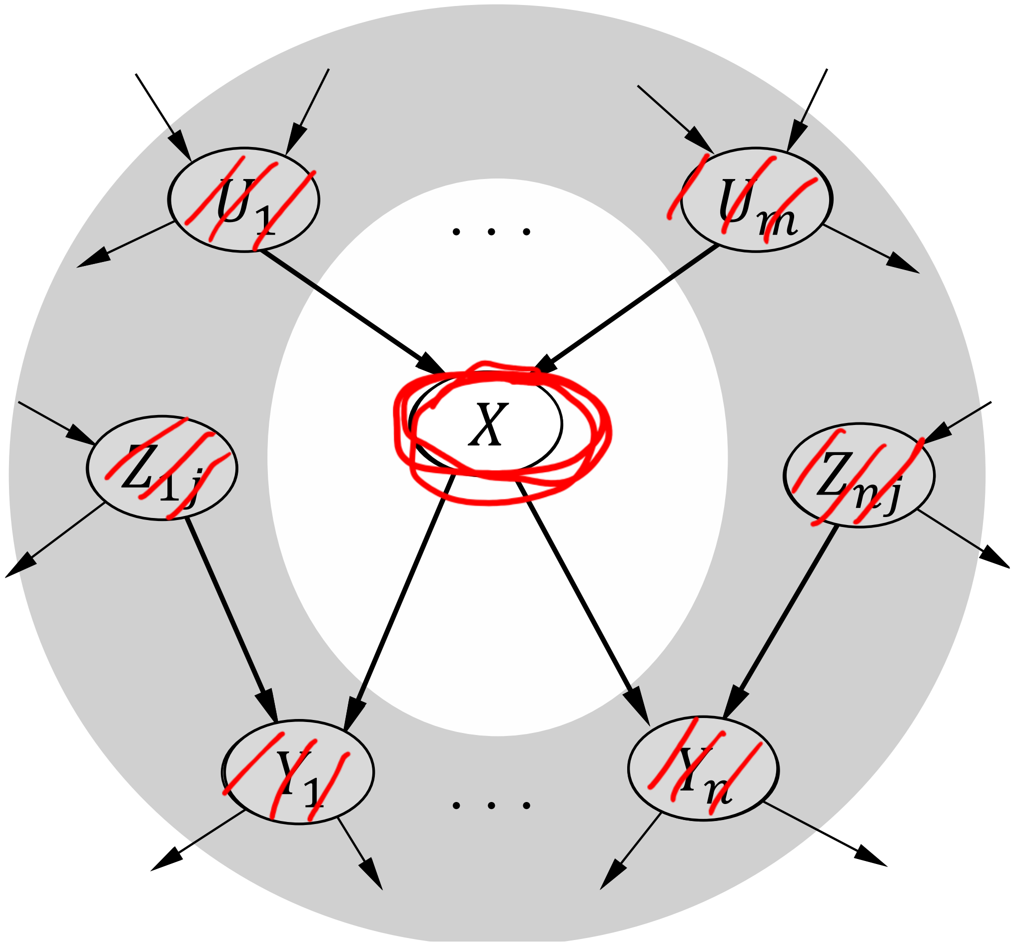
Every variable is conditionally independent of its non-descendants given its parents



Markov blanket

A variable's Markov blanket consists of parents, children, children's other parents

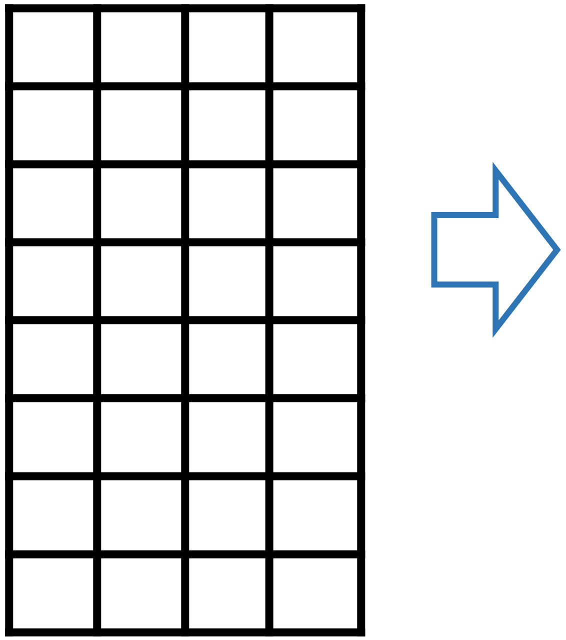
Every variable is conditionally independent of all other variables given its Markov blanket



Answer Any Query from Joint Distribution

Joint distributions are the best!

Joint



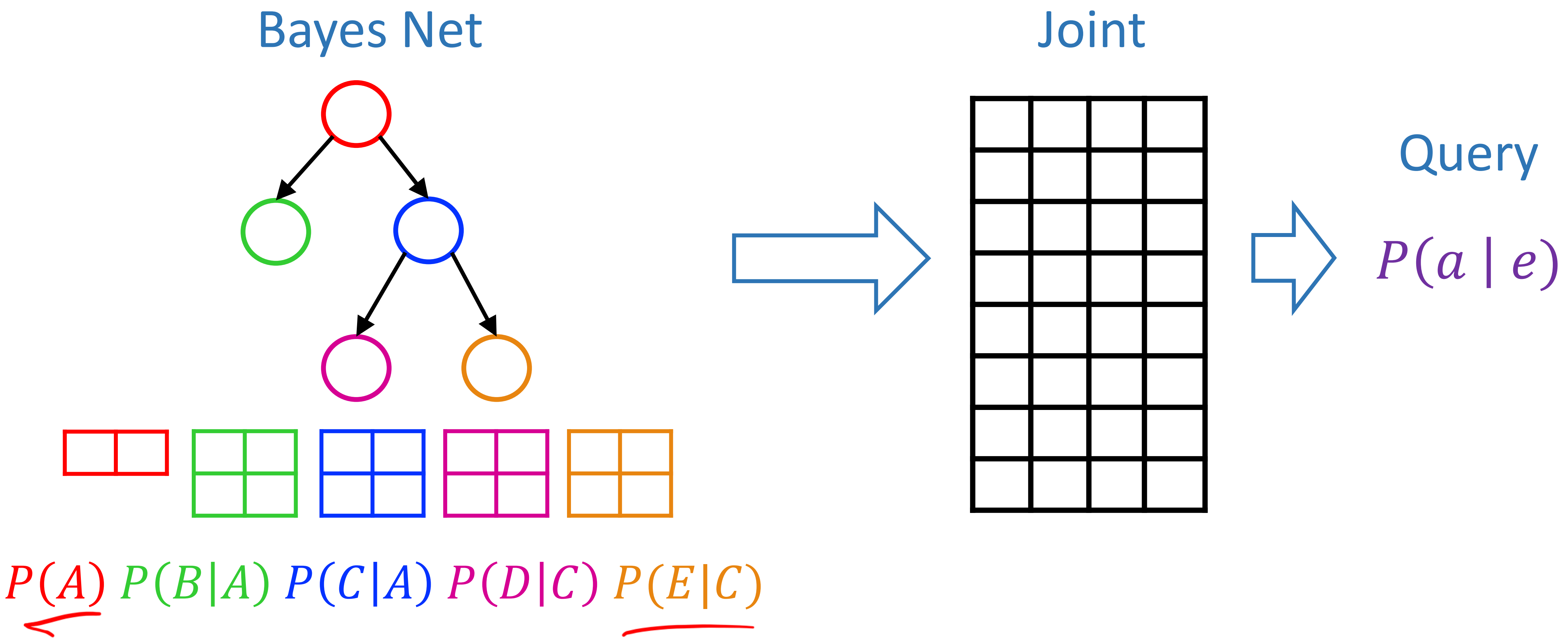
Query

$P(a \mid e)$

Problems with joints

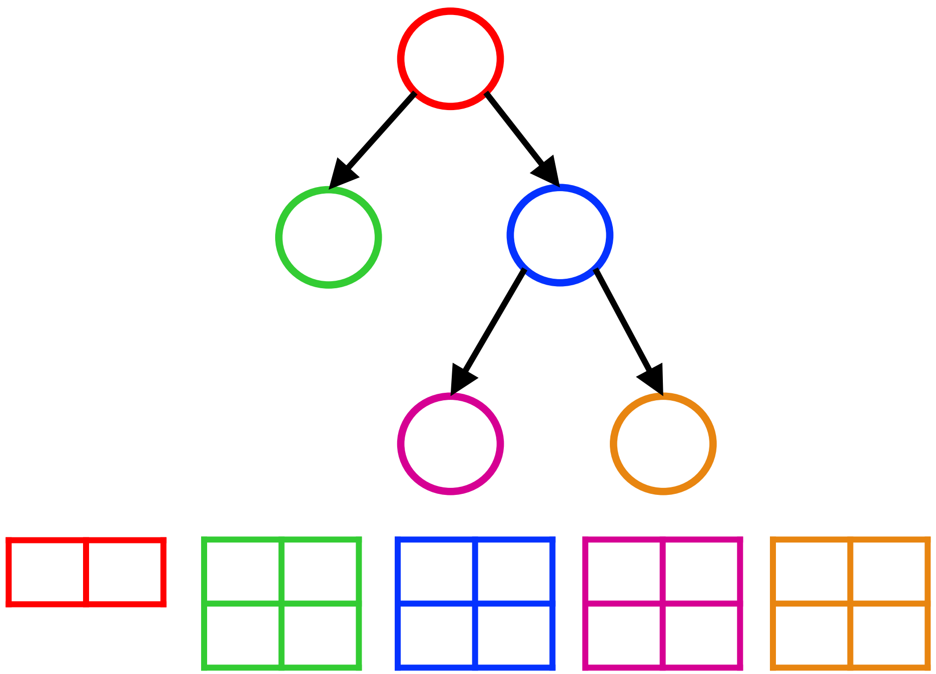
- We aren't given the joint table
  - Usually some set of conditional probability tables
- Huge
  - n variables with d values
  - d^n entries

Answer Any Query from Bayes Net



Next: Answer Any Query from Bayes Net

Bayes Net



$P(A)$ $P(B|A)$ $P(C|A)$ $P(D|C)$ $P(E|C)$

Query

$P(a \mid e)$