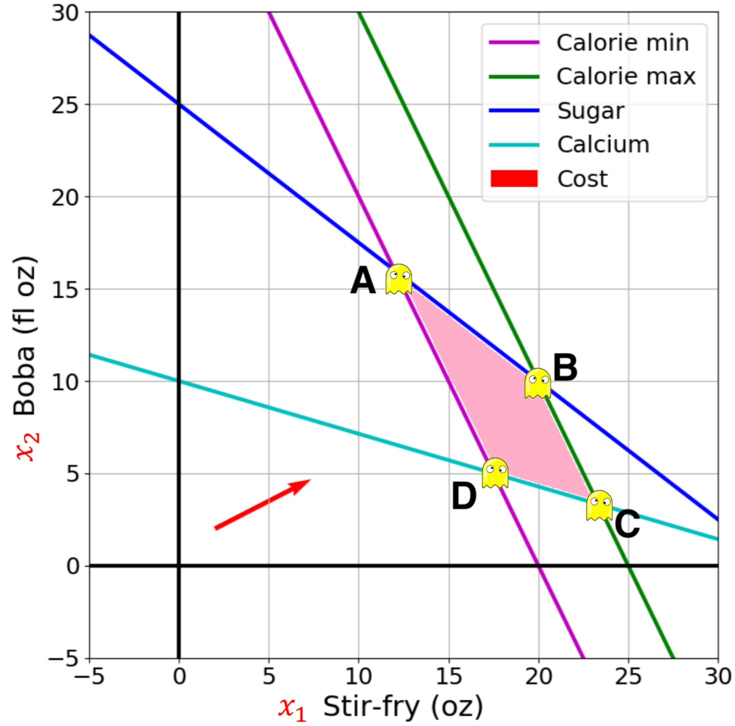
1 Warmup

In Tuesday's lecture, we went through two algorithms for solving linear programming programs - vector enumeration and the simplex algorithm.

Recall the "Healthy Squad Goals" example from Tuesday's lecture. The goal is to minimize the cost, and the cost vector (red) is perpendicular to the purple and green lines.



(a) Briefly describe both algorithms and explain how they differ. (hint: use terms such as vertices, intersections and neighbors).

Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found

Intersection is found by solving a pair of constraints (although some constraint pairs might not intersect). A neighboring intersection differs by only one constraint.

(b) Run simplex algorithm starting from point B. Now try running the algorithm starting from point C. How do their solutions differ?

Starting from point B returns a solution of point A, while starting from point C returns a solution of point D. Notice that points A and D are equally optimal.

Starting from point B, we have neighboring vertices point A and C. Point A is chosen as it is better. From point A, we have neighboring vertices point B and D. point B is worse than point A, and point D is equally optimal as point A. Thus, no better neighbor is found and the algorithm returns point A.

Starting from point C, we have neighboring vertices point B and D. Point D is chosen as it is better. From point D, the neighboring vertices A and C are either worse or equal to point D. Thus, no better neighbor is found and the algorithm returns point D.

2 Baymax's Factory

Baymax and the 281 TAs have opened a factory to produce special medicine and bandages. These are really difficult to produce and require the collaboration of robots and humans.

To produce an ounce of medicine, it takes 0.2 hours of human labor and 4 hours of robot labor. To produce an inch of bandage, it takes 0.5 hours of human labor and 2 hours of robot labor. An ounce of medicine sells for $30 and an inch of bandages sells for $30. Medicine and bandages can be sold in fractions of an ounce or inch.

We want to maximize our profit so we can buy gifts for all the students. However, the TAs are really busy so they can only devote 90 human hours. In addition, Baymax can only devote 800 robot hours because he has other obligations to tend to. How can we maximize our profit?

(a) Is this a linear, mixed or integer programming problem? Formulate and solve it.

It is a linear programming problem, as the medicine and bandages can be sold a fraction of a unit.
Let x be the ounces of medicine and y be the inches of bandages produced.

Objective: Maximize total profit:

$$\min_{x,y} -30x - 30y \tag{1}$$

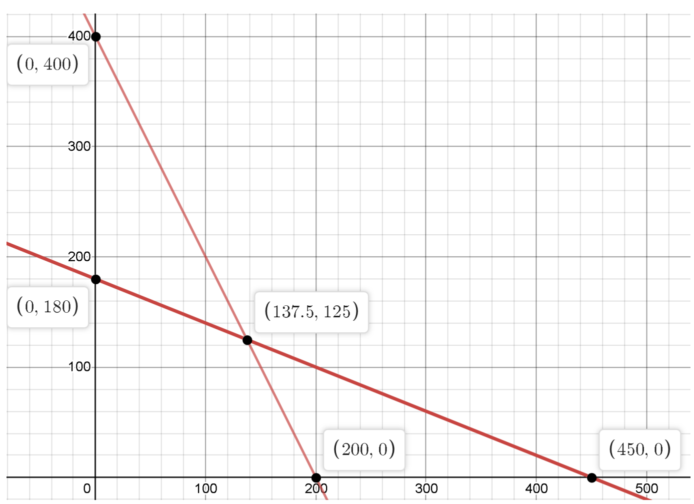
Constraints:

$$0.2x + 0.5y \leq 90$$
$$4x + 2y \leq 800$$
$$x \geq 0, y \geq 0$$

Given the two constraints, we can solve for x and y:

$$y = 180 - 0.4x$$
$$y = 400 - 2x$$

That gives us the following graph:



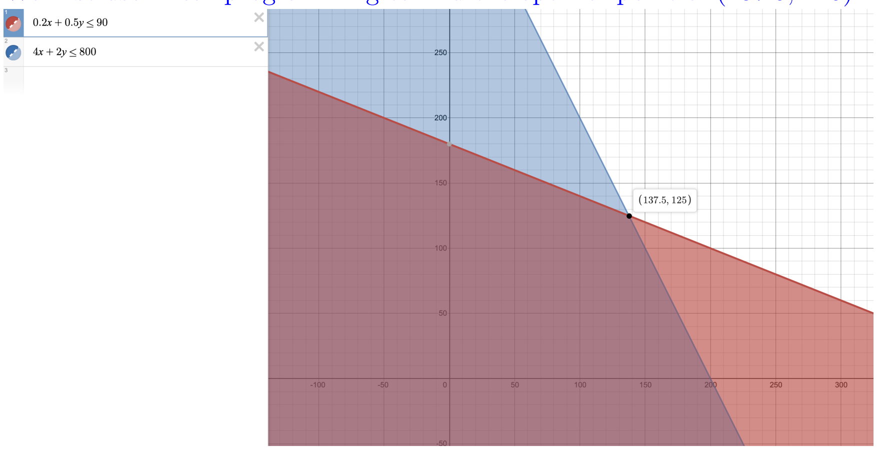
Since we want to maximize profit, we want to choose the furthest point, giving us $x = 137.5$ and $y = 125$.

Now suppose the items can only be sold in whole units (by ounce/inch).

(b) Is this a linear, mixed, or integer programming problem? Perform branch and bound for one branch level. You do not have to evaluate; writing out the constraints will suffice.

This is an integer programming problem, and the formulation is identical to part a). However, the domains of x and y are reduced to integers. We can solve the problem with branch and bound.

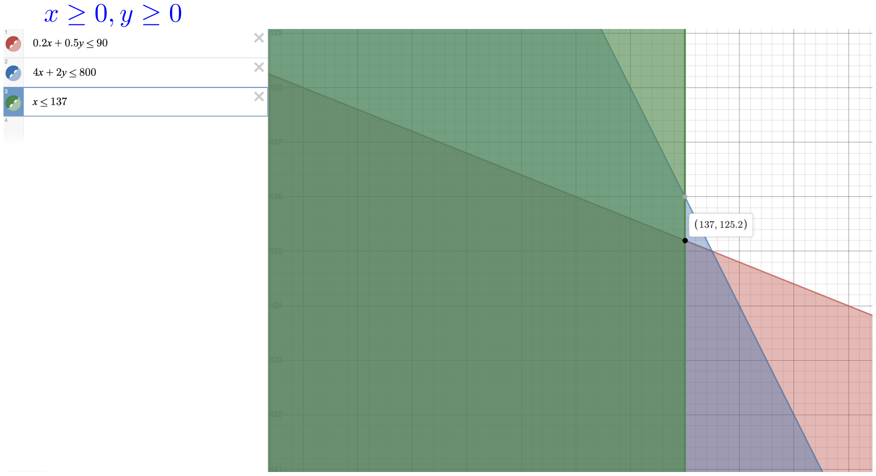
We first use linear programming to find the optimal point of (137.5, 125).



Since $x = 137.5$ is not an integer, we branch on it by adding the constraints that $x \leq 137$ or $x \geq 138$.

Left branch:

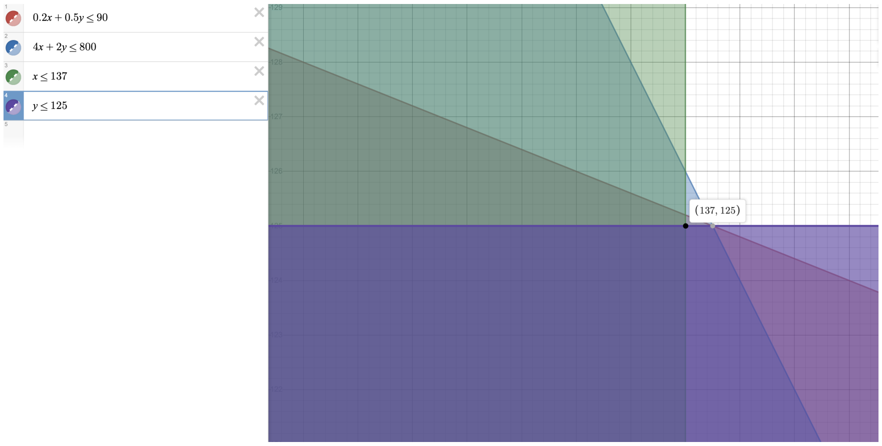
$x \leq 137$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$



The linear programming solution is (137, 125.2) so we now need to branch on the y value by adding the constraints that $y \leq 125$ or $y \geq 126$.

Left-Left branch:

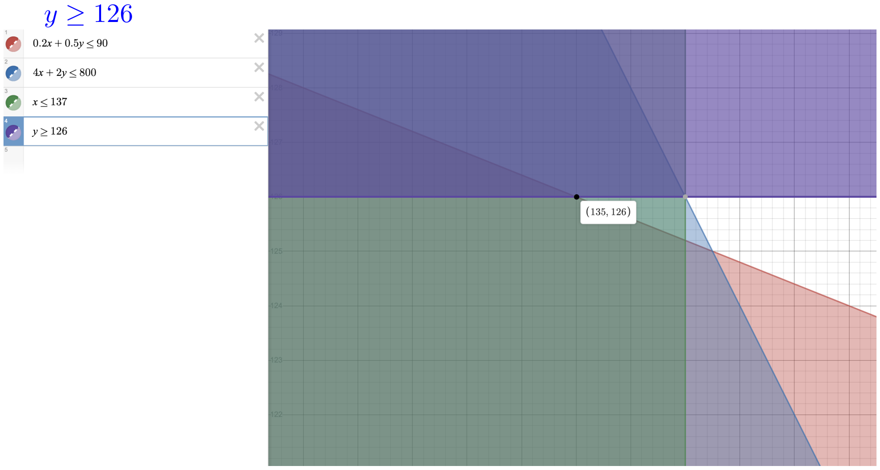
$x \leq 137$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$
$y \leq 125$



Since the linear programming solution (137, 125) is also an integer solution, we stop branching on the left-left branch and return this point along with its objective value of 7860.

Left-Right branch:

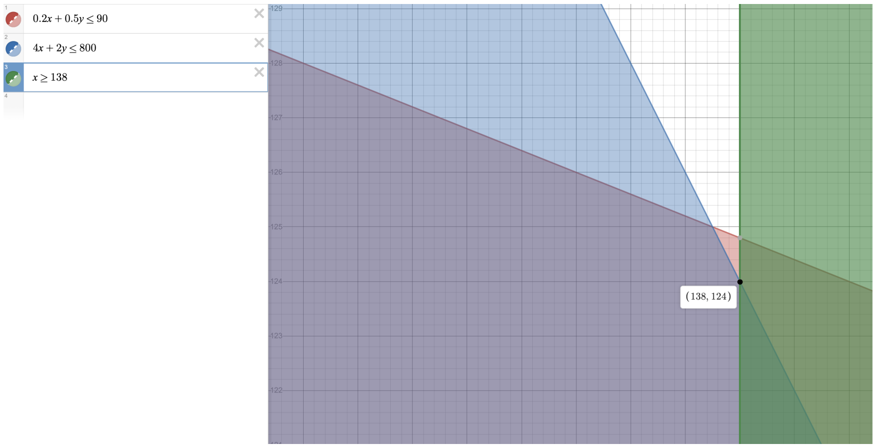
$x \leq 137$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$
$y \geq 126$



Since the linear programming solution (135, 126) is also an integer solution, we stop branching on the left-left branch and return this point along with its objective value of 7830.

Right branch:

$x \geq 138$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$



Since the linear programming solution (138, 124) is also an integer solution, we stop branching on the right branch and return this point, (138, 124), and objective value, 7860.

The final solution returned will be (137,125) with objective value 7860. Note that depending on implementation, (138,124), with objective value 7860 is also an acceptable solution.

(c) Now assume medicine can be sold in fractions but bandages can only be sold in whole units. What kind of a programming problem would this be, and how would our evaluation process differ from the problem type in part b?

This will be a mixed integer linear programming problem. We will evaluate by only branching and bounding on the number of bandages.

(d) How many optimal solutions can a LP have? How about IP?

Both LP and IP can have an infinite number of optimal solutions. Imagine a cost vector that's perpendicular to a constraint boundary. Then, we could have that constraint boundary cross infinitely many integers/real numbers (i.e. the line $x = 0$).

3 4-Queens

Recall the 4-Queens problem. The goal is to place 4 chess queens on a 4x4 chess board such at no two queens are in the same row, column and diagonal.

Formulate the 4-Queens problem as an integer programming problem.

Let our variables be x_{ij} for $0 \leq i \leq 3$, $0 \leq j \leq 3$, representing whether there is a queen in row i, column j. We want to find $\max_x \sum_i \sum_j x_{ij}$ such that $x_{ij} \in \{0, 1\}$.

Check: only one queen in each row - fix i and iterate over each column, ensuring they sum up to ≤ 1.

$$\sum_j x_{ij} = 1 \forall i \in \{0, 3\}$$

Check: only one queen in each column: fix j and iterate over each row, ensuring they sum up to ≤ 1.

$$\sum_i x_{ij} = 1 \forall j \in \{0, 3\}$$

Check: at most one queen in positive-slope diagonals (stretching from top left to bottom right):

$$\sum_{i,j:i+j=k} x_{ij} \leq 1, \forall k \in \{0, 1, 2, ..., 6\}$$

(k=0: (0, 0) | k=1: (0,1), (1,0) | k=2: (0,2), (1,1), (2,0) | k=3 ...)

Check: at most one queen in negative-slope diagonals (stretching from bottom left to top right):

$$\sum_{i,j:i-j=k} x_{ij} \leq 1, \forall k \in \{-3, -2, -1, ..., 3\}$$

Note that the equalities should all be represented as inequalities ≤ 1 and the negation of it ≤ -1.