1 Particle Filtering: Warmup

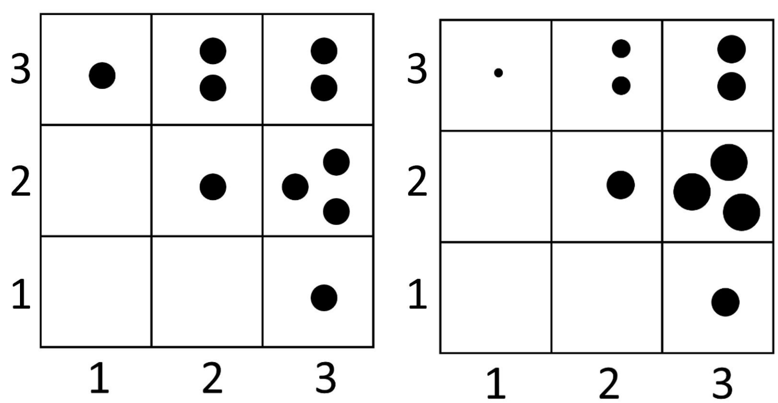
(a) **True / False:** The particle filtering algorithm is consistent since it gives correct probabilities as the number of samples N tends to infinity.

True

(b) **True / False:** The number of samples we use in the particle filtering algorithm increases from one time step to the next.

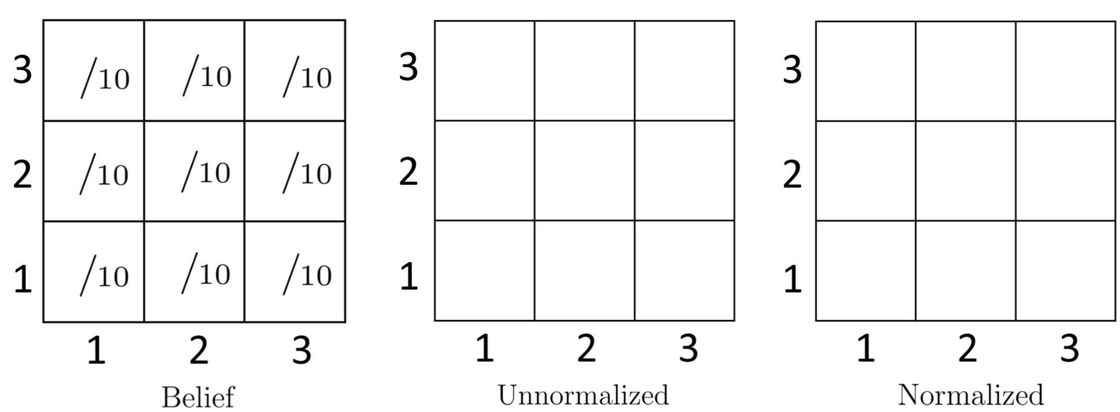
False. The number of samples stays constant from one time step to the next. The last step for each iteration of the algorithm is resampling, which builds a new population of N samples from the belief distribution updated by observation weights.

(c) The following state space contains 10 particles. The left grid shows the prior belief distribution of the particles at time t, while the grid on the right shows the particles weighted by the observations $P(e_t|S_t)$.



State	Weight
(1,3)	0.1
(2,2)	0.4
(2,3)	0.2
(3,1)	0.4
(3,2)	0.9
(3,3)	0.4

Fill in the following grids to update the belief distribution. Each square in the "Belief" grid should correspond to $\hat{P}(S_t|e_{1:t-1})$, the estimated probability of a particle being in state S at time t. Each square in the "Unnormalized" grid should correspond to the probability $P(S_t, e_t|e_{1:t-1})$. The "Normalized" grid should contain our updated belief distribution $\hat{P}(S_t|e_t, e_{1:t-1})$.



Solution: Note that states which did not appear in the weight table have a weight of 0.

3	1/10	2/10	2/10
2	0/10	1/10	3/10
1	0/10	0/10	1/10
	1	2	3

Belief

3	1/100	4/100	8/100
2	0	4/100	27/100
1	0	0	4/100
	1	2	3

Unnormalized

3	1/48	4/48	8/48
2	0	4/48	27/48
1	0	0	4/48
	1	2	3

Normalized

2 Tracking the Jabberwock

Lewis' Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers $z = (x, y) \in \mathbb{Z}^2 = \{\dots, -2, -1, 0, 1, 2, \dots\} \times \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. At each time step $t = 1, 2, 3, \dots$, the Jabberwock is in some cell $Z_t = z \in \mathbb{Z}^2$, and it moves to cell Z_{t+1} randomly as follows: with probability 1/2, it stays where it is; otherwise, it chooses one of its four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write a function for the transition probability $P(Z_{t+1} = (x', y')|Z_t = (x, y))$.

$$P(Z_{t+1} = (x', y')|Z_t = (x, y)) = \begin{cases} \frac{1}{2} & \text{if } x = x', y = y' \\ \frac{1}{8} & \text{if } |x - x'| + |y - y'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

We will use the particle filtering algorithm to track the Jabberwock. As a source of randomness use values in order from the following sequence $\{a_i\}_{1 \leq i \leq 14}$. Use these values to sample from any discrete distribution of the form $P(X)$ where X takes values in $\{1, 2, \dots, N\}$. Given $a_i \sim U[0, 1]$, return j such that $\sum_{k=1}^{j-1} P(X = k) \leq a_i < \sum_{k=1}^{j} P(X = k)$. You have to fix an ordering of the elements for this procedure to make sense.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
0.142	0.522	0.916	0.792	0.703	0.231	0.036	0.859	0.677	0.221	0.156	0.249

At each time step t you get an observation of the x coordinate R_t in which the Jabberwock sits, but it is a noisy observation. Given the true position $Z_t = (x, y)$, you observe the correct value according to the following probability:

$$P(R_t = r|Z_t = (x, y)) \propto (0.5)^{|x-r|}$$

(b) Suppose that you know that half of the time, the Jabberwock starts at $z_1 = (0, 0)$, and the other half, at $z_1 = (1, 1)$. You get the following observations: $R_1 = 1, R_2 = 0, R_3 = 1$. Fill out the table for each time step using a particle filter with 2 particles to compute an approximation to $P(Z_1, Z_2, Z_3|r_1, r_2, r_3)$. Sample transitions from the table below using the a_i's as our source of randomness. The a_i's you should use for each step habe been indicated in the last row of each table. Note that going "left" decrements the x-coordinate by 1, and going "down" decrements the y-coordinate by 1.

[0; 0.5)	Stay
[0.5; 0.625)	Up
[0.625; 0.75)	Left
[0.75; 0.875)	Right
[0.875; 1)	Down

Initial	Belief $\hat{P}(z_1)$	Weights $P(r_1\|z_1)$	Unnormalized $\hat{P}(z_1, r_1)$	Normalized $\hat{P}(z_1\|r_1)$	Resampling
$p_1 =$ (0,0) $p_2 =$ (1,1) a_1, a_2	1/2 1/2				$p_1 =$ (,) $p_2 =$ (,) a_3, a_4

Transition $P(z_2\|z_1)$	Belief $\hat{P}(z_2\|r_1)$	Weights $P(r_2\|z_2)$	Unnormalized $\hat{P}(z_2, r_2\|r_1)$	Normalized $\hat{P}(z_2\|r_1, r_2)$	Resampling
$p_2 =$ (,) $p_2 =$ (,) a_5, a_6					$p_1 =$ (,) $p_2 =$ (,) a_7, a_8

Transition $P(z_3\|z_2)$	Belief $\hat{P}(z_3\|r_1, r_2)$	Weights $P(r_3\|z_3)$	Unnormalized $\hat{P}(z_3, r_3\|r_1, r_2)$	Normalized $\hat{P}(z_3\|r_1, r_2, r_3)$	Resampling
$p_1 =$ (,) $p_2 =$ (,) a_9, a_{10}					$p_1 =$ (,) $p_2 =$ (,) a_{11}, a_{12}

For each time step, we use our random numbers a_i to sample from the prior or from the transitions. Next, we find the weight of the sample based on the observation at that time step. We update our belief distribution with the weight by taking the product $\hat{P}(z_t|r_{1:t-1})P(r_t|z_t)$ and normalizing to get $\hat{P}(z_t|r_{1:t})$. Note that since the two particles are in different locations at each time step, the belief $\hat{P}(z_t|r_{1:t-1})$ is always 1/2. Finally, we resample the particles from this updated belief distribution.

Initial	Belief $\hat{P}(z_1)$	Weights $P(r_1\|z_1)$	Unnormalized $\hat{P}(z_1, r_1)$	Normalized $\hat{P}(z_1\|r_1)$	Resampling
$p_1 =$ (0 , 0) $p_2 =$ (1 , 1) a_1, a_2	1/2 1/2	1/2 1	1/4 1/2	$4/3 * 1/4 = 1/3$ $4/3 * 1/2 = 2/3$	$p_1 =$ (1 , 1) $p_2 =$ (1 , 1) a_3, a_4

Transition $P(z_2 z_1)$	Belief $\hat{P}(z_2\|r_1)$	Weights $P(r_2\|z_2)$	Unnormalized $\hat{P}(z_2, r_2\|r_1)$	Normalized $\hat{P}(z_2\|r_1, r_2)$	Resampling
$p_1 =$ (0 , 1) $p_2 =$ (1 , 1) a_5(left), a_6(stay)	1/2 1/2	1 1/2	1/2 1/4	$4/3 * 1/2 = 2/3$ $4/3 * 1/4 = 1/3$	$p_1 =$ (0 , 1) $p_2 =$ (1 , 1) a_7, a_8

Transition $P(z_3\|z_2)$	Belief $\hat{P}(z_3\|r_1, r_2)$	Weights $P(r_3\|z_3)$	Unnormalized $\hat{P}(z_3, r_3\|r_1, r_2)$	Normalized $\hat{P}(z_3\|r_1, r_2, r_3)$	Resampling
$p_1 = (-1 , 1)$ $p_2 =$ (1 , 1) a_9(left), a_{10}(stay)	1/2 1/2	1/4 1	1/8 1/2	$8/5 * 1/8 = 1/5$ $8/5 * 1/2 = 4/5$	$p_1 = (-1, 1)$ $p_2 =$ (1 , 1) a_{11}, a_{12}

(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of Z_3 is different than the column of Z_3, i.e. $X_3 \neq Y_3$.

Out of the two unweighted particles in the last step, exactly one satisfies $X_3 = Y_3$, so the estimate is 1/2.

(e) What is the problem of using the elimination algorithm instead of a particle filter for tracking Jabberwock?

The state space is infinite, so factors of infinite size (distributions over all points on the plane) would need to be computed and stored when using the variable elimination algorithm.

3 Game Theory: Equilibrium

(a) What is a Nash Equilibrium?

A Nash Equilibrium is a set of strategies where no player benefits from changing their strategy alone.

(b) What is a zero-sum game?

A game in which the sum of the utilities(payoffs) is zero.

(c) What is the difference between a weakly dominant and strictly dominant strategy?

A strategy is weakly dominant if the player's payoff is at least as good as the opponent's (no matter the opponent's strategy). A strategy is strictly dominant if the player's payoff is always strictly better than the opponent's.

(d) What is the difference between pure and mixed strategies?

A pure strategy is deterministic, while a mixed strategy incorporates a randomized action selection based on a distribution.

(e) Consider rock paper scissors where Player 1's strategy is to always play rock, and Player 2's strategy is to play scissors or paper with equal probability. Is this a Nash Equilibrium? What strategy would be best for Player 1 given Player 2's current strategy? What strategy would be best for Player 2 given Player 1's current strategy?

This is not a Nash Equilibrium. Player 1 would benefit from changing their strategy to always playing scissors. Player 2 would benefit from changing their strategy to always playing paper.

(f) Recursively remove dominated strategies to find the Nash Equilibrium of the following game. The order of utilities in each cell is the roman numeral player then the alphabet player.

	A	B	C
i	3,0	0,-5	0,-4
ii	1,-1	3,3	-2,4
iii	2,4	4,1	-1,8

Firstly, strategy ii is dominated by strategy iii. Also, strategy B is dominated by strategy C. This leaves the four corners of the table to be considered. Considering only the corners, strategy i dominates strategy iii. This leaves only the top two corners, in which case strategy A dominates strategy C and we are left with (3,0) as the Nash Equilibrium.

(g) Bert and Ernie have a worrying disregard for their own safety. For their entertainment, they have designed a game as follows: They drive towards each other while in the same lane (A) of a 2-lane road, and right before they are to meet, each decides to stay in lane A or move to the other lane B. If they meet while in the same lane (either both A or both B), they crash and have to get new lambos. If they pick different lanes, they get the positive reward of an adrenaline rush. The utility table is as follows, with Bert as the row and Ernie as the column:

Bert, Ernie	A	B
A	-5,-5	3,3
B	3,3	-5,-5

(i) What are the pure Nash Equilibria of this problem?

The two pure Nash Equilibria are when they pick different lanes. Neither player benefits by switching into the same lane as the other, so this is an equilibrium.

(ii) We will now investigate the possibility of a mixed Nash equilibrium. Recall that in a mixed Nash Equilibrium, the utilities of the weighted actions are equal. Let p be the probability that Ernie picks lane A.

(1) What is the expected value of action A for Bert?

$-5p + 3(1 - p) = 3 - 8p$

(2) What is the expected value of action B for Bert?

$3p - 5(1 - p) = -5 + 8p$

(3) What value of p makes these two expected values the same?

$3 - 8p = -5 + 8p$
$8 = 16p$
$p = 1/2$

(4) Since the table is symmetric, the probability that equalizes the value of action A and B for Ernie is also 1/2. What is the expected utility for both Bert and Ernie? How does this utility compare to the equilibria from (a)?

$3 - 8(1/2) = -5 + 8(1/2) = -1$. This equilibrium is worse since the expected utility for both is -1, which is less than 3.