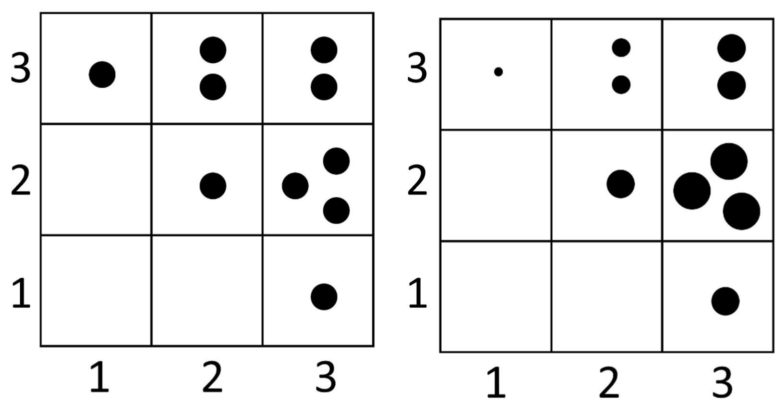
1 Particle Filtering: Warmup

(a) **True / False:** The particle filtering algorithm is consistent since it gives correct probabilities as the number of samples N tends to infinity.

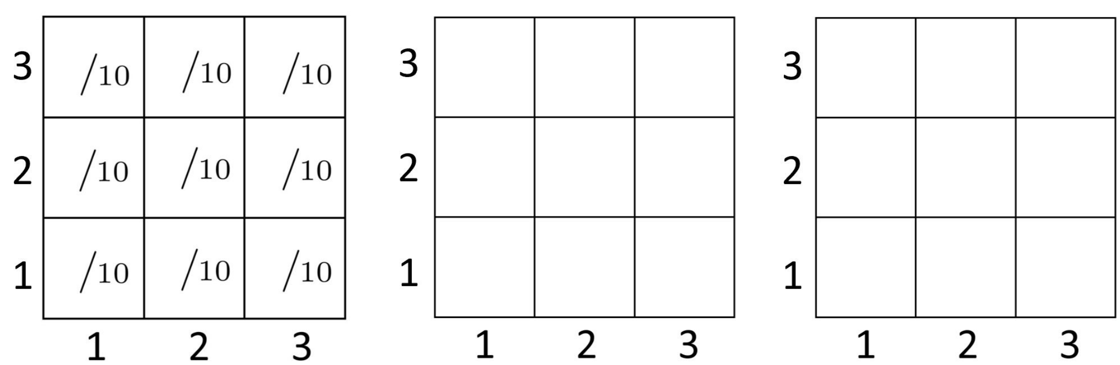
(b) **True / False:** The number of samples we use in the particle filtering algorithm increases from one time step to the next.

(c) The following state space contains 10 particles. The left grid shows the prior belief distribution of the particles at time t, while the grid on the right shows the particles weighted by the observations $P(E_t|S_t)$.



State	Weight
(1,3)	0.1
(2,2)	0.4
(2,3)	0.2
(3,1)	0.4
(3,2)	0.9
(3,3)	0.4

Fill in the following grids to update the belief distribution. Each square in the "Belief" grid should correspond to $\hat{P}(S_t|e_{1:t-1})$, the estimated probability of a particle being in state S at time t. Each square in the "Unnormalized" grid should correspond to the probability $P(S_t, e_t|e_{1:t-1})$. The "Normalized" grid should contain our updated belief distribution $\hat{P}(S_t|e_t, e_{1:t-1})$.



Belief Unnormalized Normalized

2 Tracking the Jabberwock

Lewis' Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers $z = (x, y) \in \mathbb{Z}^2 = \{\dots, -2, -1, 0, 1, 2, \dots\} \times \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. At each time step $t = 1, 2, 3, \dots$, the Jabberwock is in some cell $Z_t = z \in \mathbb{Z}^2$, and it moves to cell Z_{t+1} randomly as follows: with probability 1/2, it stays where it is; otherwise, it chooses one of its four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write a function for the transition probability $P(Z_{t+1} = (x', y') | Z_t = (x, y))$.

We will use the particle filtering algorithm to track the Jabberwock. As a source of randomness use values in order from the following sequence $\{a_i\}_{1 \leq i \leq 14}$. Use these values to sample from any discrete distribution of the form $P(X)$ where X takes values in $\{1, 2, \dots, N\}$. Given $a_i \sim U[0, 1]$, return j such that $\sum_{k=1}^{j-1} P(X = k) \leq a_i < \sum_{k=1}^{j} P(X = k)$. You have to fix an ordering of the elements for this procedure to make sense.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
0.142	0.522	0.916	0.792	0.703	0.231	0.036	0.859	0.677	0.221	0.156	0.249

At each time step t you get an observation of the x coordinate R_t in which the Jabberwock sits, but it is a noisy observation. Given the true position $Z_t = (x, y)$, you observe the correct value according to the following probability:

$$P(R_t = r | Z_t = (x, y)) \propto (0.5)^{|x-r|}$$

(b) Suppose that you know that half of the time, the Jabberwock starts at $z_1 = (0, 0)$, and the other half, at $z_1 = (1, 1)$. You get the following observations: $R_1 = 1, R_2 = 0, R_3 = 1$. Fill out the table for each time step using a particle filter with 2 particles to compute an approximation to $P(Z_1, Z_2, Z_3 | r_1, r_2, r_3)$. Sample transitions from the table below using the a_i's as our source of randomness. The a_i's you should use for each step habe been indicated in the last row of each table. Note that going "left" decrements the x-coordinate by 1, and going "down" decrements the y-coordinate by 1.

$[0; 0.5)$	Stay
$[0.5; 0.625)$	Up
$[0.625; 0.75)$	Left
$[0.75; 0.875)$	Right
$[0.875; 1)$	Down

Initial	Belief $\hat{P}(z_1)$	Weights $P(r_1\|z_1)$	Unnormalized $\hat{P}(z_1, r_1)$	Normalized $\hat{P}(z_1\|r_1)$	Resampling
$p_1 =$ (0,0)	1/2				$p_1 =$ (,)
$p_2 =$ (1,1)	1/2				$p_2 =$ (,)
a_1, a_2					a_3, a_4

Transition $P(z_2\|z_1)$	Belief $\hat{P}(z_2\|r_1)$	Weights $P(r_2\|z_2)$	Unnormalized $\hat{P}(z_2, r_2\|r_1)$	Normalized $\hat{P}(z_2\|r_1, r_2)$	Resampling
$p_2 =$ (,)					$p_1 =$ (,)
$p_2 =$ (,)					$p_2 =$ (,)
a_5, a_6					a_7, a_8

Transition $P(z_3\|z_2)$	Belief $\hat{P}(z_3\|r_1, r_2)$	Weights $P(r_3\|z_3)$	Unnormalized $\hat{P}(z_3, r_3\|r_1, r_2)$	Normalized $\hat{P}(z_3\|r_1, r_2, r_3)$	Resampling
$p_1 =$ (,)					$p_1 =$ (,)
$p_2 =$ (,)					$p_2 =$ (,)
a_9, a_{10}					a_{11}, a_{12}

(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of Z_3 is different than the column of Z_3, i.e. $X_3 \neq Y_3$.

(e) What is the problem of using the elimination algorithm instead of a particle filter for tracking Jabberwock?

3 Game Theory: Equilibrium

(a) What is a Nash Equilibrium?

(b) What is a zero-sum game?

(c) What is the difference between a weakly dominant and strictly dominant strategy?

(d) What is the difference between pure and mixed strategies?

(e) Consider rock paper scissors where Player 1's strategy is to always play rock, and Player 2's strategy is to play scissors or paper with equal probability. Is this a Nash Equilibrium? What strategy would be best for Player 1 given Player 2's current strategy? What strategy would be best for Player 2 given Player 1's current strategy?

(f) Recursively remove dominated strategies to find the Nash Equilibrium of the following game. The order of utilities in each cell is the roman numeral player then the alphabet player.

	A	B	C
i	3,0	0,-5	0,-4
ii	1,-1	3,3	-2,4
iii	2,4	4,1	-1,8

(g) Bert and Ernie have a worrying disregard for their own safety. For their entertainment, they have designed a game as follows: They drive towards each other while in the same lane (A) of a 2-lane road, and right before they are to meet, each decides to stay in lane A or move to the other lane B. If they meet while in the same lane (either both A or both B), they crash and have to get new lambos. If they pick different lanes, they get the positive reward of an adrenaline rush. The utility table is as follows, with Bert as the row and Ernie as the column:

Bert, Ernie	A	B
A	-5,-5	3,3
B	3,3	-5,-5

(i) What are the pure Nash Equilibria of this problem?

(ii) We will now investigate the possibility of a mixed Nash equilibrium. Recall that in a mixed Nash Equilibrium, the utilities of the weighted actions are equal. Let p be the probability that Ernie picks lane A.

(1) What is the expected value of action A for Bert?

(2) What is the expected value of action B for Bert?

(3) What value of p makes these two expected values the same?

(4) Since the table is symmetric, the probability that equalizes the value of action A and B for Ernie is also 1/2. What is the expected utility for both Bert and Ernie? How does this utility compare to the equilibria from (a)?