Warm-up:

What is the relationship between number of constraints and number of possible solutions?

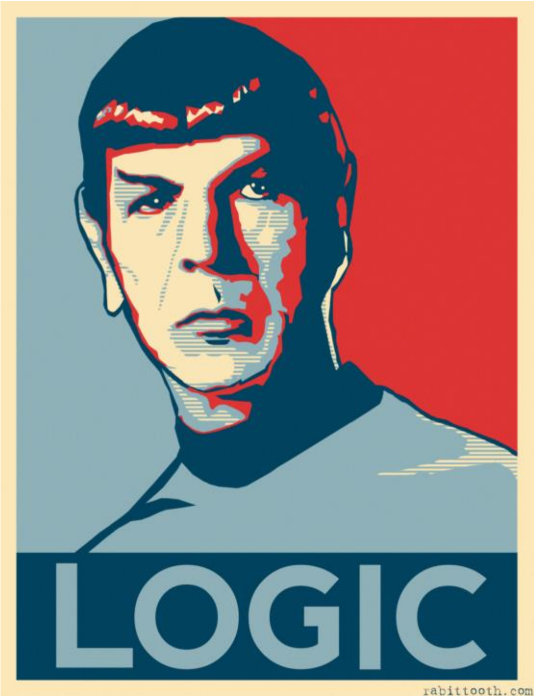
In other words, as the number of the constraints increases,

does the number of possible solutions:

A) Increase
B) Decrease
C) Stay the same

AI: Representation and Problem Solving

Propositional Logic

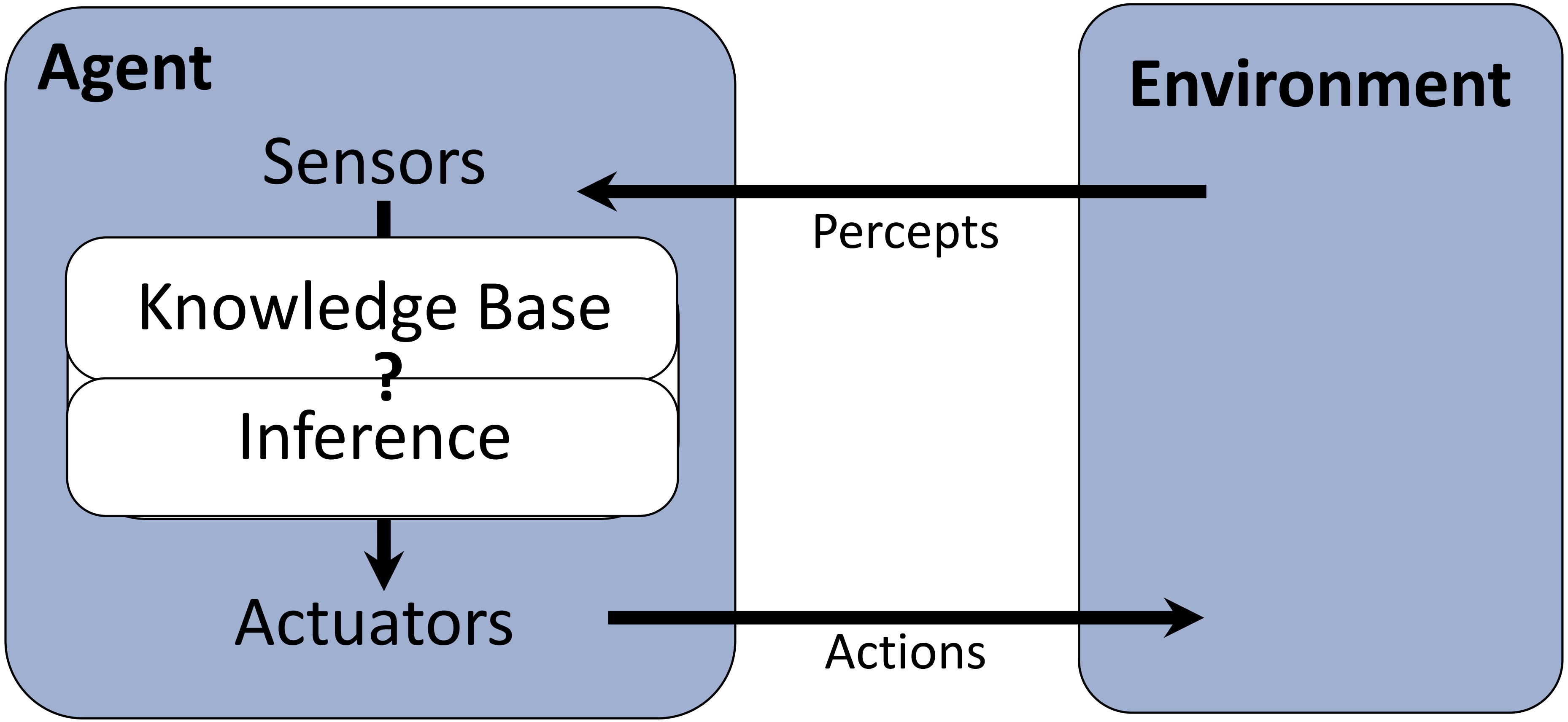


Instructors: Pat Virtue & Fei Fang

Slide credits: CMU AI, http://ai.berkeley.edu

Logical Agents

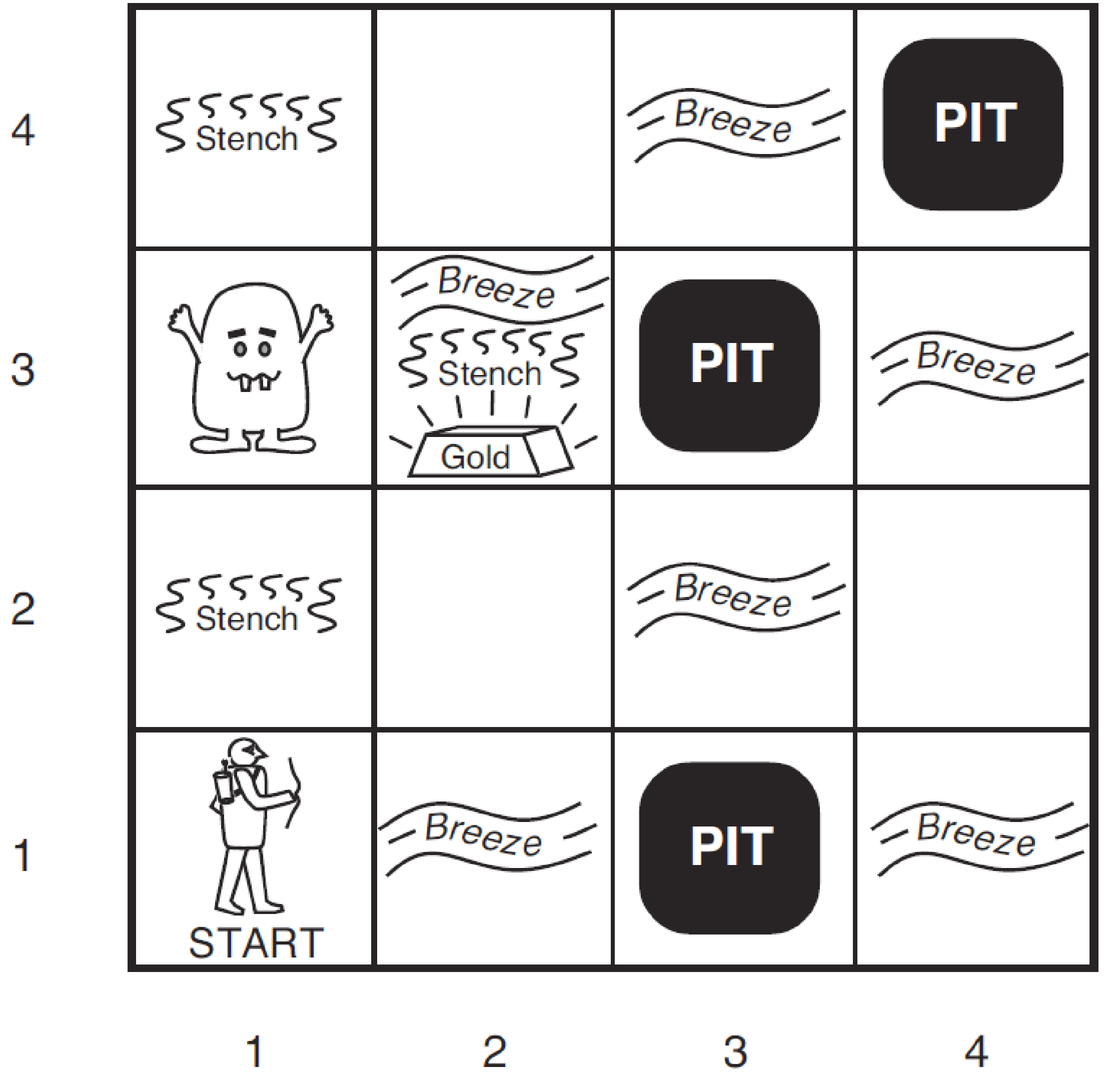
Logical agents and environments



Wumpus World

Logical Reasoning as a CSP

- B_{ij} = breeze felt
- S_{ij} = stench smelt
- P_{ij} = pit here
- W_{ij} = wumpus here
- G = gold



http://thiagodnf.github.io/wumpus-world-simulator/

A Knowledge-based Agent

function KB-AGENT(percept) returns an action

persistent: KB, a knowledge base

t, an integer, initially 0

TELL(KB, PROCESS-PERCEPT(percept, t))

action ← ASK(KB, PROCESS-QUERY(t))

TELL(KB, PROCESS-RESULT(action, t))

t←t+1

return action

Logical Agents

So what do we TELL our knowledge base (KB)?

- Facts (sentences)
  - The grass is green
  - The sky is blue
- Rules (sentences)
  - Eating too much candy makes you sick
  - When you’re sick you don’t go to school
- Percepts and Actions (sentences)
  - Pat ate too much candy today

What happens when we ASK the agent?

- Inference – new sentences created from old
  - Pat is not going to school today

Logical Agents

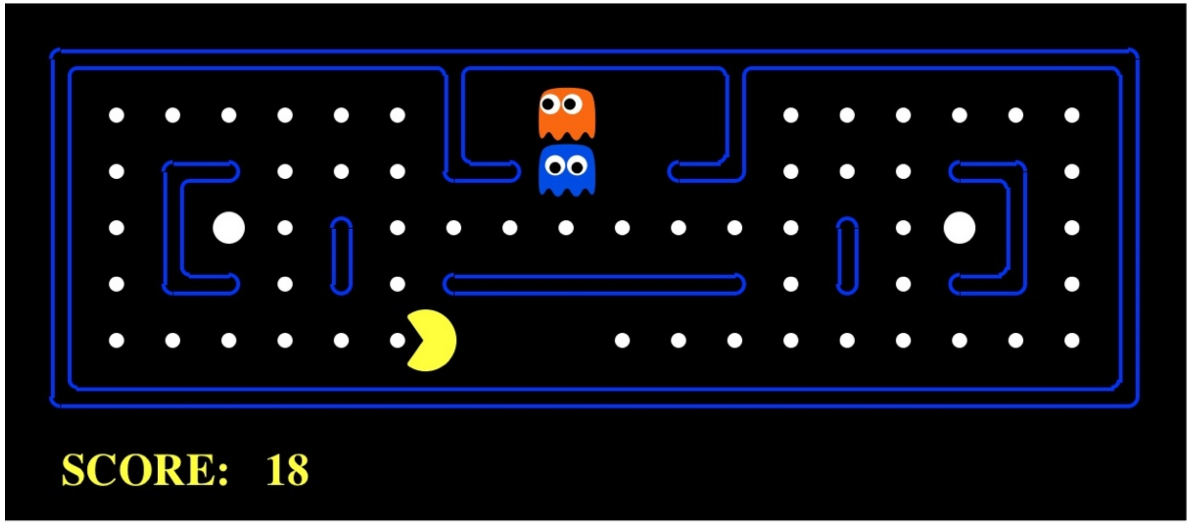
Sherlock Agent

- Really good knowledge base
  - Evidence
  - Understanding of how the world works (physics, chemistry, sociology)
- Really good inference
  - Skills of deduction
  - "It's elementary my dear Watson"

Dr. Strange?
Alan Turing?
Kahn?

Worlds

What are we trying to figure out?



- Who, what, when, where, why
- Time: past, present, future

- Actions, strategy
- Partially observable? Ghosts, Walls

Which world are we living in?

Models

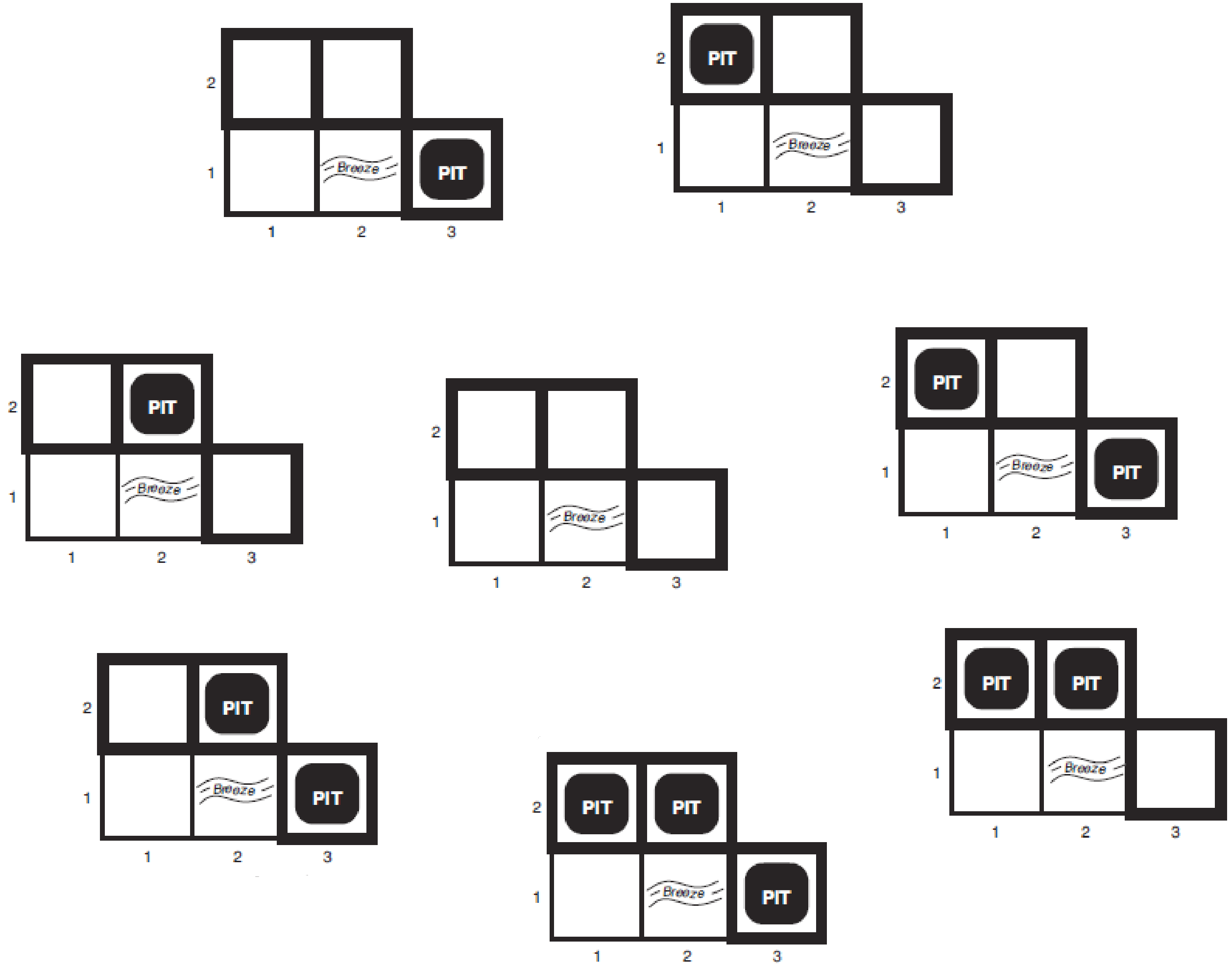
How do we represent possible worlds with models and knowledge bases?

How do we then do inference with these representations?

Wumpus World

Possible Models

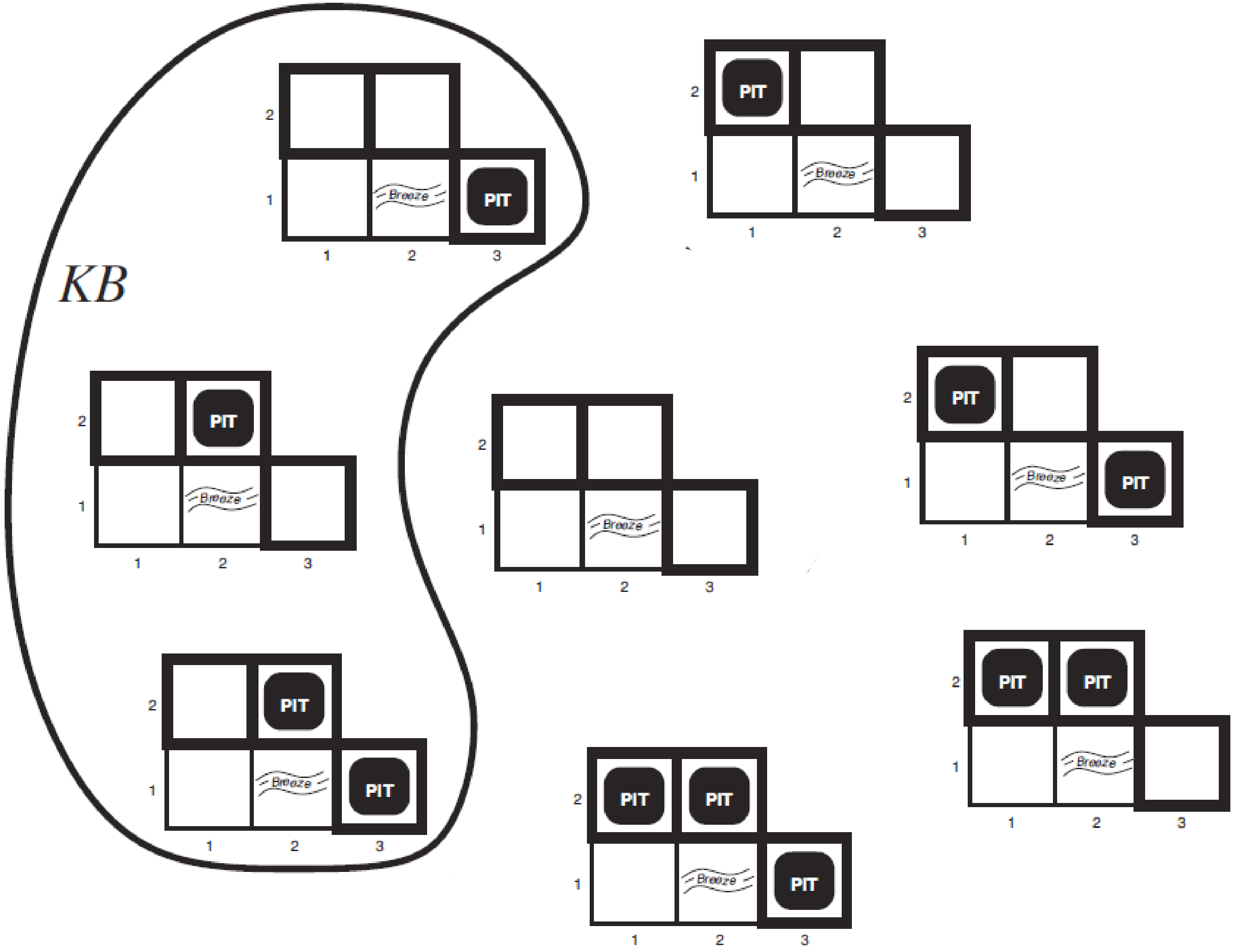
- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$



Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
  - Nothing in [1,1]
  - Breeze in [2,1]



Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
  - Nothing in [1,1]
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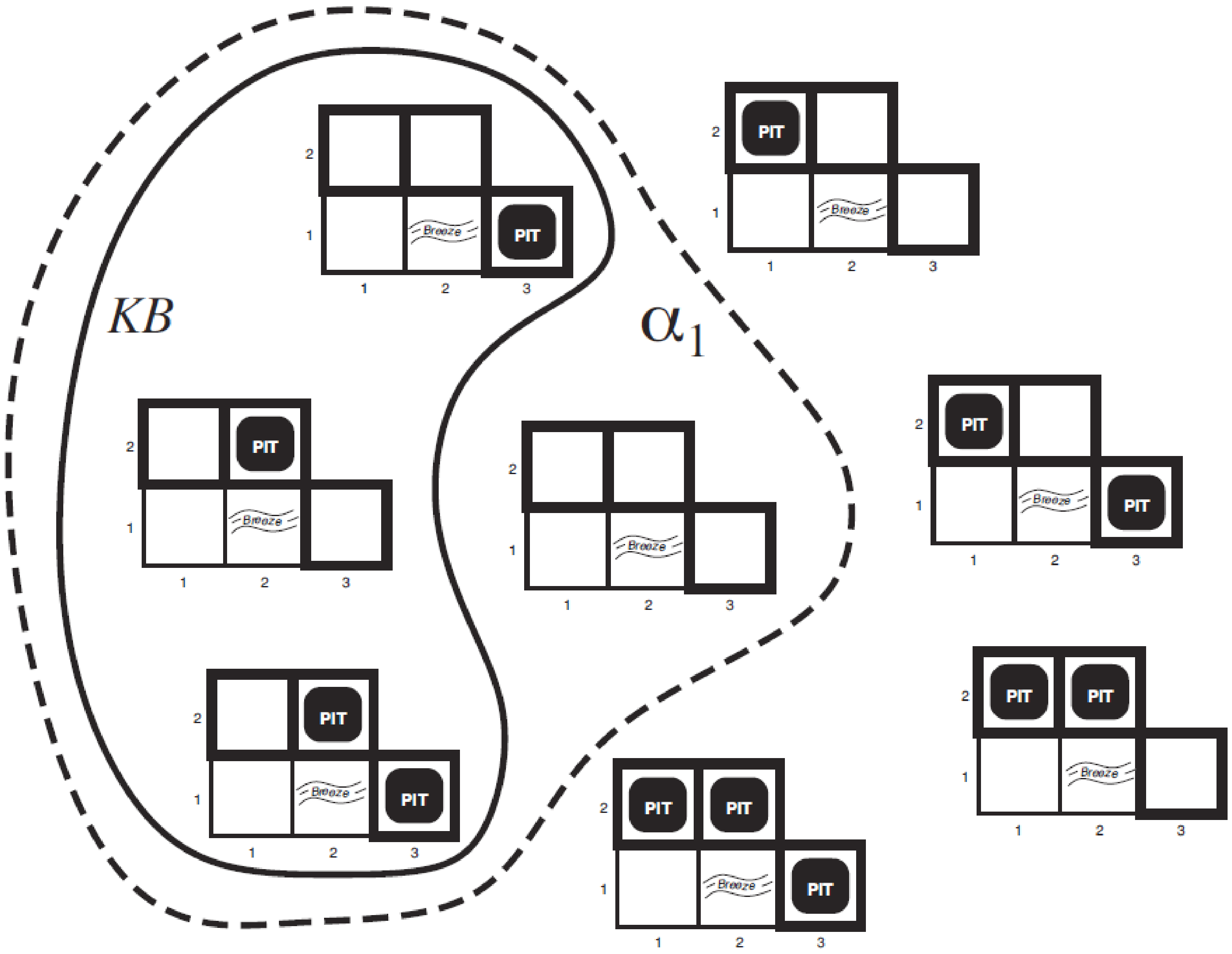
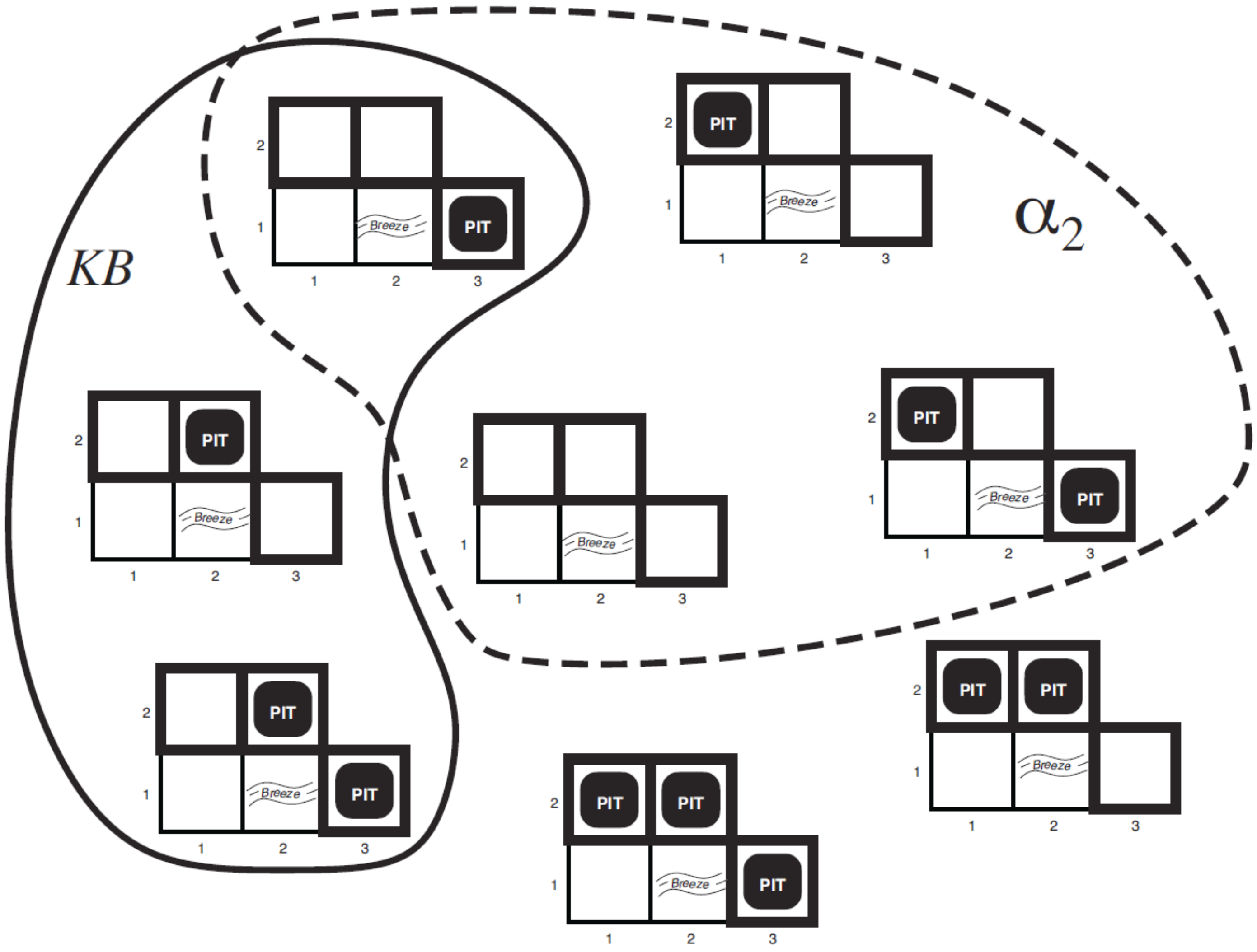
- Query α_1:
  - No pit in [1,2]



Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
  - Nothing in [1,1]
  - Breeze in [2,1]
- Query α_2:
  - No pit in [2,2]



Logic Language

Natural language?

Propositional logic

- Syntax: $P \vee (\neg Q \wedge R)$; $X_1 \Leftrightarrow (\text{Raining} \Rightarrow \text{Sunny})$
- Possible world: {P=true, Q=true, R=false, S=true} or 1101
- Semantics: $\alpha \wedge \beta$ is true in a world iff is α true and β is true (etc.)

First-order logic

- Syntax: $\forall x\ \exists y\ P(x,y) \wedge \neg Q(Joe,f(x)) \Rightarrow f(x)=f(y)$
- Possible world: Objects o_1, o_2, o_3; P holds for $<o_1,o_2>$; Q holds for $<o_3>$; $f(o_1)=o_1$; $Joe=o_3$; etc.
- Semantics: $\phi(\sigma)$ is true in a world if $\sigma=o_j$ and ϕ holds for o_j; etc.

Propositional Logic

Propositional Logic

Symbol:

- Variable that can be true or false
- We'll try to use capital letters, e.g. A, B, $P_{1,2}$
- Often include True and False

Operators:

- $\neg A$: not A
- $A \wedge B$: A and B (conjunction)
- $A \vee B$: A or B (disjunction) Note: this is not an "exclusive or"
- $A \Rightarrow B$: A implies B (implication). If A then B
- $A \Leftrightarrow B$: A if and only if B (biconditional)

Sentences

Propositional Logic Syntax

Given: a set of proposition symbols $\{X_1, X_2, \ldots, X_n\}$

- (we often add True and False for convenience)

X_i is a sentence

If α is a sentence then $\neg\alpha$ is a sentence

If α and β are sentences then $\alpha \wedge \beta$ is a sentence

If α and β are sentences then $\alpha \vee \beta$ is a sentence

If α and β are sentences then $\alpha \Rightarrow \beta$ is a sentence

If α and β are sentences then $\alpha \Leftrightarrow \beta$ is a sentence

And p.s. there are no other sentences!

Notes on Operators

$\boldsymbol{\alpha} \lor \boldsymbol{\beta}$ is <u>inclusive or</u>, not exclusive

Truth Tables

$\alpha \vee \beta$ is inclusive or, not exclusive

α	β	$\alpha \wedge \beta$
F	F	F
F	T	F
T	F	F
T	T	T

α	β	$\alpha \vee \beta$
F	F	F
F	T	T
T	F	T
T	T	T

Notes on Operators

$\alpha \vee \beta$ is inclusive or, not exclusive

$\alpha \Rightarrow \beta$ is equivalent to $\neg\alpha \vee \beta$

- Says who?

Truth Tables

$\alpha \Rightarrow \beta$ is equivalent to $\neg\alpha \vee \beta$

α	β	$\alpha \Rightarrow \beta$	$\neg\alpha$	$\neg\alpha \vee \beta$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

Notes on Operators

$\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ is inclusive or, not exclusive

$\boldsymbol{\alpha} \Rightarrow \boldsymbol{\beta}$ is equivalent to $\neg\boldsymbol{\alpha} \vee \boldsymbol{\beta}$

- Says who?

$\boldsymbol{\alpha} \Leftrightarrow \boldsymbol{\beta}$ is equivalent to $(\boldsymbol{\alpha} \Rightarrow \boldsymbol{\beta}) \wedge (\boldsymbol{\beta} \Rightarrow \boldsymbol{\alpha})$

- Prove it!

Truth Tables

$\alpha \Leftrightarrow \beta$ is equivalent to $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$

α	β	$\alpha \Leftrightarrow \beta$	$\alpha \Rightarrow \beta$	$\beta \Rightarrow \alpha$	$(\alpha\Rightarrow\beta) \land (\beta\Rightarrow\alpha)$
F	F	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
T	T	T	T	T	T

Equivalence: it's true in all models. Expressed as a logical sentence:

$$(\alpha \Leftrightarrow \beta) \Leftrightarrow [(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)]$$

Propositional Logic

Check if sentence is true in given model

In other words, does the model *satisfy* the sentence?

function PL-TRUE?(α,model) returns true or false

if α is a symbol then return Lookup(α, model)

if Op(α) = $\neg$ then return **not**(PL-TRUE?(Arg1(α),model))

if Op(α) = $\wedge$ then return **and**(PL-TRUE?(Arg1(α),model),

PL-TRUE?(Arg2(α),model))

etc.

(Sometimes called "recursion over syntax")

Propositional Logical Vocab

Vocab Alert!

Literal

- Atomic sentence: True, False, Symbol, ¬Symbol

Clause

- Disjunction of literals: $A \vee B \ \vee \ \neg C$

Definite clause

- Disjunction of literals, *exactly one* is positive
- $\neg A \vee B \ \vee \ \neg C$

Horn clause

- Disjunction of literals, *at most one* is positive
- All definite clauses are Horn clauses

Monty Python Inference

There are ways of telling whether she is a witch

Sentences as Constraints

Adding a sentence to our knowledge base constrains the number of possible models:

KB: Nothing

Possible Models

P	Q	R
false	false	false
false	false	true
false	true	false
false	true	true
true	false	false
true	false	true
true	true	false
true	true	true

Sentences as Constraints

Adding a sentence to our knowledge base constrains the number of possible models:

KB: Nothing

KB: $[(P \wedge \neg Q) \vee (Q \wedge \neg P)] \Rightarrow R$

Possible Models

P	Q	R
false	false	false
false	false	true
false	true	false
false	true	true
true	false	false
true	false	true
true	true	false
true	true	true

Sentences as Constraints

Adding a sentence to our knowledge base constrains the number of possible models:

KB: Nothing

KB: $[(P \land \neg Q) \lor (Q \land \neg P)] \Rightarrow R$

KB: R, $[(P \land \neg Q) \lor (Q \land \neg P)] \Rightarrow R$

Possible Models

P	Q	R
false	false	false
false	false	true
false	true	false
false	true	true
true	false	false
true	false	true
true	true	false
true	true	true

Sherlock Entailment

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth" – *Sherlock Holmes via Sir Arthur Conan Doyle*

- Knowledge base and inference allow us to remove impossible models, helping us to see what is true in all of the remaining models

Entailment

Entailment: $\alpha \models \beta$ ("α entails β" or "β follows from α") iff in every world where α is true, β is also true

- I.e., the α-worlds are a subset of the β-worlds [*models*(α) $\subseteq$ *models*(β)]

Usually we want to know if $KB \models query$

- *models*(*KB*) $\subseteq$ *models*(*query*)
- In other words
  - *KB* removes all impossible models (any model where *KB* is false)
  - If β is true in all of these remaining models, we conclude that β must be true

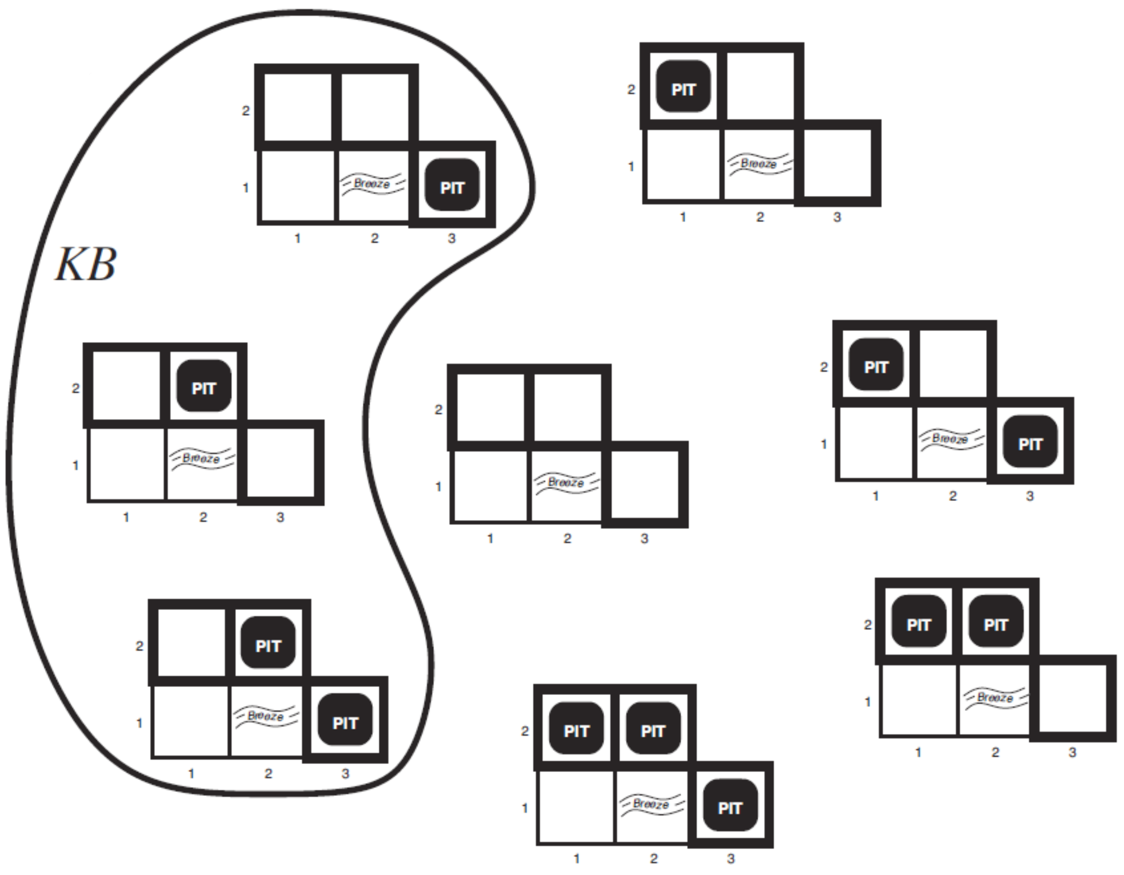
Entailment and implication are very much related

- However, entailment relates two sentences, while an implication is itself a sentence (usually derived via inference to show entailment)

Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
  - Nothing in [1,1]
  - Breeze in [2,1]



Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
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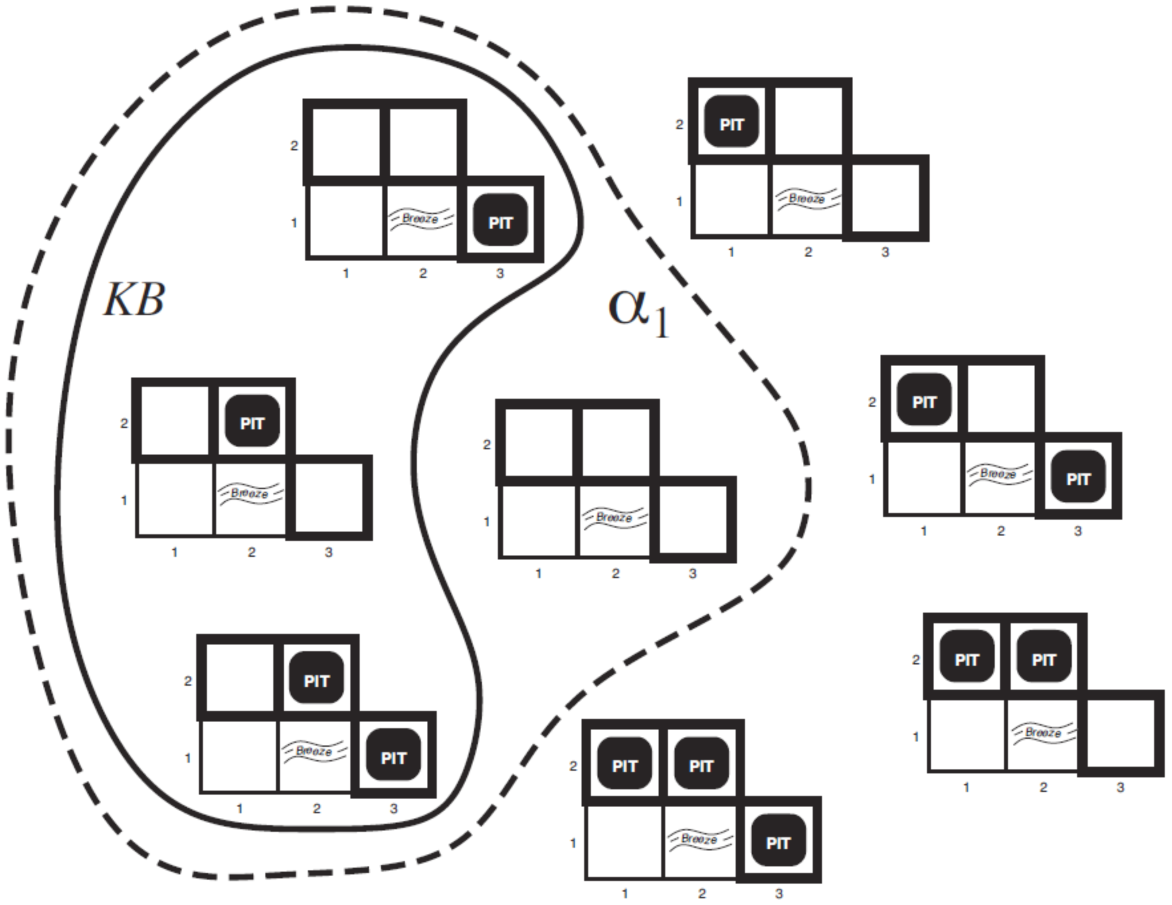
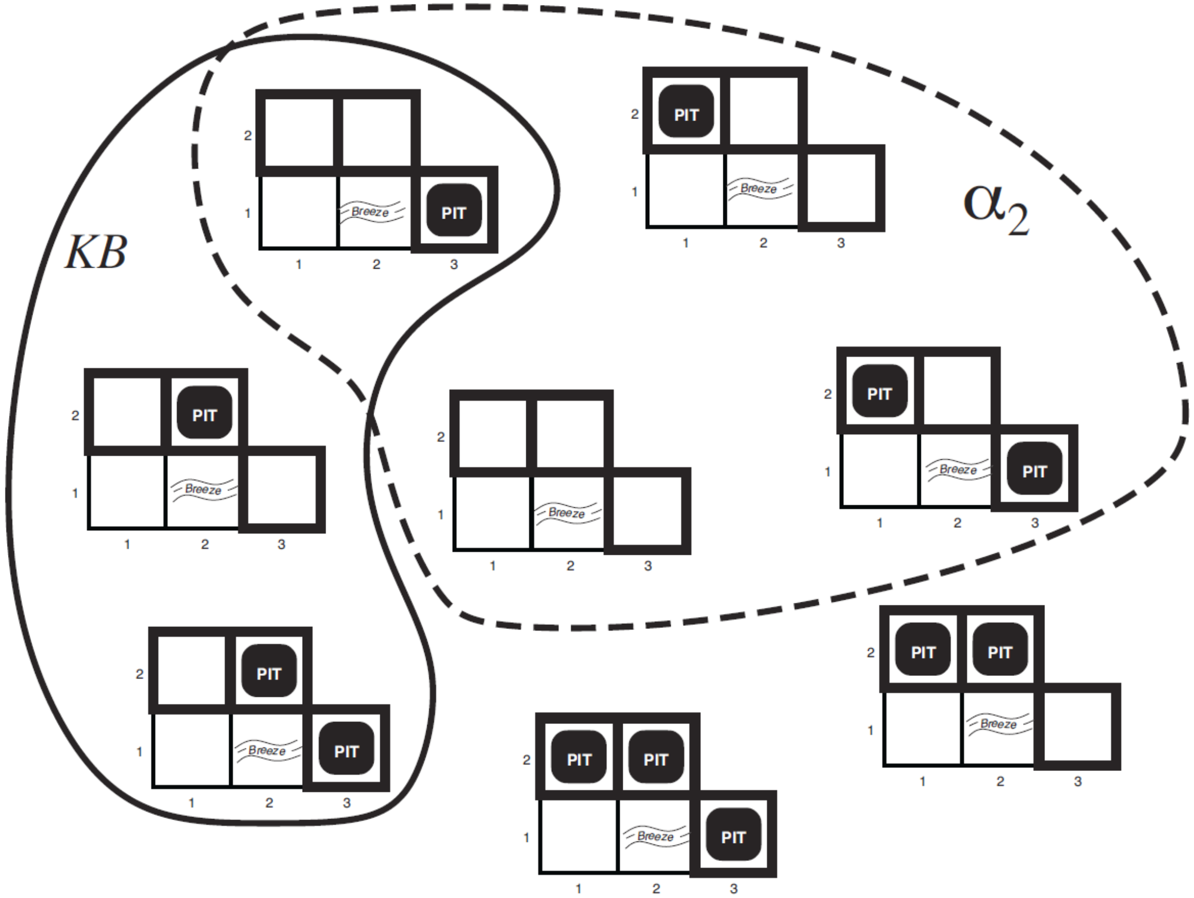


Wumpus World

Possible Models

- $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
- Knowledge base
  - Nothing in [1,1]
  - Breeze in [2,1]
- Query α_2:
  - No pit in [2,2]



Propositional Logic Models

All Possible Models

Model Symbols

A	0	0	0	0	1	1	1	1
B	0	0	1	1	0	0	1	1
C	0	1	0	1	0	1	0	1

Entailment

How do we implement a logical agent that proves entailment?

- Logic language
  - Propositional logic
  - First order logic

- Inference algorithms
  - Theorem proving
  - Model checking

Simple Model Checking

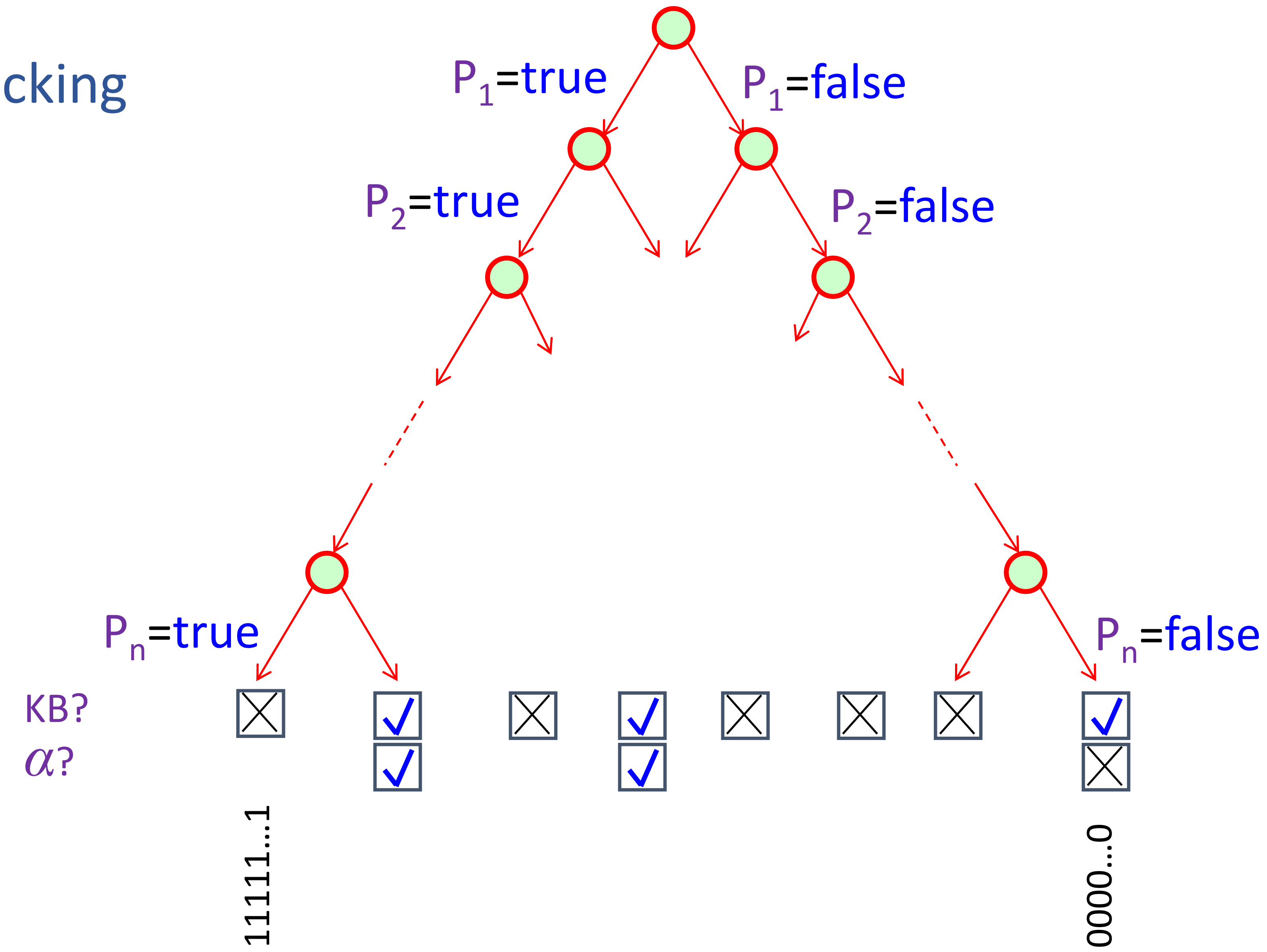
function TT-ENTAILS?(KB, α) returns true or false

Simple Model Checking, contd.

Same recursion as backtracking

$O(2^n)$ time, linear space

We can do much better!



Simple Model Checking

```
function TT-ENTAILS?(KB, α) returns true or false
   return TT-CHECK-ALL(KB, α, symbols(KB) U symbols(α),{})

function TT-CHECK-ALL(KB, α, symbols,model) returns true or false
   if empty?(symbols) then
         if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
         else return true
   else
         P ← first(symbols)
         rest ← rest(symbols)
         return  and (TT-CHECK-ALL(KB, α, rest, model ∪ {P = true})
                            TT-CHECK-ALL(KB, α, rest, model ∪ {P = false }))
```

Inference: Proofs

A proof is a *demonstration* of entailment between α and β

Method 1: *model-checking*

- For every possible world, if α is true make sure that is β true too
- OK for propositional logic (finitely many worlds); not easy for first-order logic

Method 2: *theorem-proving*

- Search for a sequence of proof steps (applications of *inference rules*) leading from α to β
- E.g., from $P \wedge (P \Rightarrow Q)$, infer Q by *Modus Ponens*

Properties

- *Sound* algorithm: everything it claims to prove is in fact entailed
- *Complete* algorithm: every sentence that is entailed can be proved

Simple Theorem Proving: Forward Chaining

Forward chaining applies Modus Ponens to generate new facts:

- Given $X_1 \wedge X_2 \wedge \dots X_n \Rightarrow Y$ and $X_1, X_2, \dots, X_n$
- Infer Y

Forward chaining keeps applying this rule, adding new facts, until nothing more can be added

Requires KB to contain only *definite clauses*:

- (Conjunction of symbols) $\Rightarrow$ symbol; or
- A single symbol (note that X is equivalent to $True \Rightarrow X$)

Forward Chaining Algorithm

function PL-FC-ENTAILS?(KB, q) returns true or false

count ← a table, where count[c] is the number of symbols in c's premise

inferred ← a table, where inferred[s] is initially false for all s

agenda ← a queue of symbols, initially symbols known to be true in KB

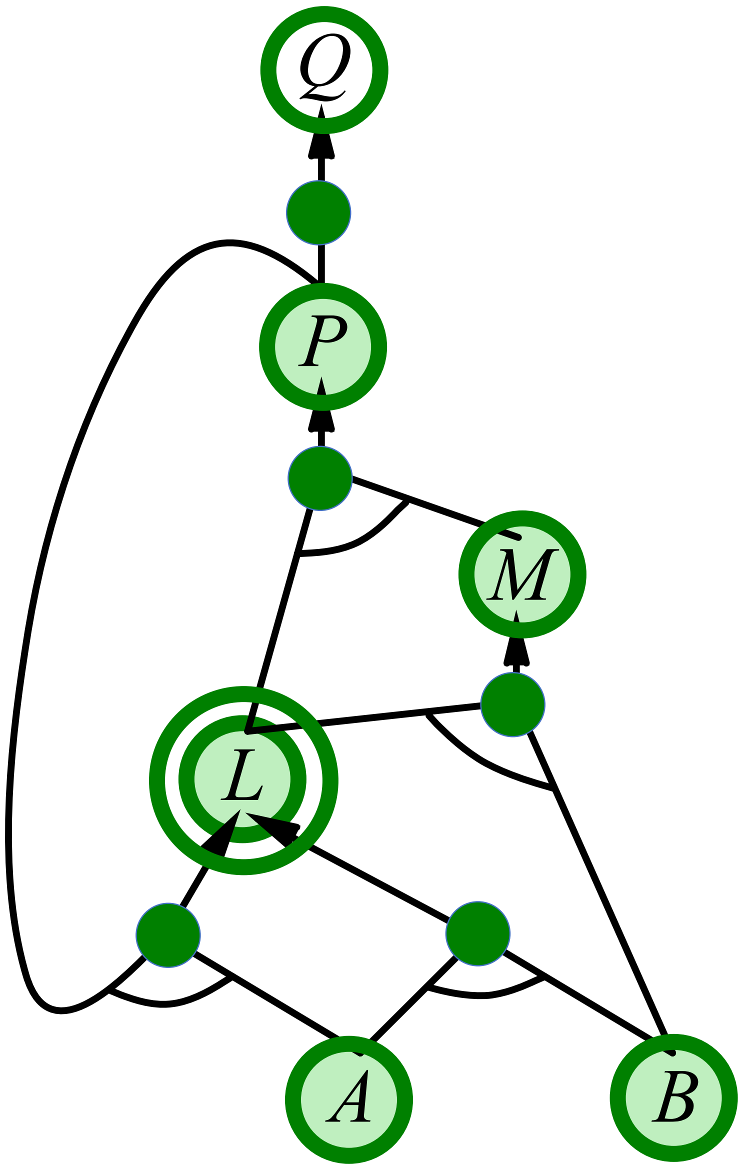
CLAUSES	COUNT	INFERRED	AGENDA
$P \Rightarrow Q$	1	A false	
$L \wedge M \Rightarrow P$	2	B false	
$B \wedge L \Rightarrow M$	2	L false	
$A \wedge P \Rightarrow L$	2	M false	
$A \wedge B \Rightarrow L$	2	P false	
A	0	Q false	
B	0		

Forward Chaining Example: Proving Q

CLAUSES	COUNT	INFERRED
$P \Rightarrow Q$	~~1~~ 0	A ~~false~~ true
$L \wedge M \Rightarrow P$	~~2~~ ~~1~~ 0	B ~~false~~ true
$B \wedge L \Rightarrow M$	~~2~~ ~~1~~ 0	L ~~false~~ true
$A \wedge P \Rightarrow L$	~~2~~ ~~1~~ 0	M ~~false~~ true
$A \wedge B \Rightarrow L$	~~2~~ ~~1~~ 0	P ~~false~~ true
A	0	Q ~~false~~ true
B	0	

AGENDA

~~A~~ ~~B~~ ~~L~~ ~~M~~ ~~P~~ ~~L~~ ~~Q~~



Forward Chaining Algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  count ← a table, where count[c] is the number of symbols in c's premise
  inferred ← a table, where inferred[s] is initially false for all s
  agenda ← a queue of symbols, initially symbols known to be true in KB

  while agenda is not empty do
      p ← Pop(agenda)
      if p = q then return true
      if inferred[p] = false then
          inferred[p]←true
          for each clause c in KB where p is in c.premise do
              decrement count[c]
              if count[c] = 0 then add c.conclusion to agenda
  return false
```

Properties of forward chaining

Theorem: FC is sound and complete for definite-clause KBs

Soundness: follows from soundness of Modus Ponens (easy to check)

Completeness proof:

1. FC reaches a fixed point where no new atomic sentences are derived
2. Consider the final *inferred* table as a model ***m***, assigning true/false to symbols
3. Every clause in the original KB is true in ***m***

   Proof: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in ***m***
   Then $a_1 \wedge \ldots \wedge a_k$ is true in ***m*** and b is false in ***m***
   Therefore the algorithm has not reached a fixed point!
4. Hence ***m*** is a model of KB
5. If KB |= q, q is true in every model of KB, including ***m***

A ~~false~~ true

B ~~false~~ true

L ~~false~~ true

M ~~false~~ true

P ~~false~~ true

Q ~~false~~ true

Inference Rules

Notation Alert!

Modus Ponens

$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

Unit Resolution

$$\frac{a \vee b, \quad \neg b \vee c}{a \vee c}$$

General Resolution

$$\frac{a_1 \vee \cdots \vee a_m \vee b, \quad \neg b \vee c_1 \vee \cdots \vee c_n}{a_1 \vee \cdots \vee a_m \vee c_1 \vee \cdots \vee c_n}$$

Resolution