

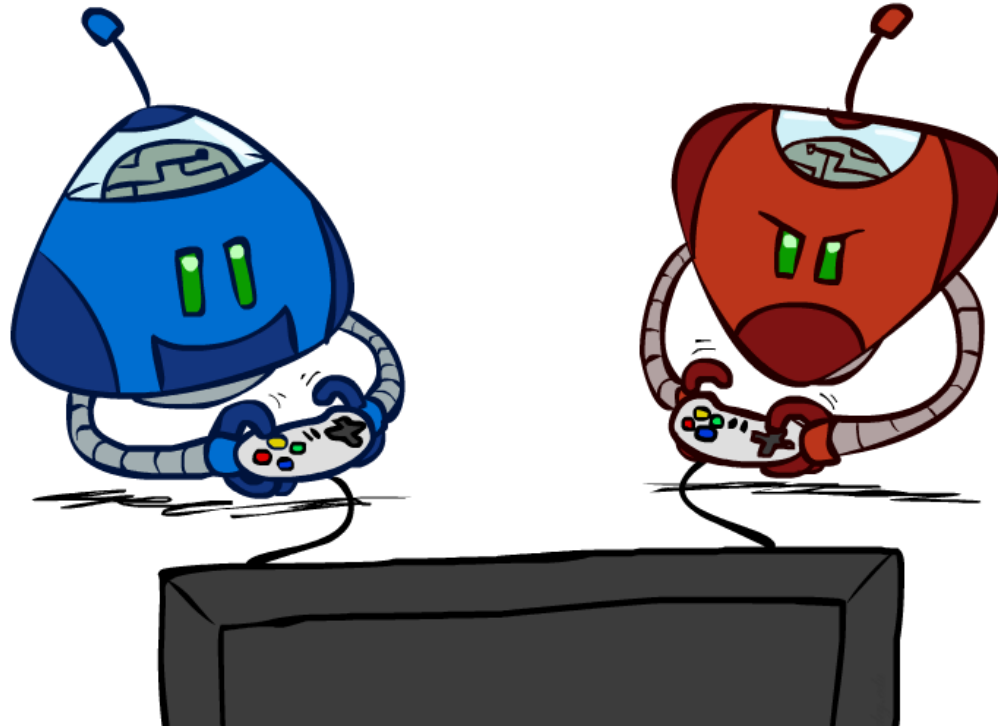
Announcements

Assignments:

- P0: Python & Autograder Tutorial
 - Due Thu 9/5, 10 pm
- HW2 (written)
 - Due Tue 9/10, 10 pm
 - No slip days. Up to 24 hours late, 50 % penalty
- P1: Search & Games
 - Due Thu 9/12, 10 pm
 - Recommended to work in pairs
 - Submit to Gradescope early and often

AI: Representation and Problem Solving

Adversarial Search



Instructors: Pat Virtue & Fei Fang

Slide credits: CMU AI, <http://ai.berkeley.edu>

Outline

History / Overview

Zero-Sum Games (Minimax)

Evaluation Functions

Search Efficiency (α - β Pruning)

Games of Chance (Expectimax)



Game Playing State-of-the-Art

Checkers:

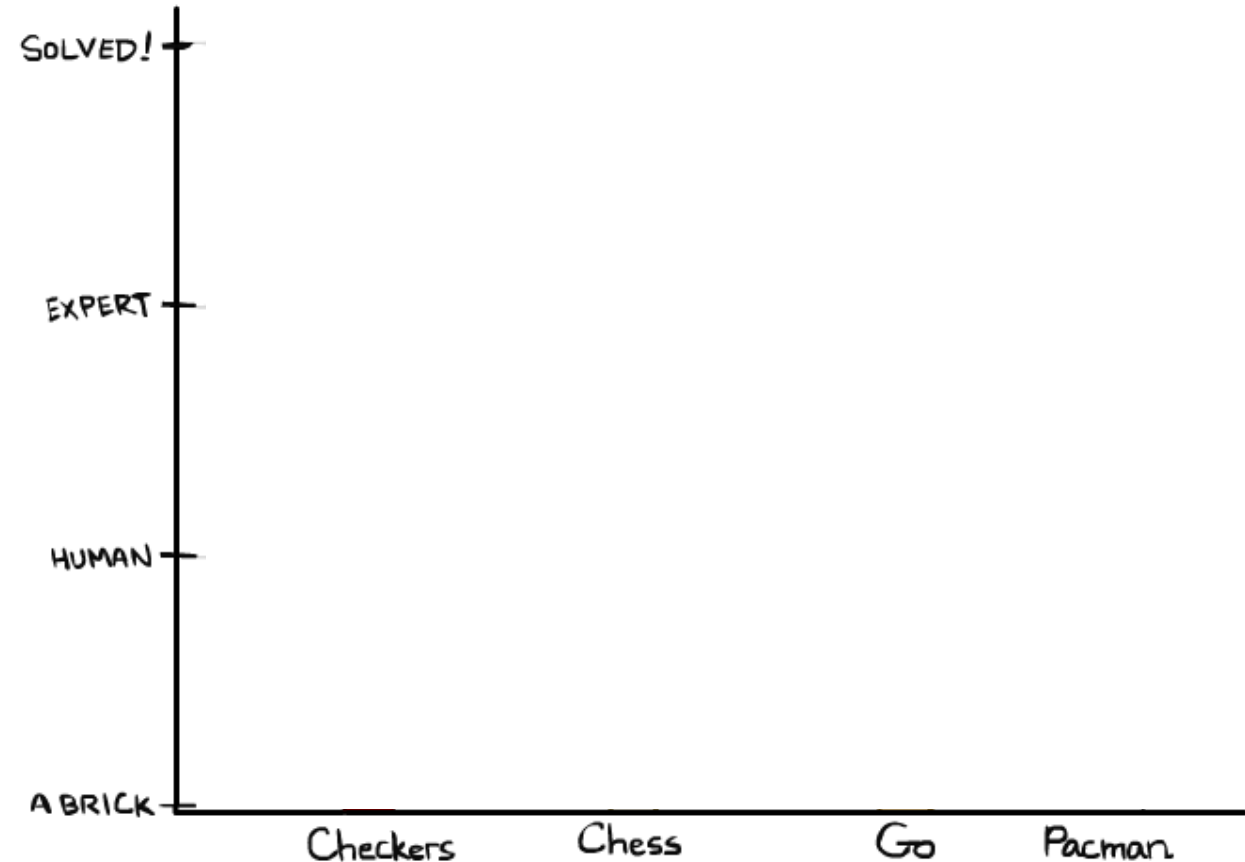
- 1950: First computer player.
- 1959: Samuel's self-taught program.
- 1994: First computer world champion: Chinook ended 40-year-reign of human champion Marion Tinsley using complete 8-piece endgame.
- 2007: Checkers solved! Endgame database of 39 trillion states

Chess:

- 1945-1960: Zuse, Wiener, Shannon, Turing, Newell & Simon, McCarthy.
- 1960s onward: gradual improvement under "standard model"
- 1997: special-purpose chess machine Deep Blue defeats human champion Gary Kasparov in a six-game match. Deep Blue examined 200M positions per second and extended some lines of search up to 40 ply. Current programs running on a PC rate > 3200 (vs 2870 for Magnus Carlsen).

Go:

- 1968: Zobrist's program plays legal Go, barely (b>300!)
- 2005-2014: Monte Carlo tree search enables rapid advances: current programs beat strong amateurs, and professionals with a 3-4 stone handicap.



Game Playing State-of-the-Art

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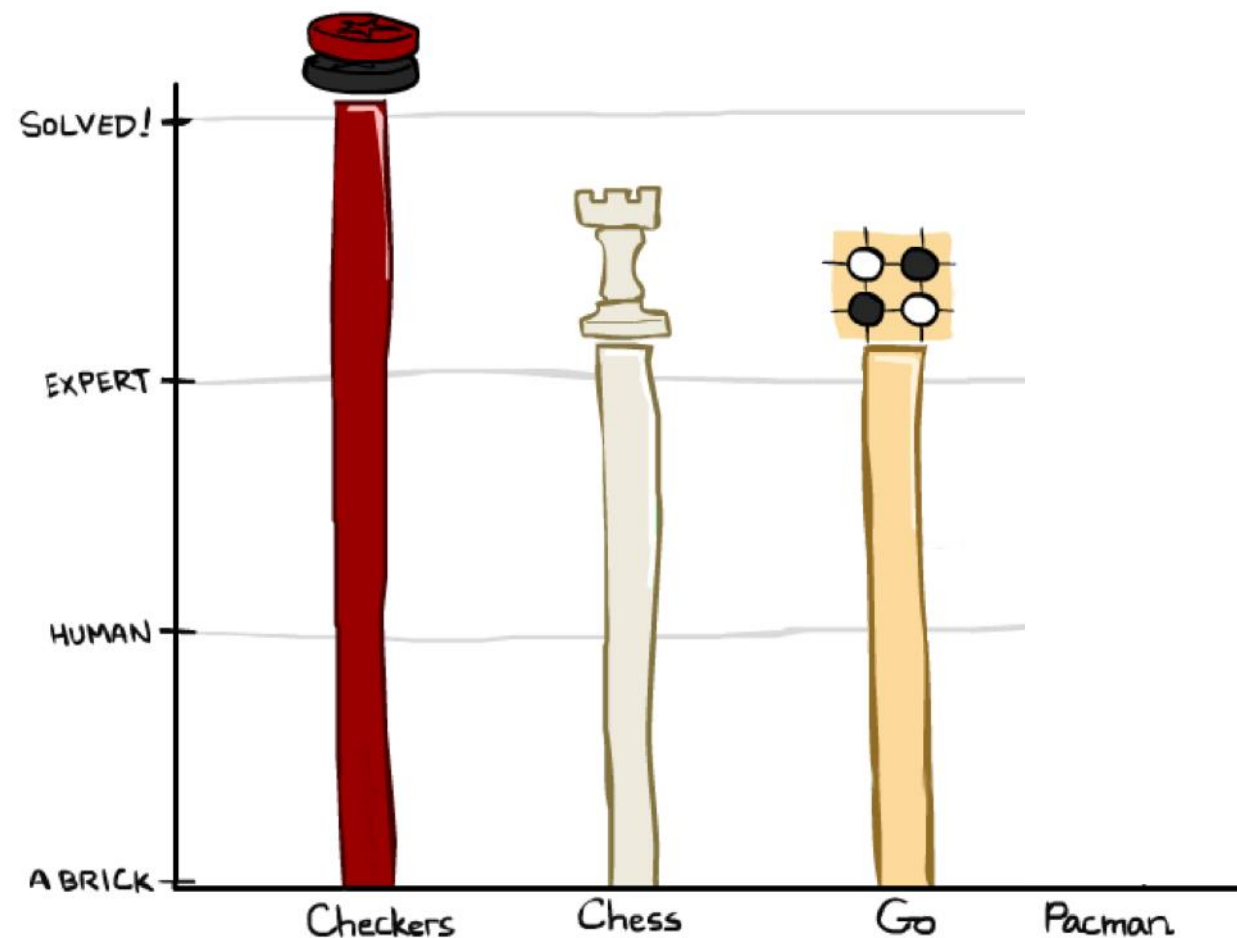
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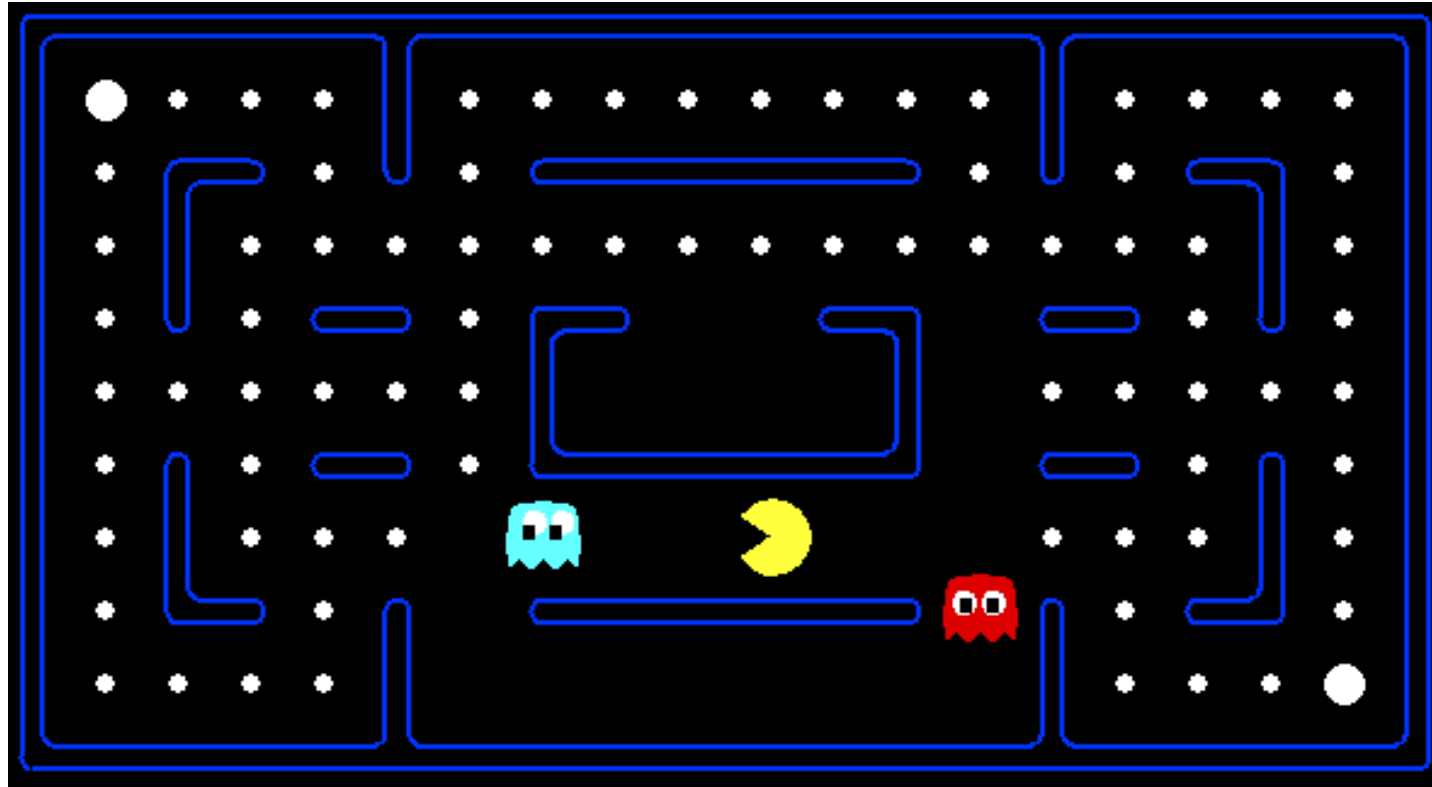
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Go:

- 1968: Zobrist's program plays legal Go, barely ($b > 300!$)
- 2005-2014: Monte Carlo tree search enables rapid advances: current programs beat strong amateurs, and professionals with a 3-4 stone handicap.
- 2015: AlphaGo from DeepMind beats Lee Sedol



Behavior from Computation



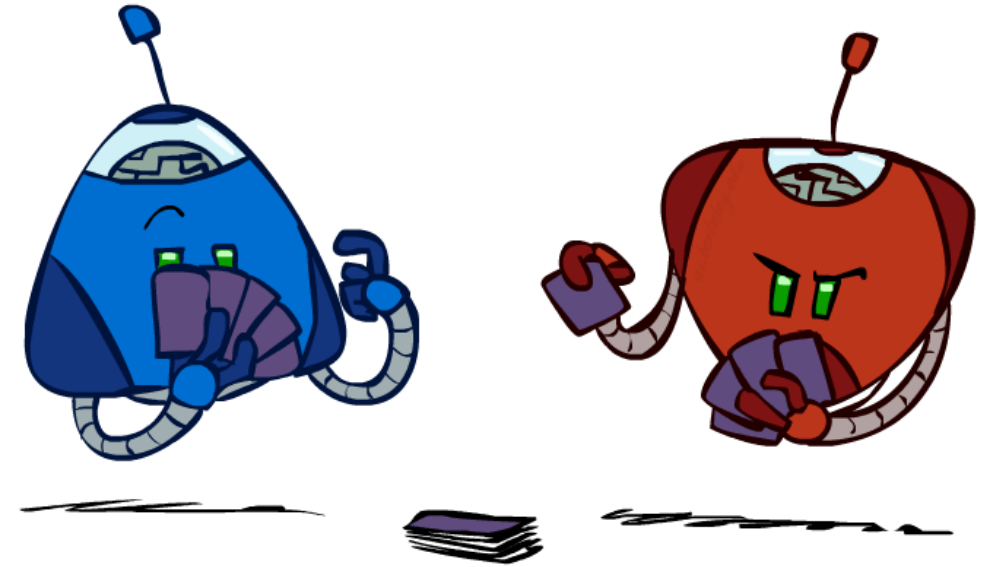
[Demo: mystery pacman (L6D1)]

Types of Games

Many different kinds of games!

Axes:

- Deterministic or stochastic?
- Perfect information (fully observable)?
- One, two, or more players?
- Turn-taking or simultaneous?
- Zero sum?



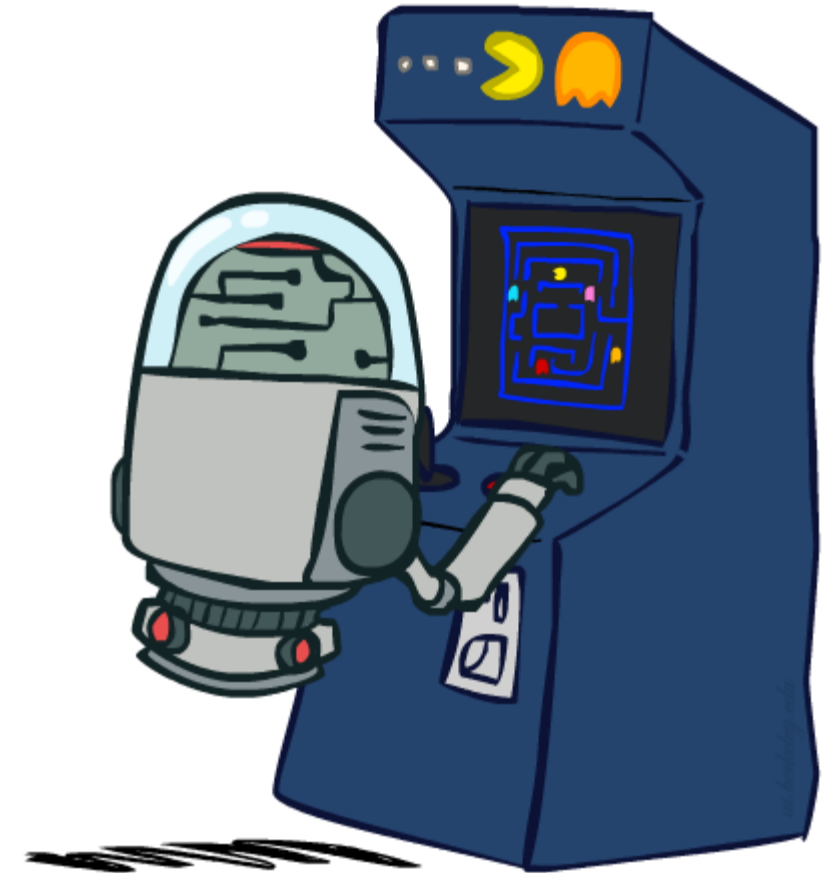
Want algorithms for calculating a *contingent plan* (a.k.a. **strategy** or **policy**) which recommends a move for every possible eventuality

“Standard” Games

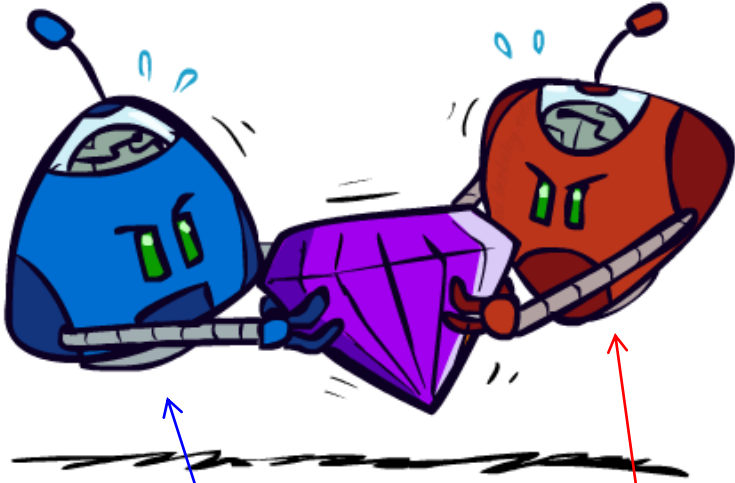
Standard games are deterministic, observable, two-player, turn-taking, zero-sum

Game formulation:

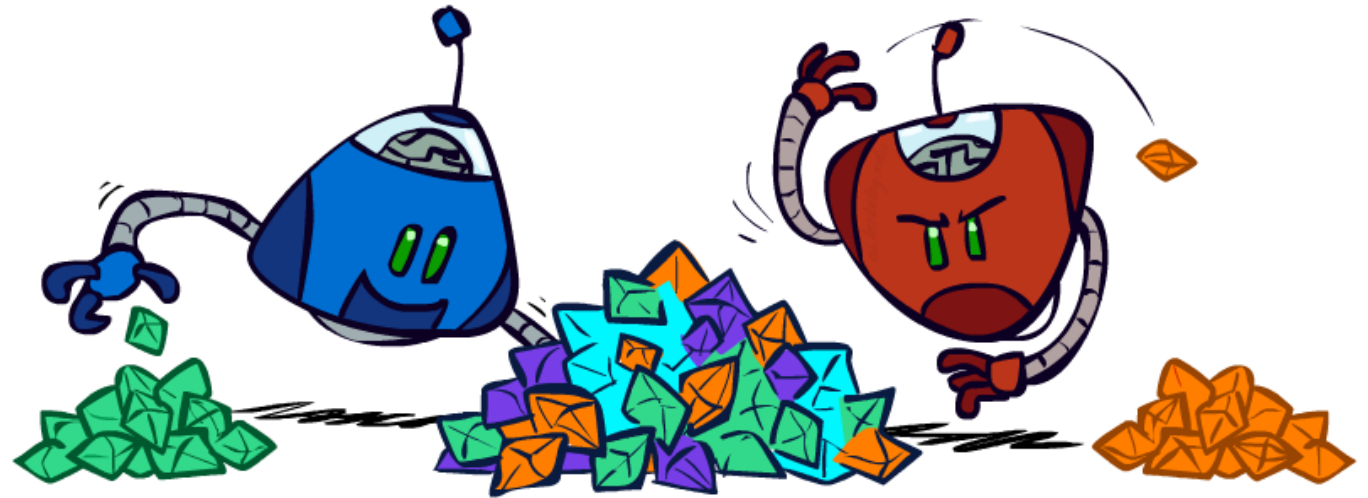
- Initial state: s_0
- Players: Player(s) indicates whose move it is
- Actions: Actions(s) for player on move
- Transition model: Result(s,a)
- Terminal test: Terminal-Test(s)
- Terminal values: Utility(s,p) for player p
 - Or just Utility(s) for player making the decision at root



Zero-Sum Games

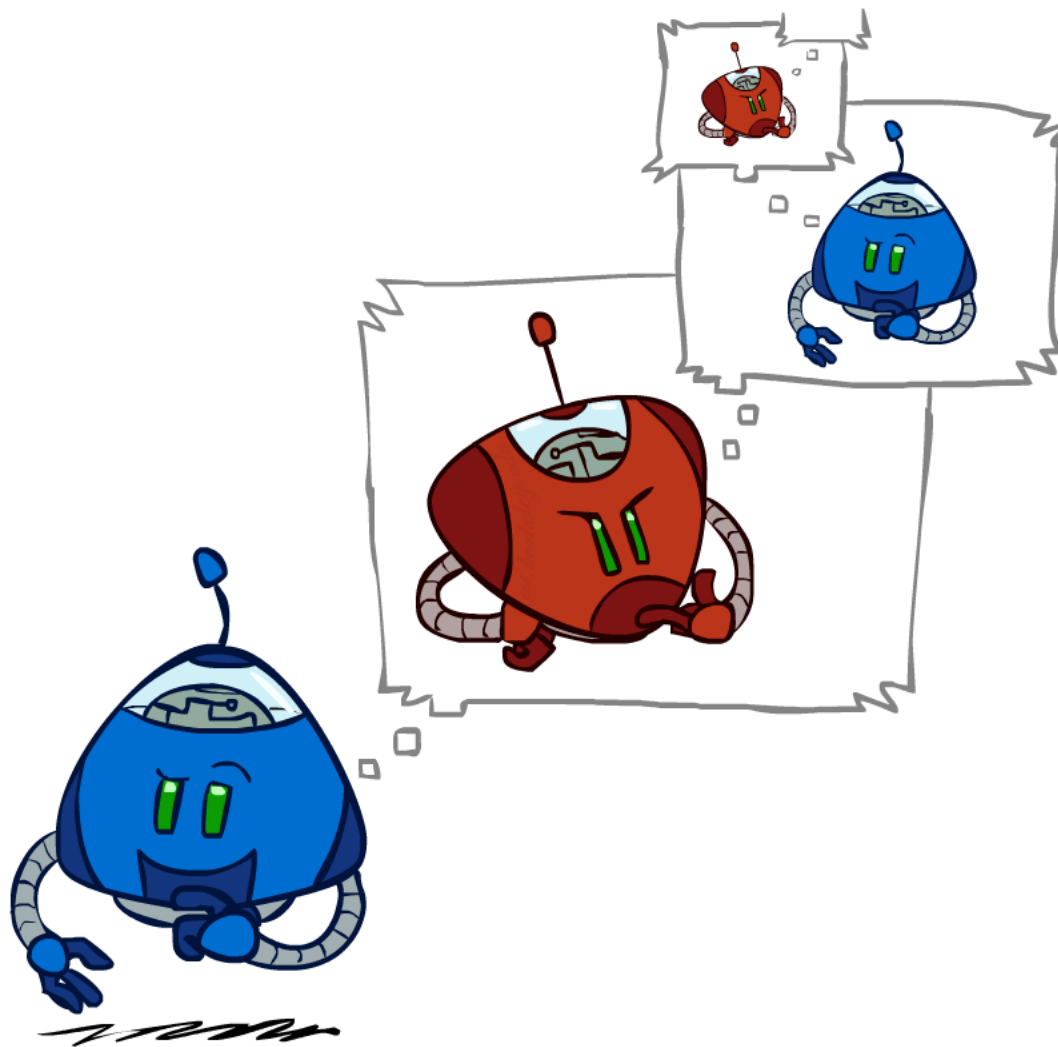


- Zero-Sum Games
 - Agents have **opposite** utilities
 - Pure competition:
 - One **maximizes**, the other **minimizes**

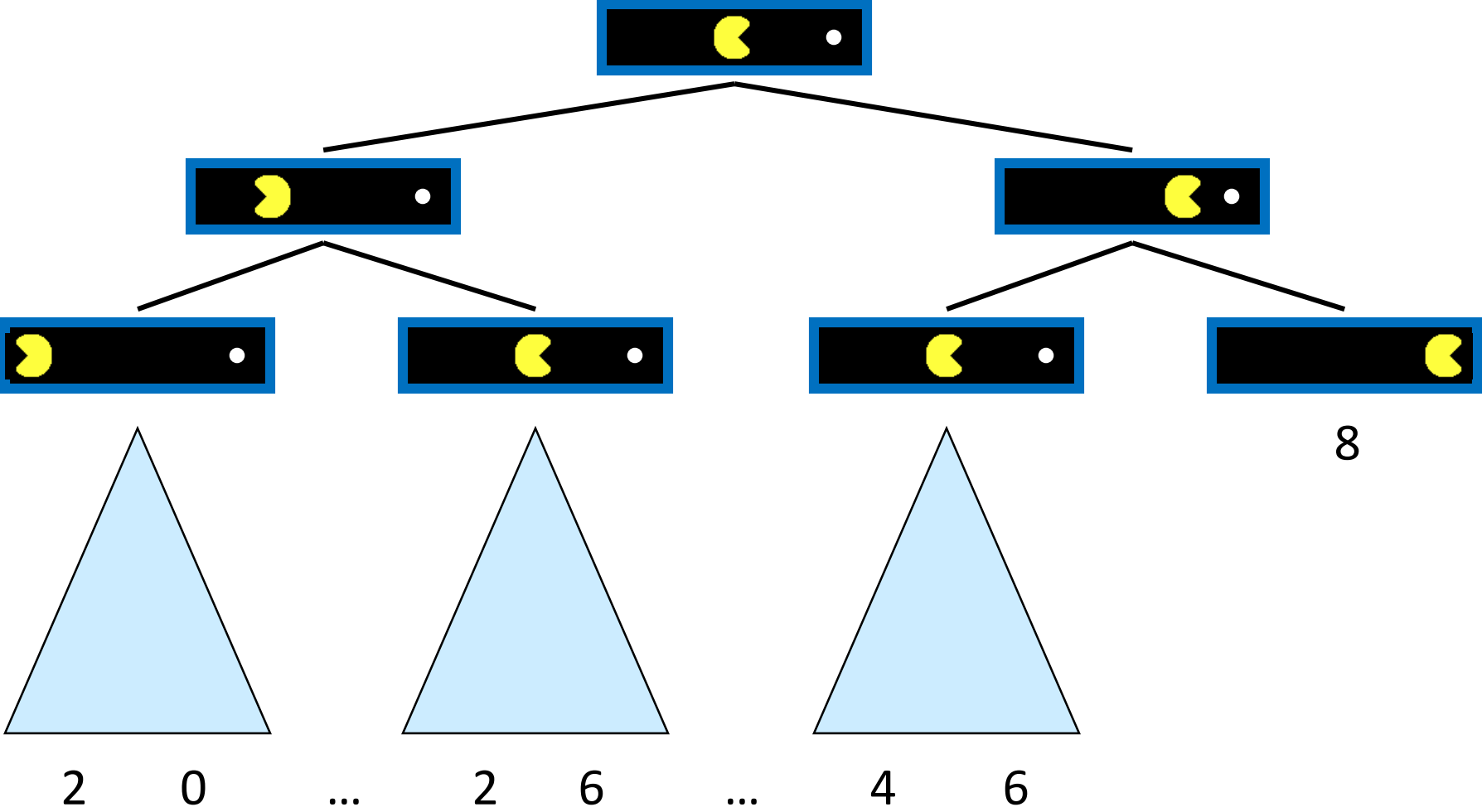


- General Games
 - Agents have **independent** utilities
 - Cooperation, indifference, competition, shifting alliances, and more are all possible

Adversarial Search



Single-Agent Trees



Minimax

States

Actions

Values



MAX (X)



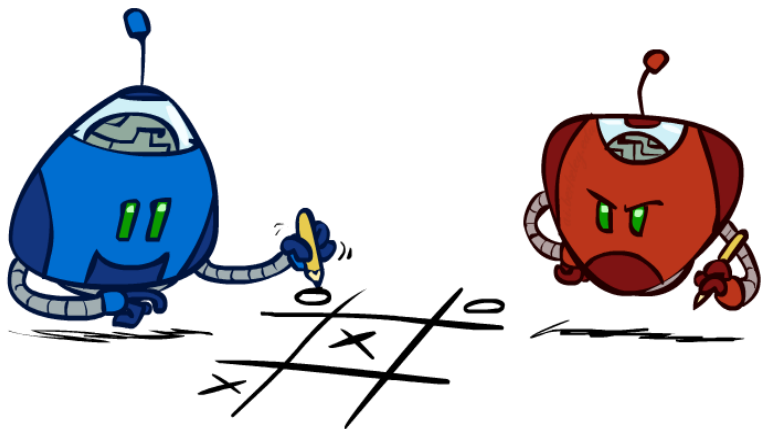
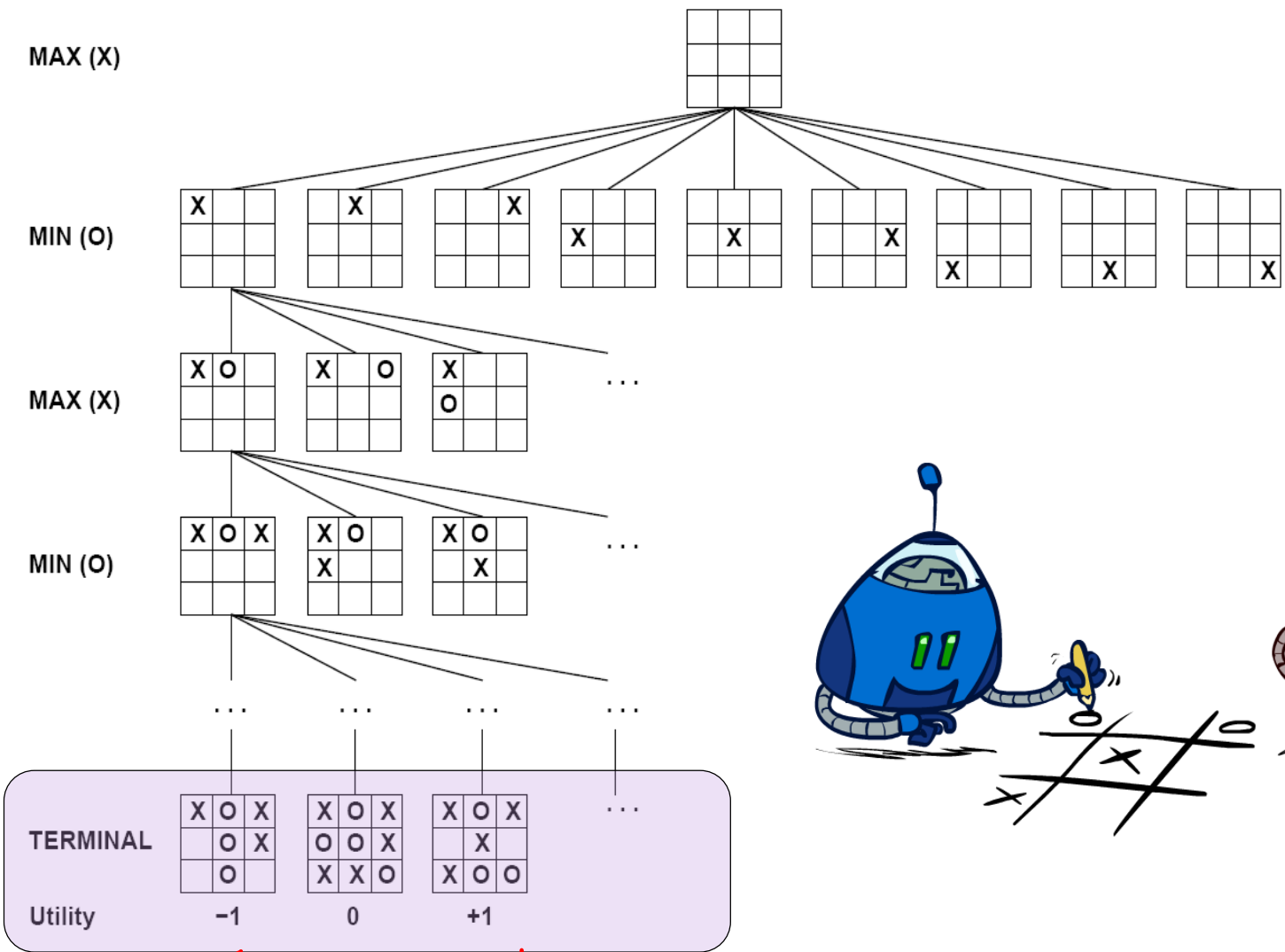
MIN (O)



MAX (X)



MIN (O)

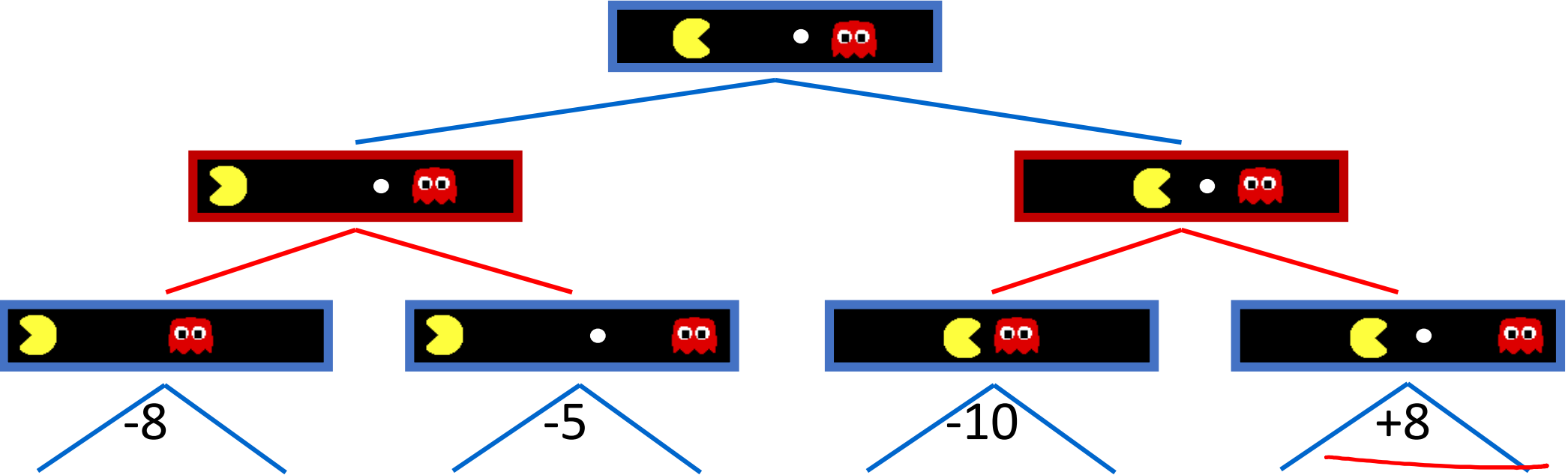


Minimax

States

Actions

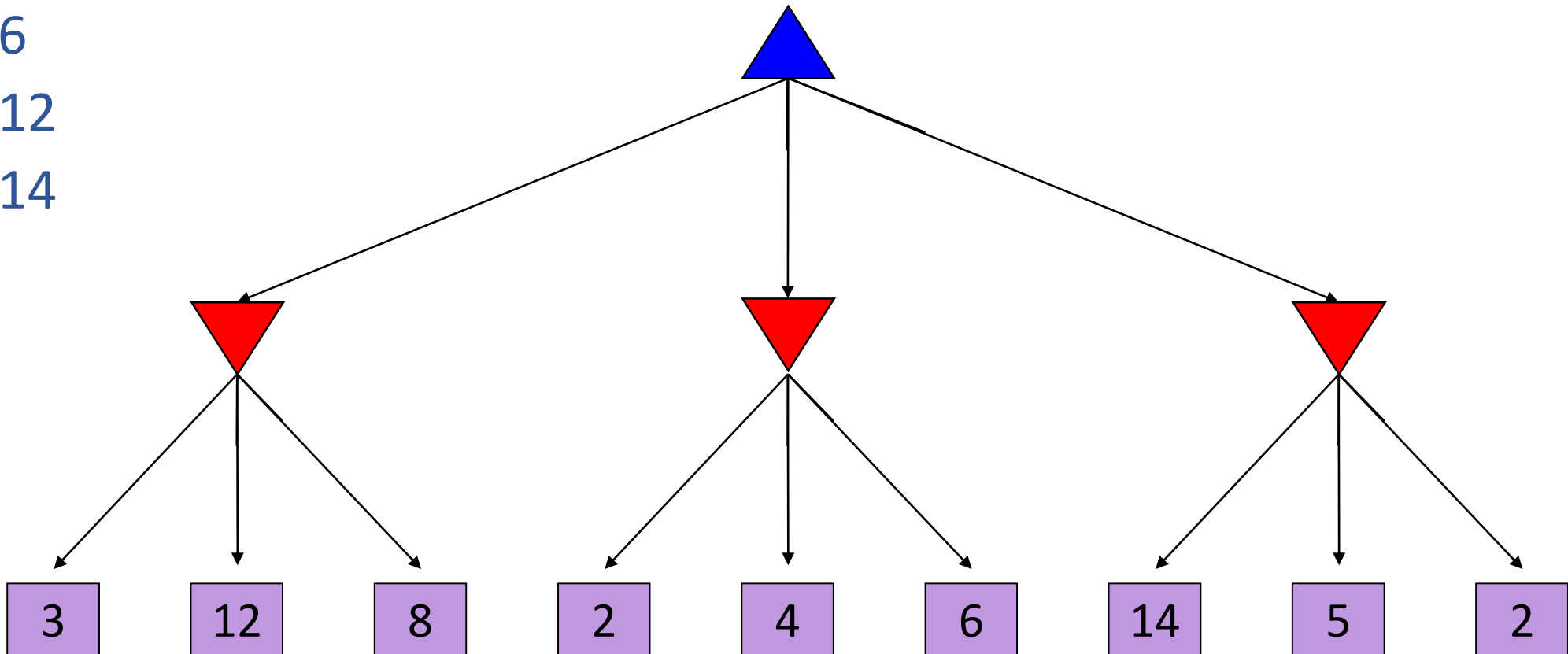
Values



Piazza Poll 1

What is the minimax value at the root?

- A) 2
- B) 3
- C) 6
- D) 12
- E) 14



Piazza Poll 1

What is the minimax value at the root?

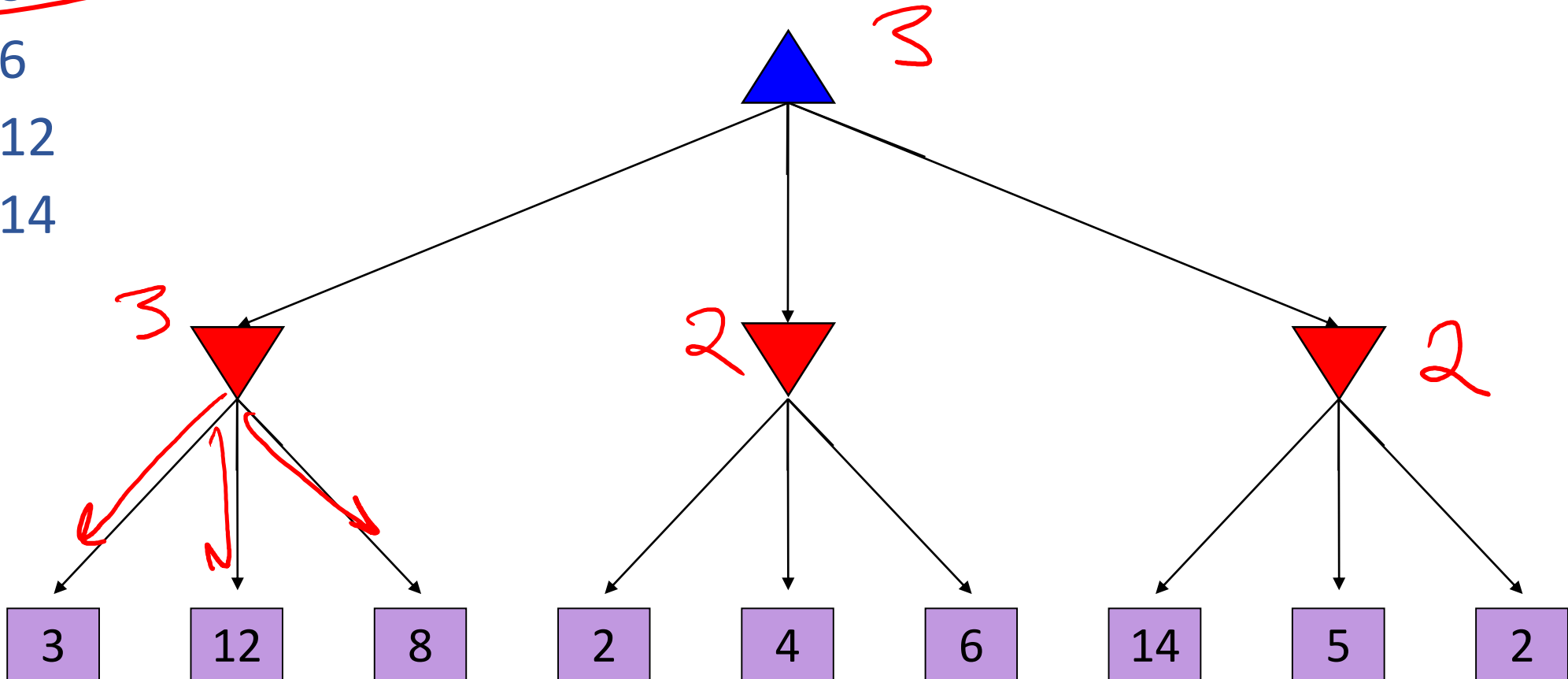
A) 2

B) 3

C) 6

D) 12

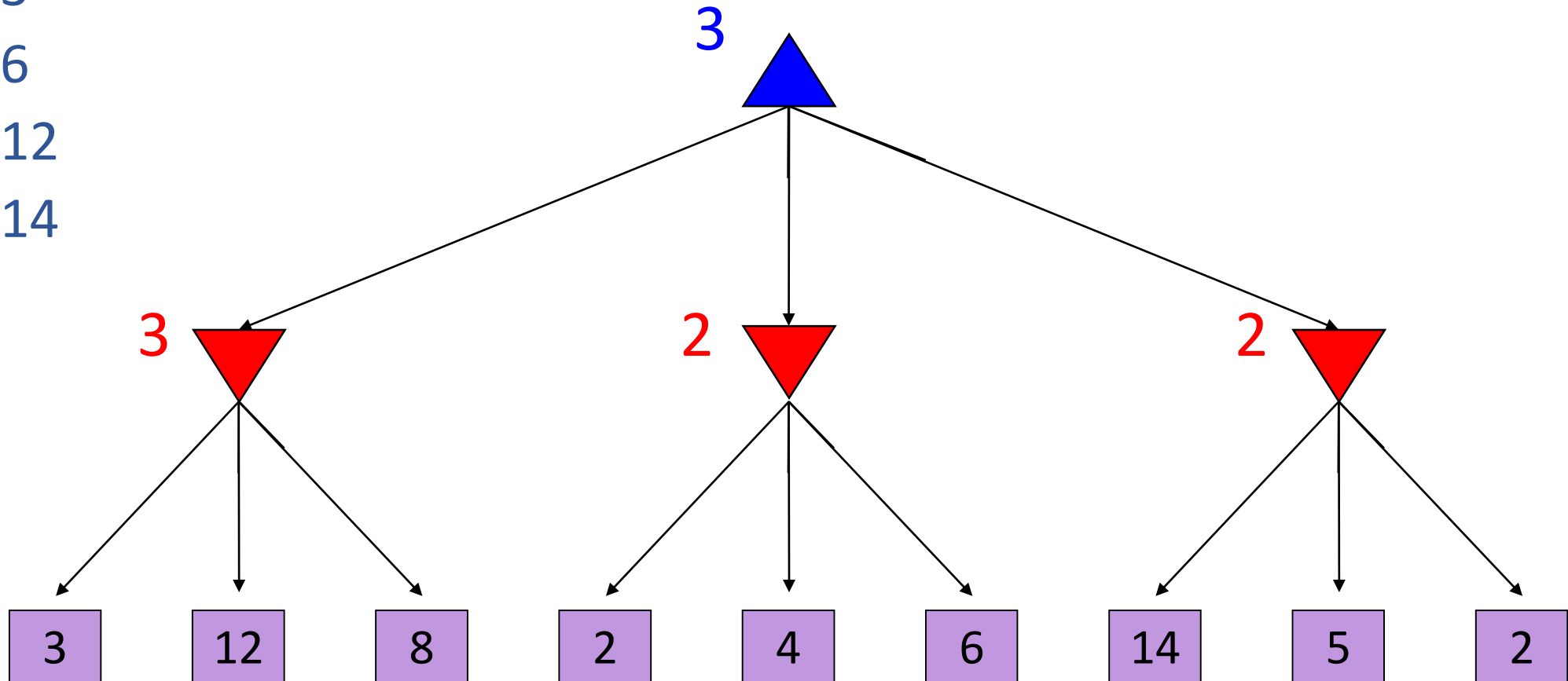
E) 14



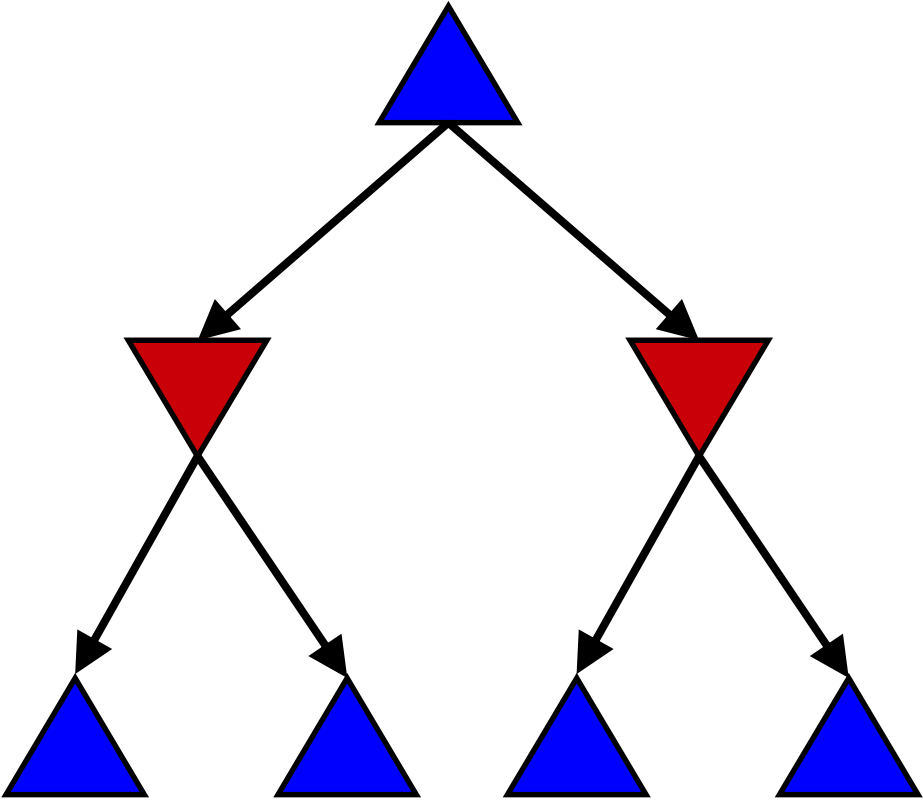
Piazza Poll 1

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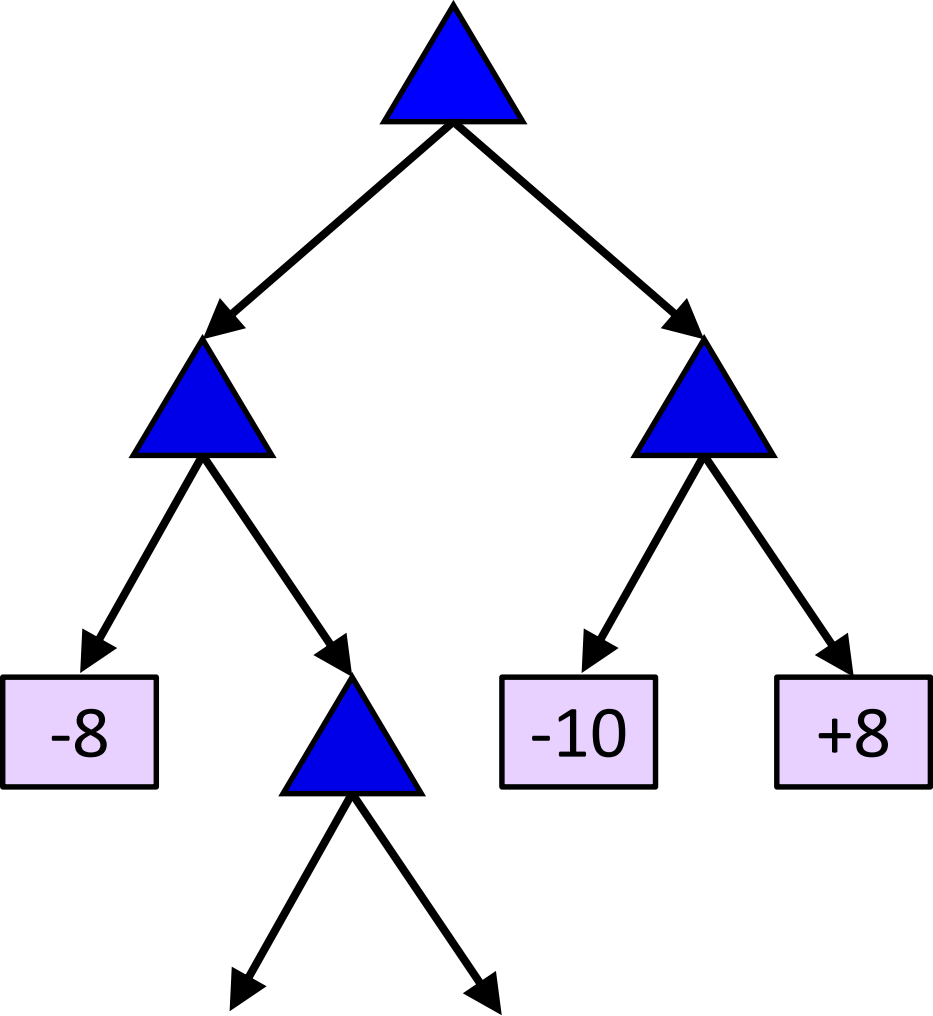
- A) 2
- B) 3
- C) 6
- D) 12
- E) 14



Minimax Code



Max Code



Max Code

```
def max_value(state):
```

```
    if state.is_leaf:
```

```
        return state.value
```

```
    # TODO Also handle depth limit
```

```
→ best_value = -10000000
```

```
    for action in state.actions:
```

```
        next_state = state.result(action)
```

```
        next_value = max_value(next_state)
```

```
        if next_value > best_value:
```

```
            best_value = next_value
```

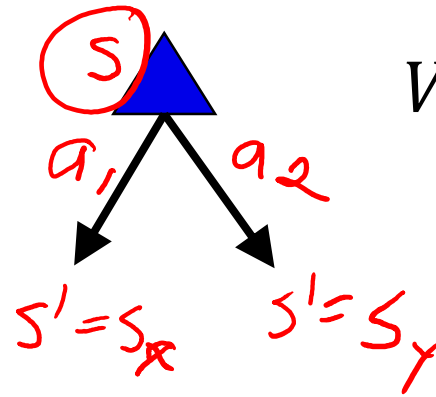
```
    return best_value
```

Minimax Code

```
def max_value(state):  
    if state.is_leaf:  
        return state.value  
    # TODO Also handle depth limit  
  
    best_value = -10000000  
  
    for action in state.actions:  
        next_state = state.result(action)  
  
        next_value = min_value(next_state)  
  
        if next_value > best_value:  
            best_value = next_value  
  
    return best_value  
  
def min_value(state):
```

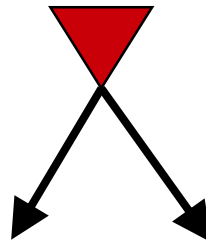
Minimax Notation

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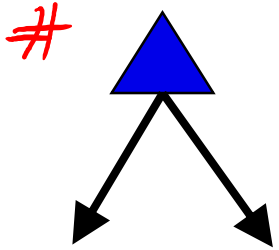


$$V(s) = \max_a V(s'),$$

where s' = *result*(s, a)

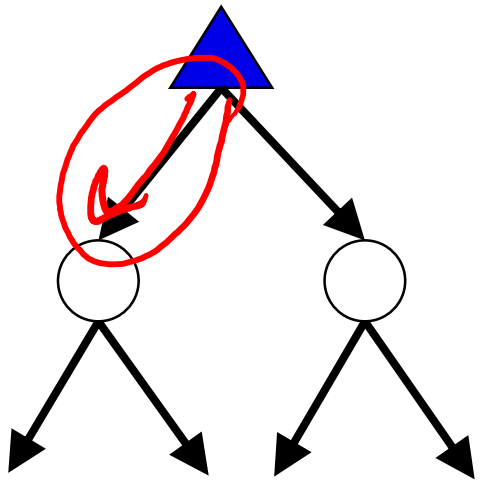


Minimax Notation



$$V(s) = \max_a V(s'),$$

where $s' = \text{result}(s, a)$



$$\hat{a} = \operatorname{argmax}_a V(s'),$$

where $s' = \text{result}(s, a)$

Generic Game Tree Pseudocode

```
function minimax_decision( state )
```

```
    return argmaxa in state.actions value( state.result(a) )
```

```
function value( state )
```

```
    if state.is_leaf
```

```
        return state.value
```

```
    if state.player is MAX
```

```
        return maxa in state.actions value( state.result(a) )
```

```
    if state.player is MIN
```

```
        return mina in state.actions value( state.result(a) )
```

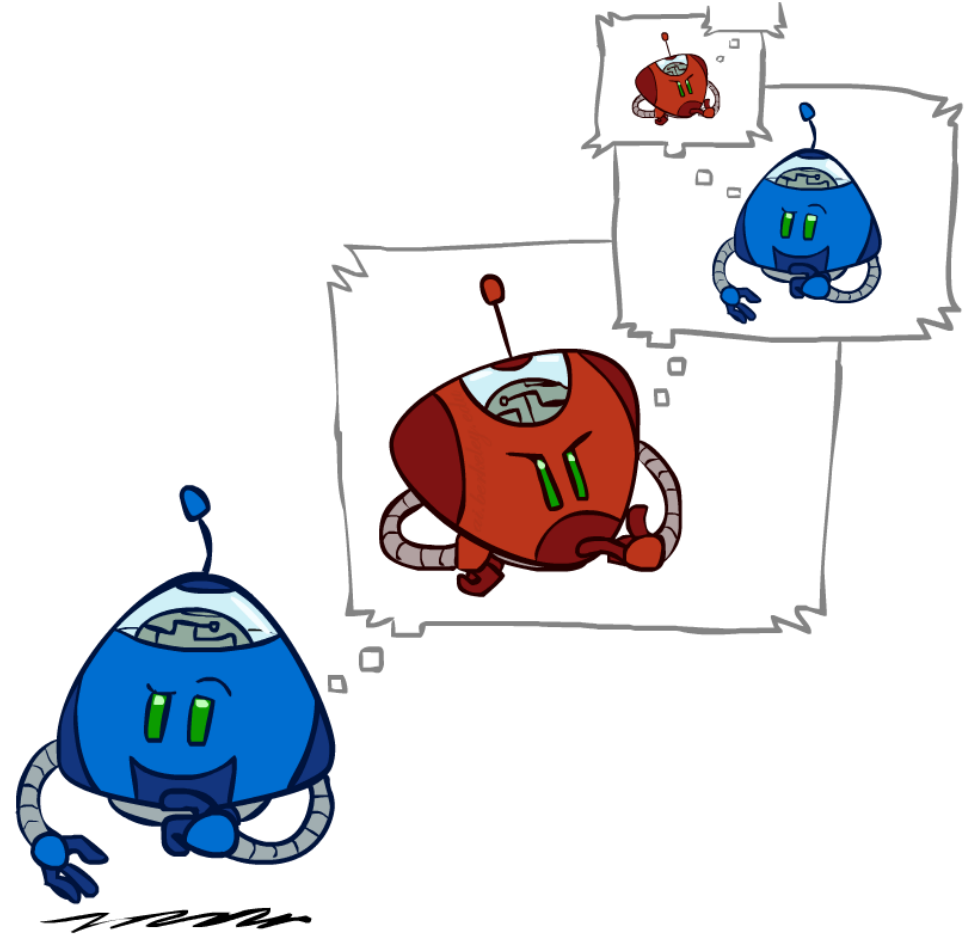
Minimax Efficiency

How efficient is minimax?

- Just like (exhaustive) DFS
- Time: $O(b^m)$
- Space: $O(bm)$

Example: For chess, $b \approx 35$, $m \approx 100$

- Exact solution is completely infeasible
- Humans can't do this either, so how do we play chess?
- **Bounded rationality** – Herbert Simon



Resource Limits



Resource Limits

Problem: In realistic games, cannot search to leaves!

Solution 1: Bounded lookahead

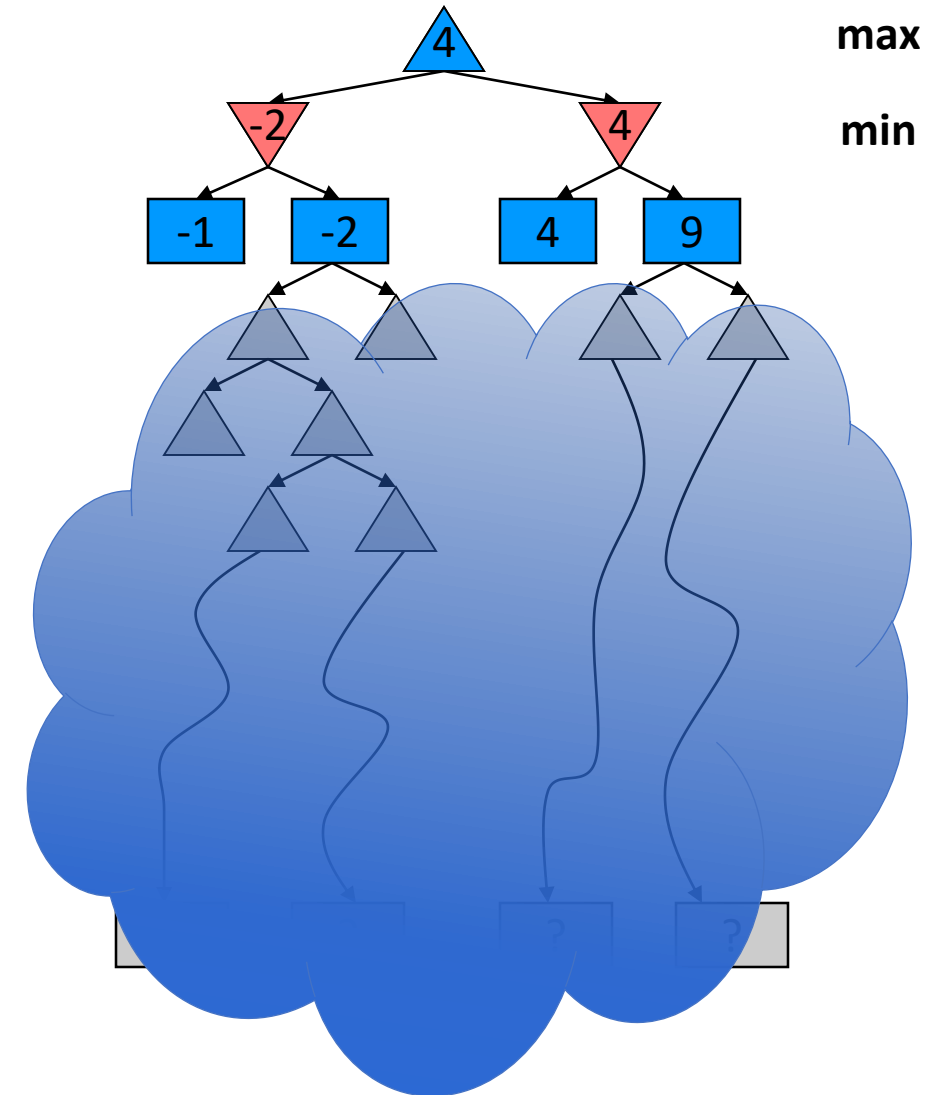
- Search only to a preset **depth limit** or **horizon**
- Use an **evaluation function** for non-terminal positions

Guarantee of optimal play is gone

More plies make a BIG difference

Example:

- Suppose we have 100 seconds, can explore 10K nodes / sec
- So can check 1M nodes per move
- For chess, $b \sim 35$ so reaches about depth 4 – not so good



Depth Matters

Evaluation functions are always imperfect

Deeper search => better play (usually)

Or, deeper search gives same quality of play with a less accurate evaluation function

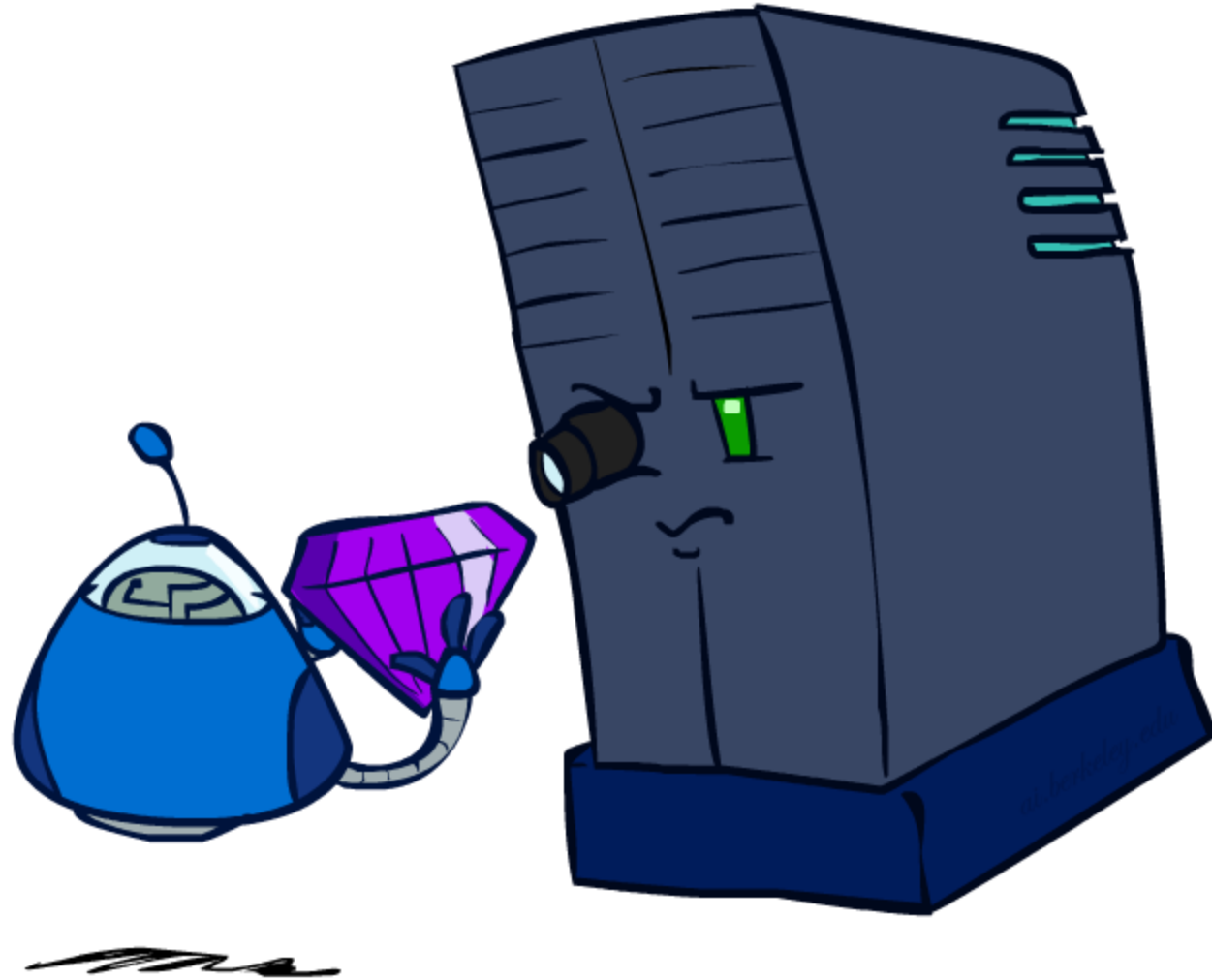
An important example of the tradeoff between complexity of features and complexity of computation



Demo Limited Depth (2)

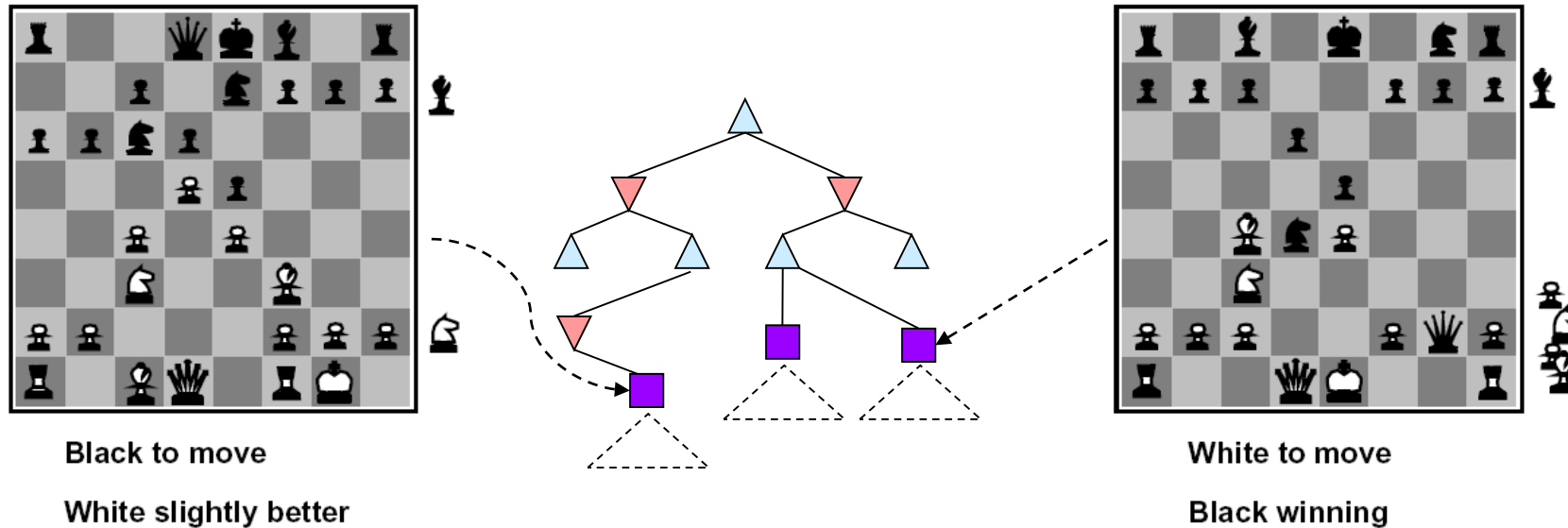
Demo Limited Depth (10)

Evaluation Functions



Evaluation Functions

Evaluation functions score non-terminals in depth-limited search

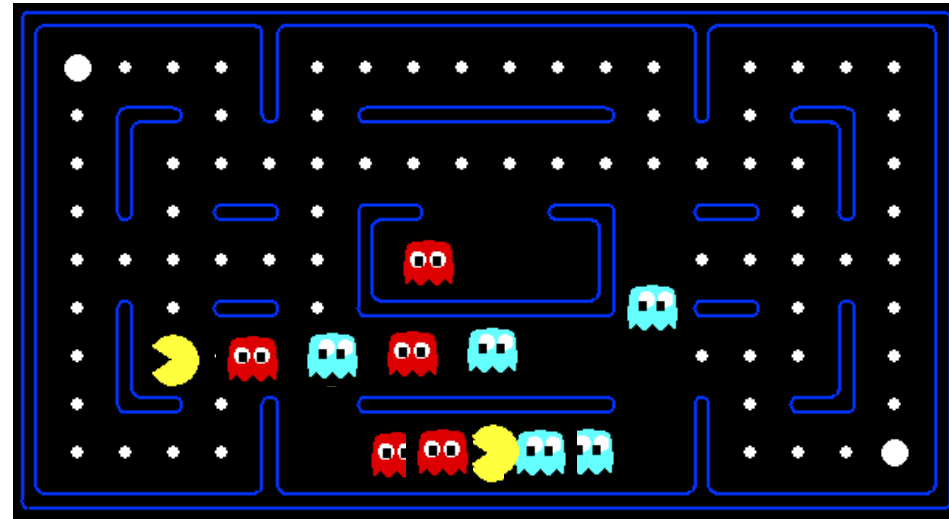


Ideal function: returns the actual minimax value of the position

In practice: typically weighted linear sum of features:

- $EVAL(s) = w_1 f_1(s) + w_2 f_2(s) + \dots + w_n f_n(s)$
- E.g., $w_1 = 9$, $f_1(s) = (\text{num white queens} - \text{num black queens})$, etc.

Evaluation for Pacman

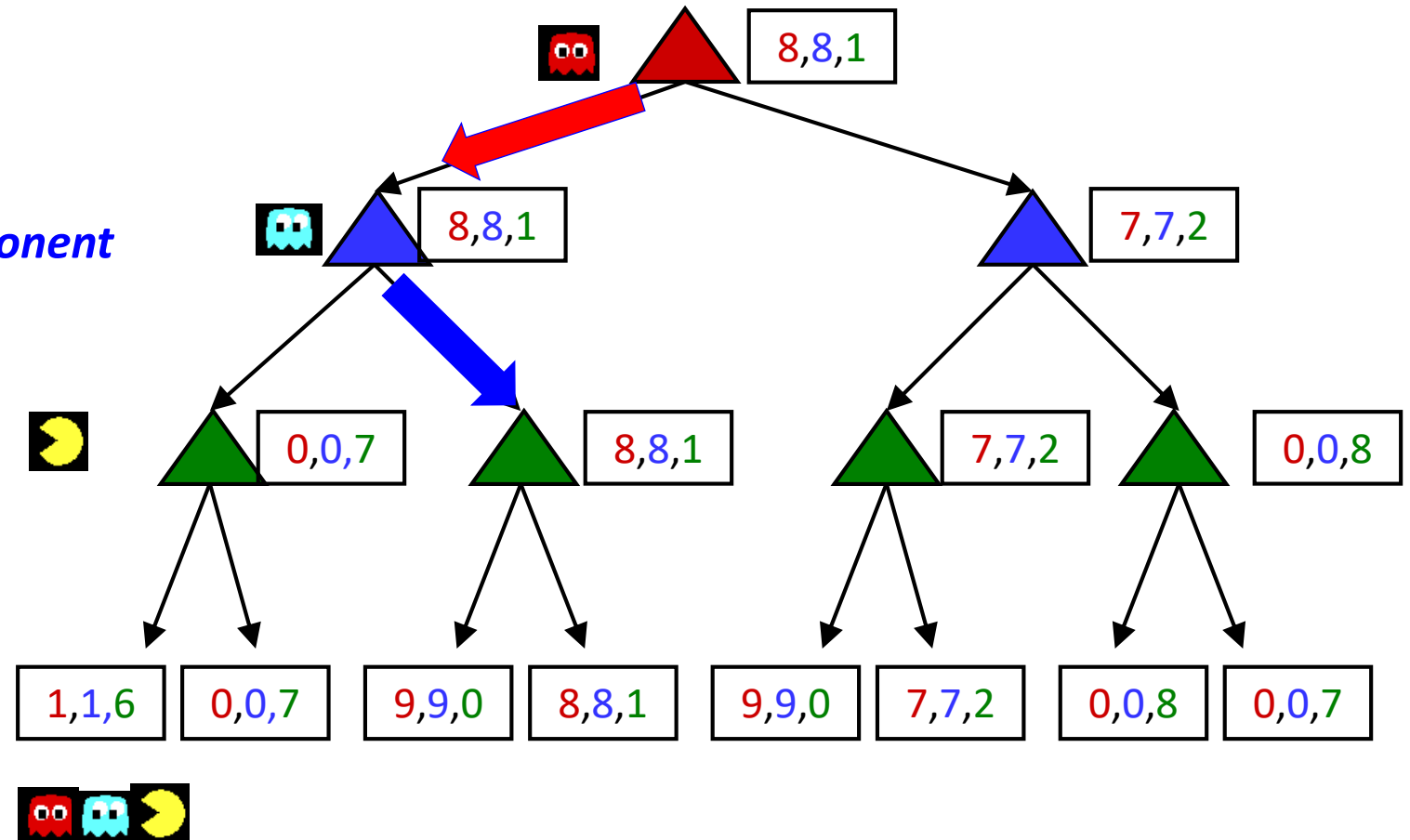
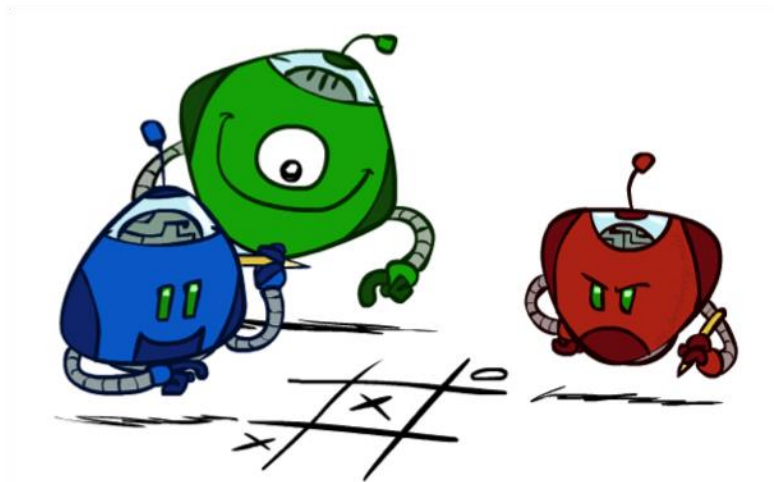


Generalized minimax

What if the game is not zero-sum, or has multiple players?

Generalization of minimax:

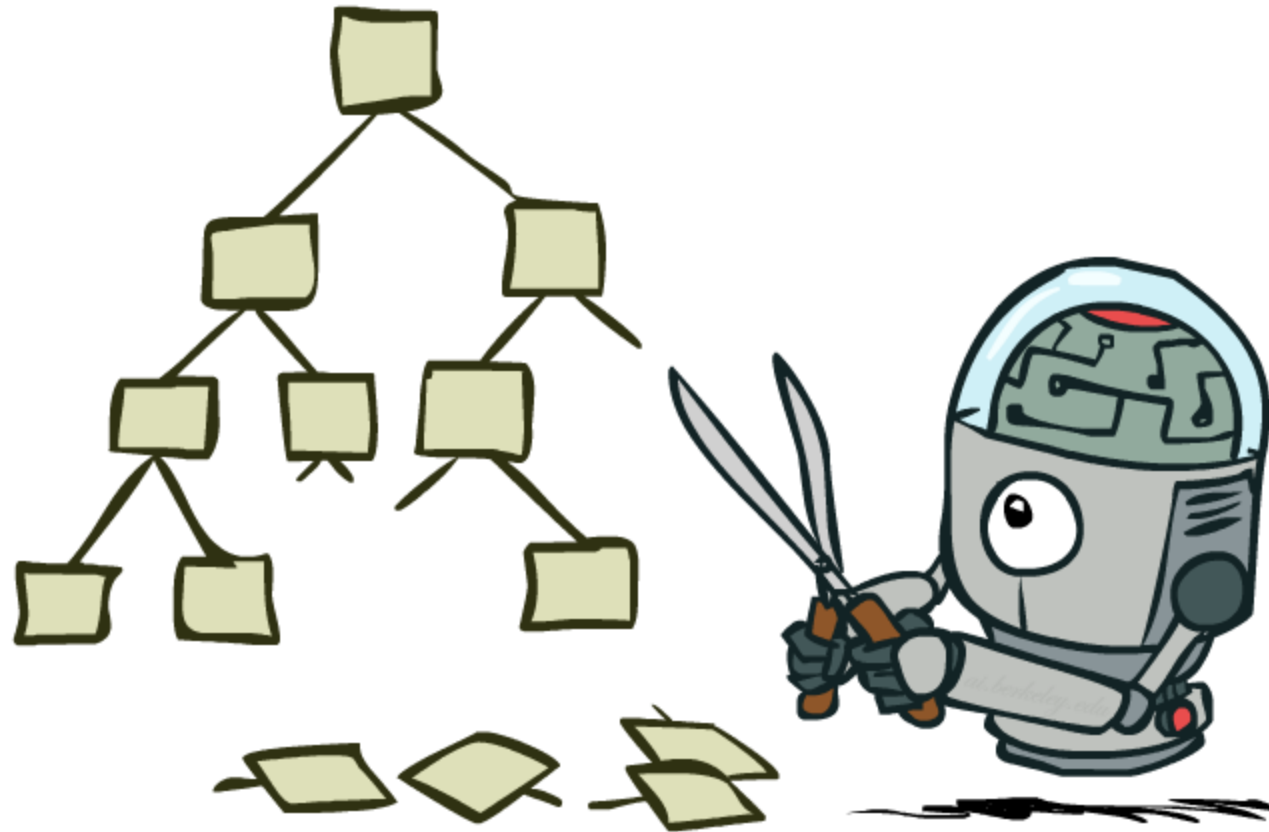
- Terminals have **utility tuples**
- Node values are also utility tuples
- **Each player maximizes its own component**
- Can give rise to cooperation and competition dynamically...



Generalized minimax

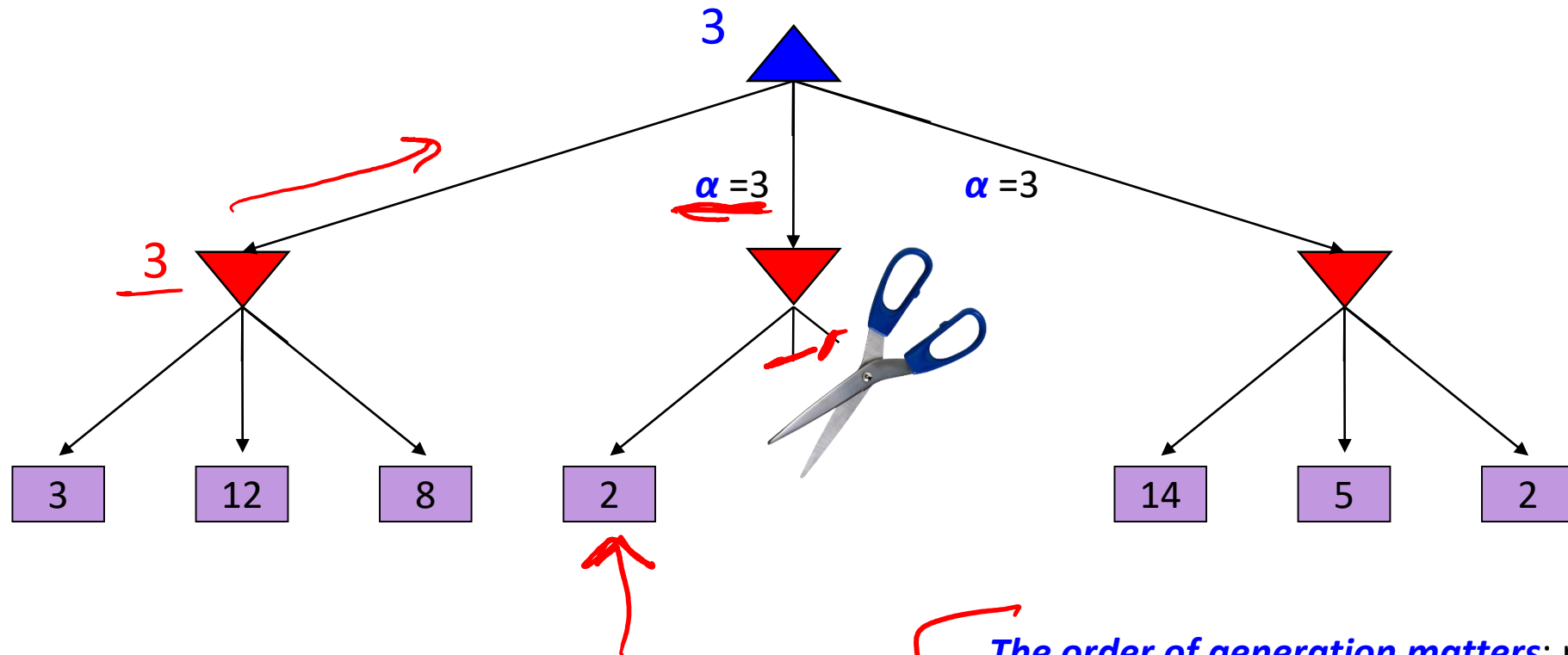
Three Person Chess

Game Tree Pruning



Alpha-Beta Example

α = best option so far from any
MAX node on this path



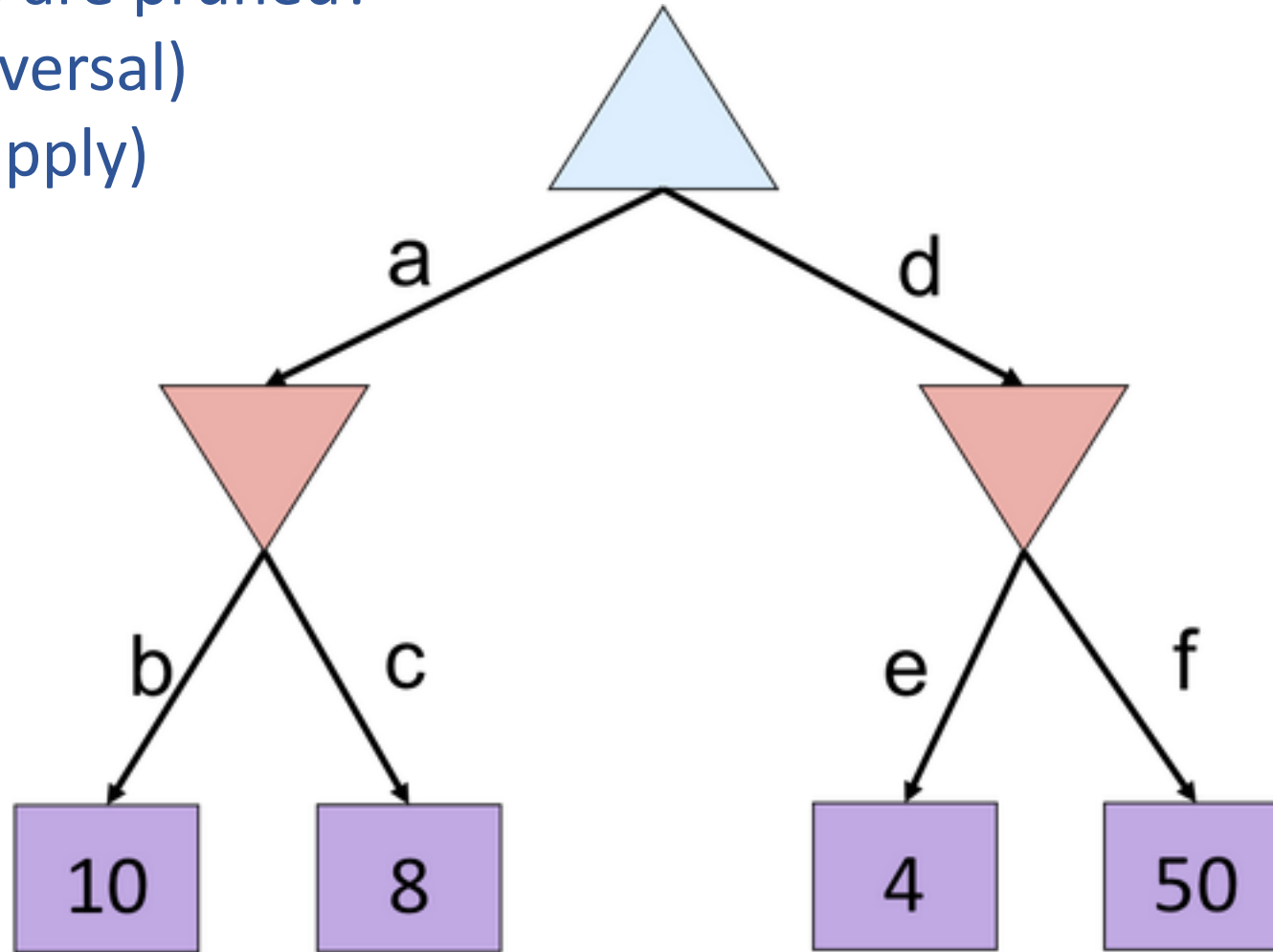
The order of generation matters: more pruning
is possible if good moves come first

Piazza Poll 2

Which branches are pruned?

(Left to right traversal)

(Select all that apply)

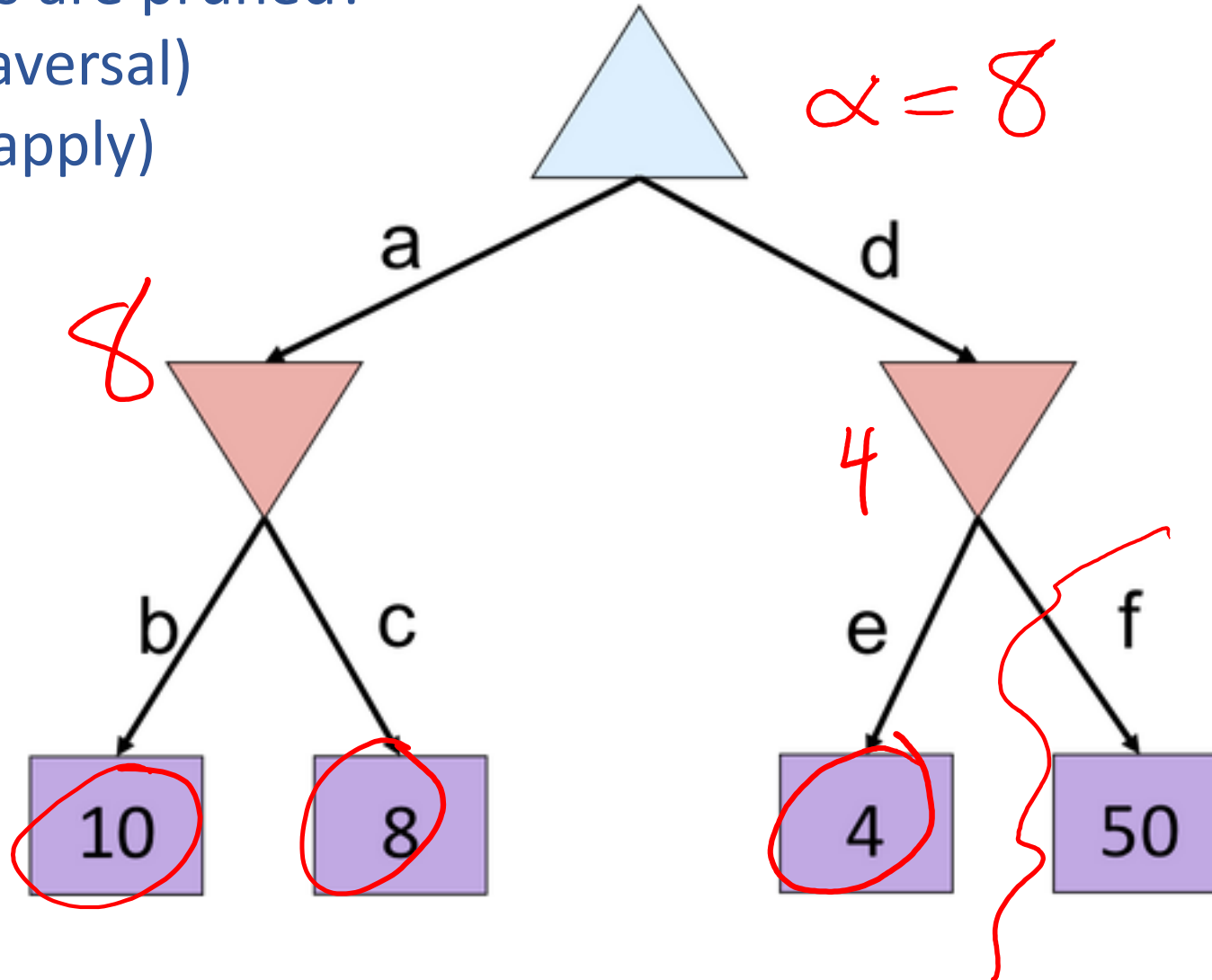


Piazza Poll 2

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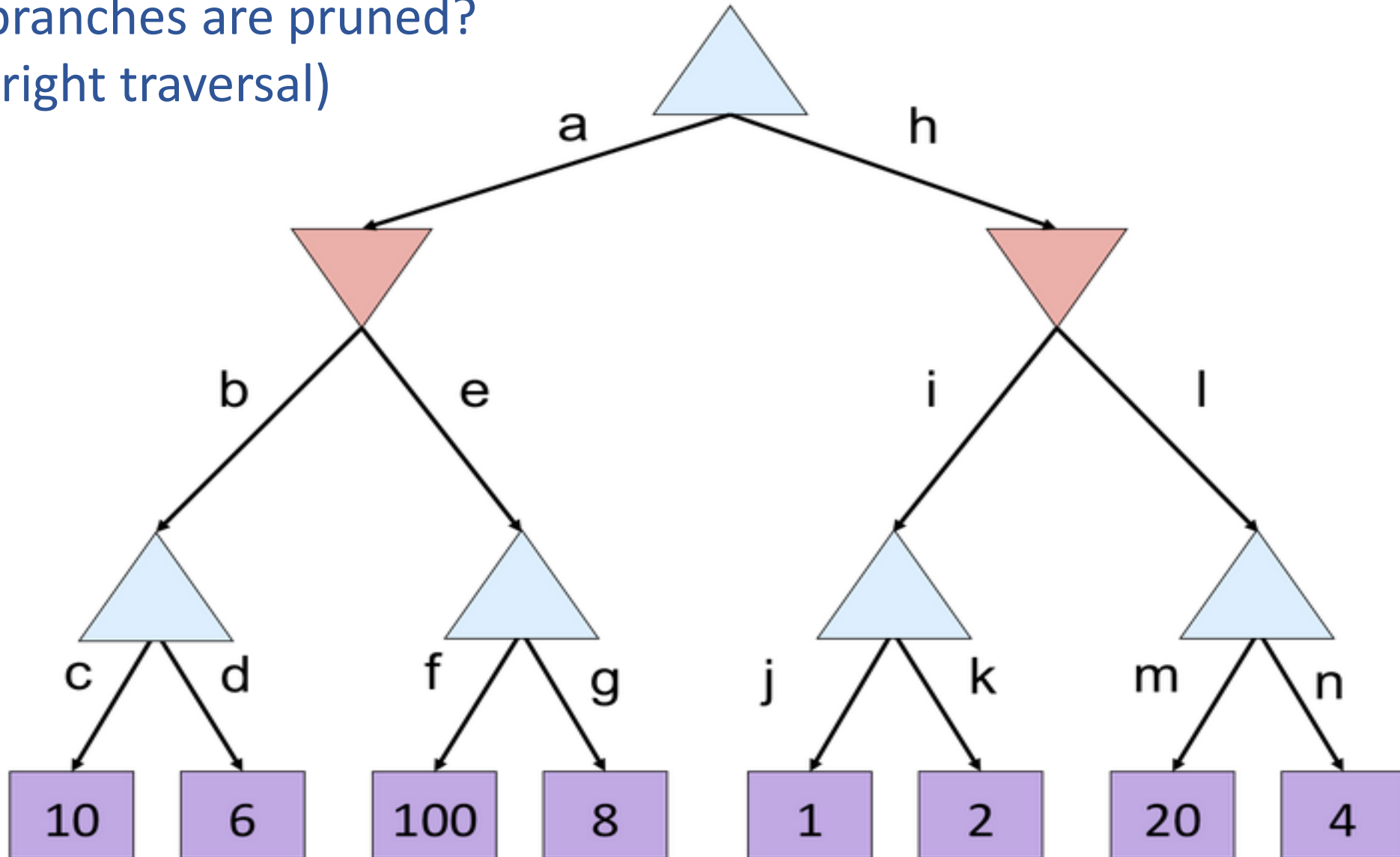
(Select all that apply)



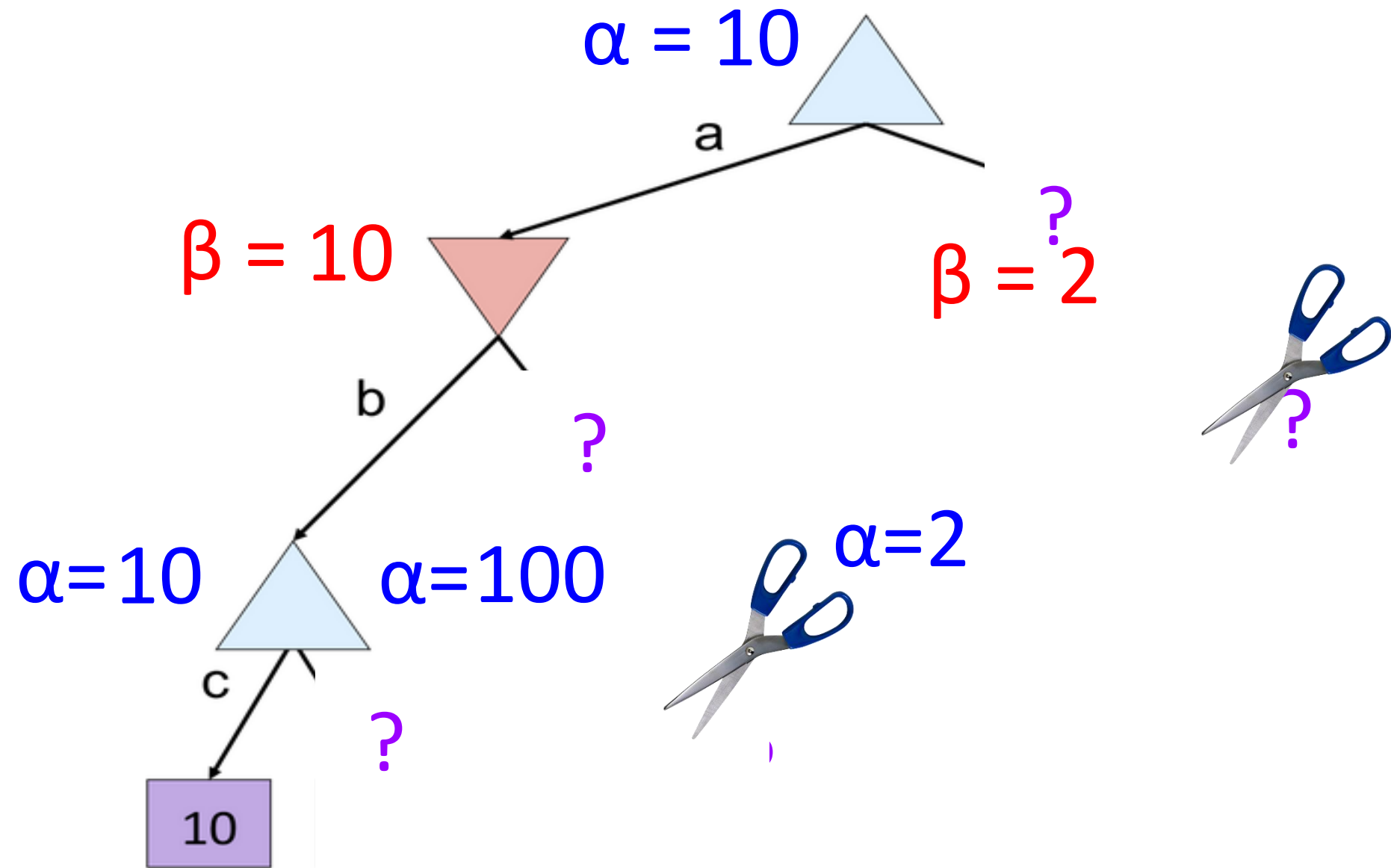
Piazza Poll 3

Which branches are pruned?
(Left to right traversal)

- A) e, l
- B) g, l
- C) g, k, l
- D) g, n



Alpha-Beta Quiz 2



Alpha-Beta Implementation

α : MAX's best option on path to root
 β : MIN's best option on path to root

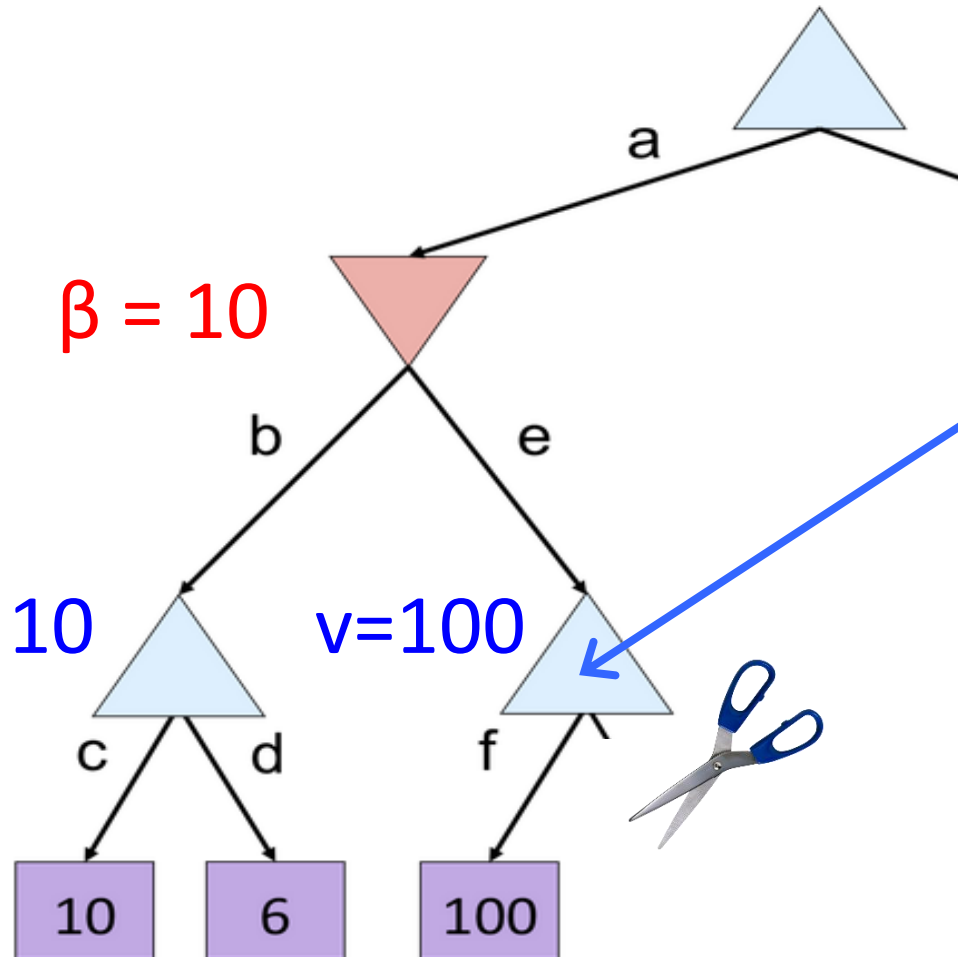
```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v = \max(v, \text{value}(\text{successor}, \alpha, \beta))$   
        if  $v \geq \beta$   
            return  $v$   
         $\alpha = \max(\alpha, v)$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    initialize  $v = +\infty$   
    for each successor of state:  
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        if  $v \leq \alpha$   
            return  $v$   
         $\beta = \min(\beta, v)$   
    return  $v$ 
```

Alpha-Beta Quiz 2

α : MAX's best option on path to root

β : MIN's best option on path to root



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def max-value(state,  $\alpha$ ,  $\beta$ ):
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    initialize  $v = -\infty$ 
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```
    for each successor of state:
```

```
         $v = \max(v, \text{value}(\text{successor}, \alpha, \beta))$ 
```

```
        if  $v \geq \beta$ 
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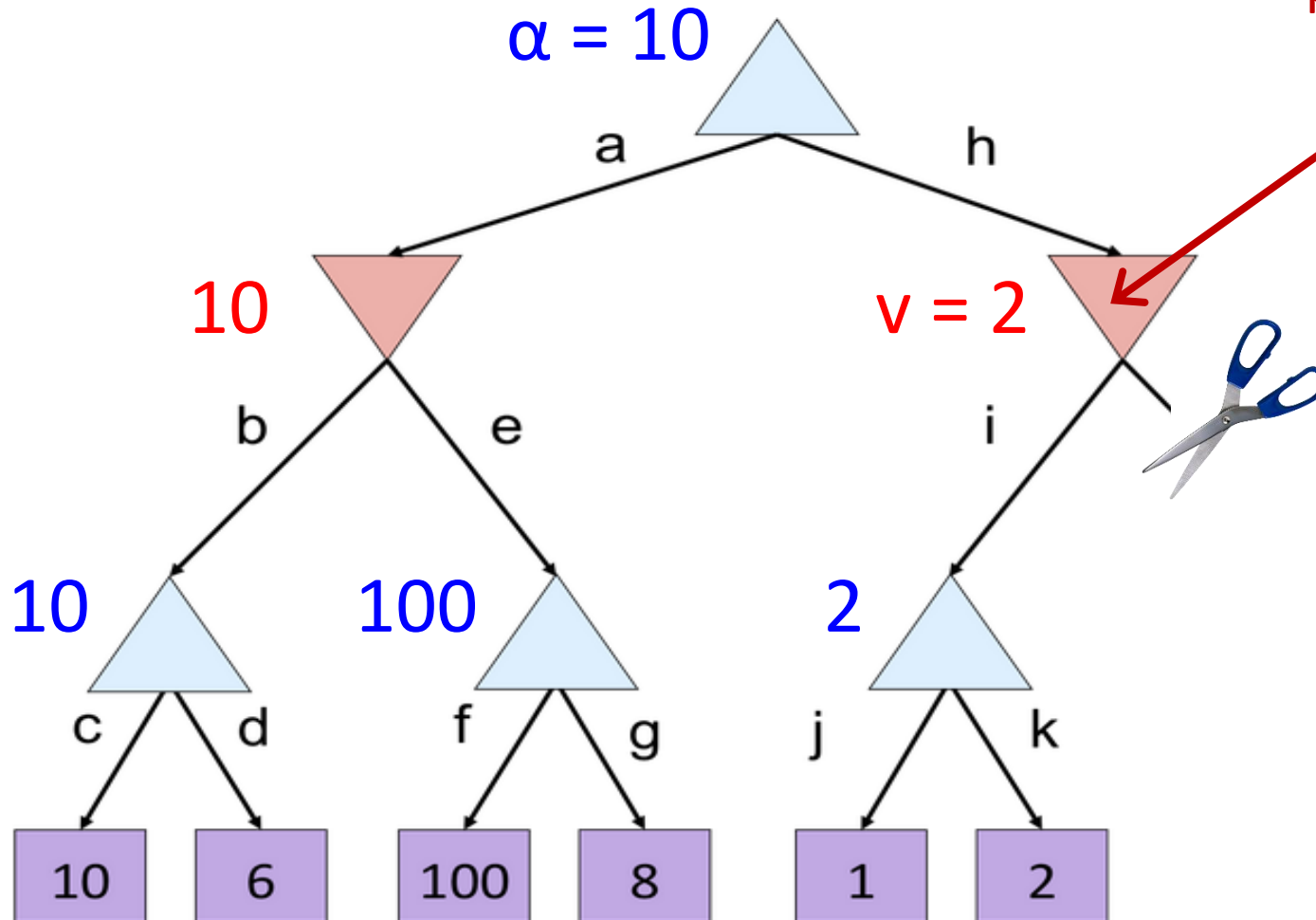
```
            return  $v$ 
```

```
         $\alpha = \max(\alpha, v)$ 
```

```
    return  $v$ 
```

Alpha-Beta Quiz 2

α : MAX's best option on path to root
 β : MIN's best option on path to root



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Alpha-Beta Pruning Properties

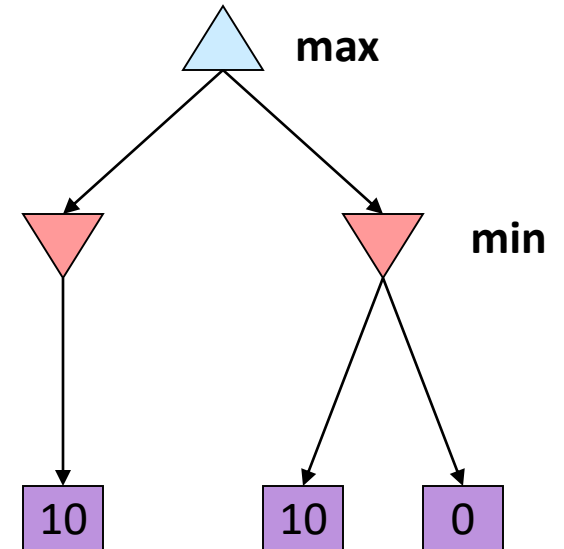
Theorem: This pruning has no effect on minimax value computed for the root!

→ Good child ordering improves effectiveness of pruning

- Iterative deepening helps with this

With “perfect ordering”:

- Time complexity drops to $O(b^{m/2})$
- Doubles solvable depth!
- 1M nodes/move => depth=8, respectable



This is a simple example of **metareasoning** (computing about what to compute)

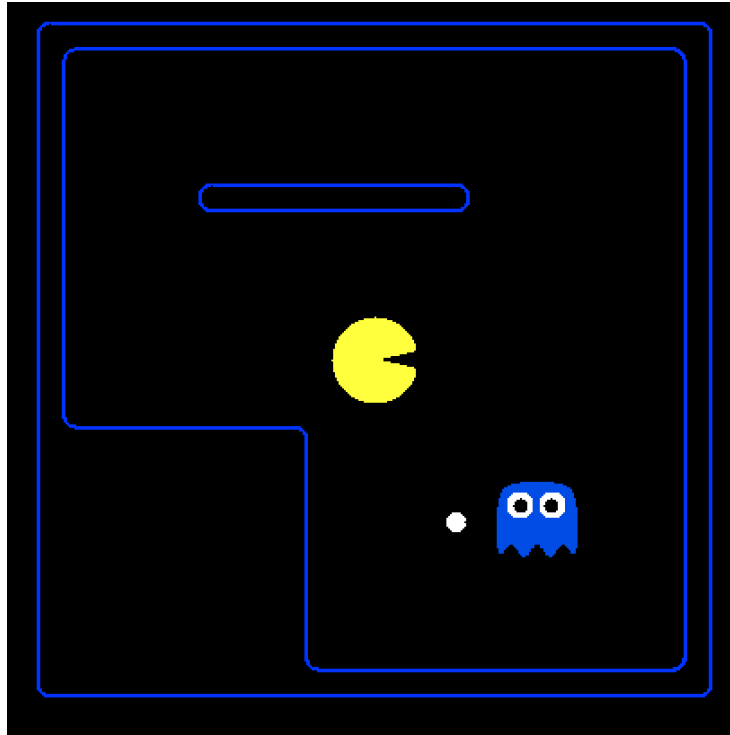
Minimax Demo

Points

+500 win

-500 lose

-1 each move



Fine print

- Pacman: uses depth 4 minimax
- Ghost: uses depth 2 minimax

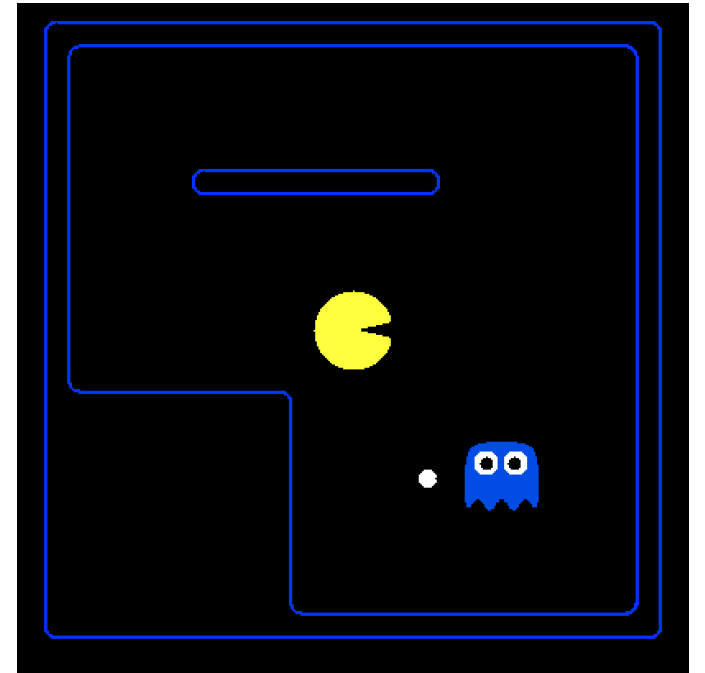
Piazza Poll 4

How well would a **minimax Pacman** perform against a **ghost that moves randomly**?

- A. **Better** than against a minimax ghost
- B. **Worse** than against a minimax ghost
- C. **Same** as against a minimax ghost

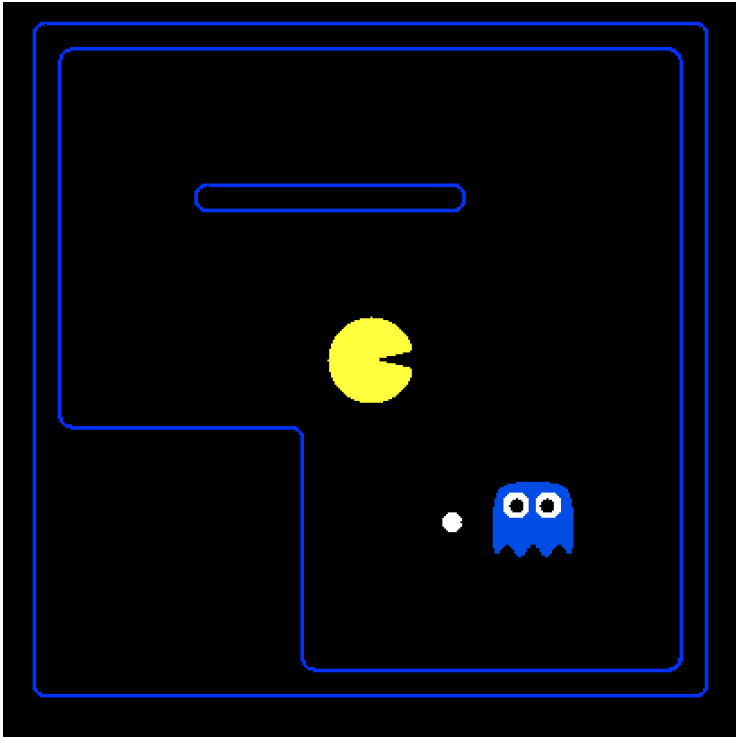
Fine print

- Pacman: uses depth 4 minimax as before
- Ghost: moves randomly



Demo

Assumptions vs. Reality

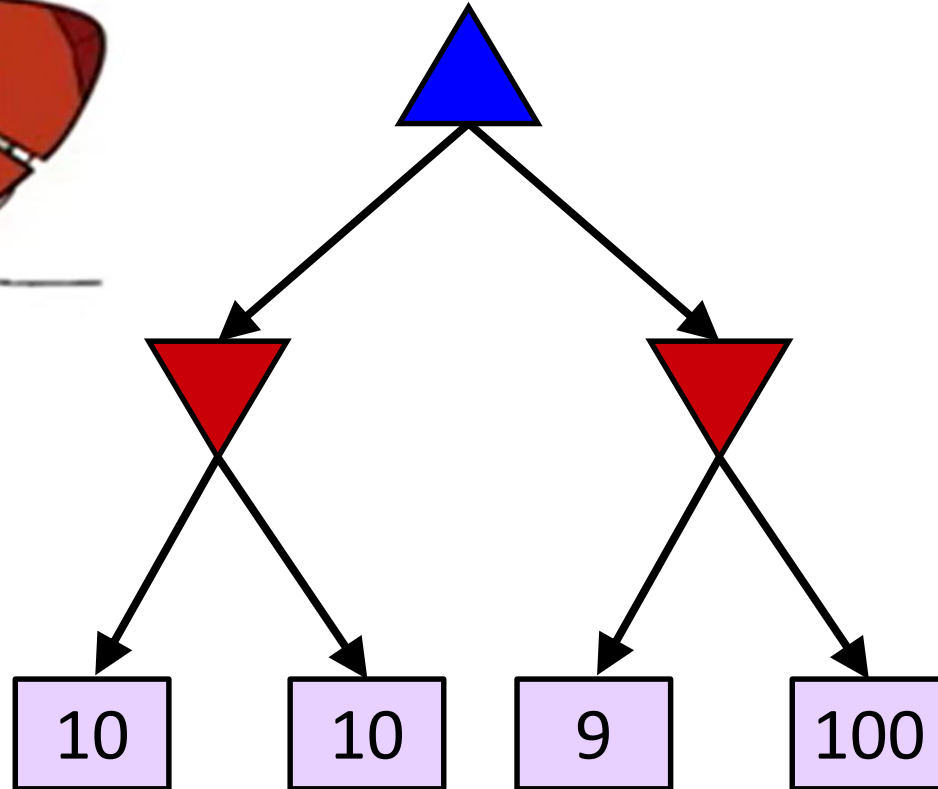
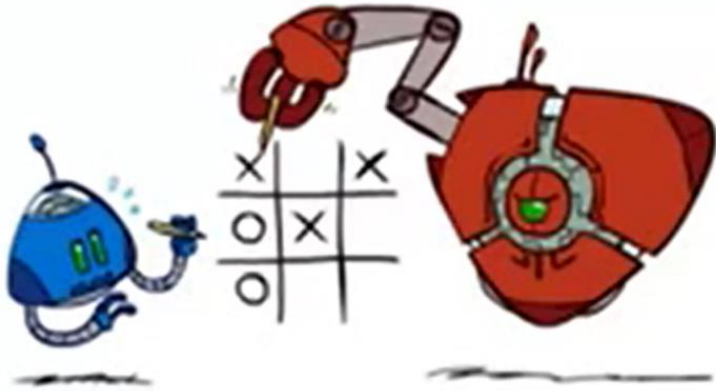


	Minimax Ghost	Random Ghost
Minimax Pacman		

Results from playing 5 games

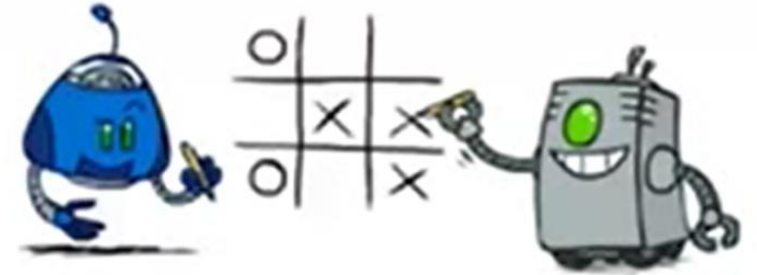
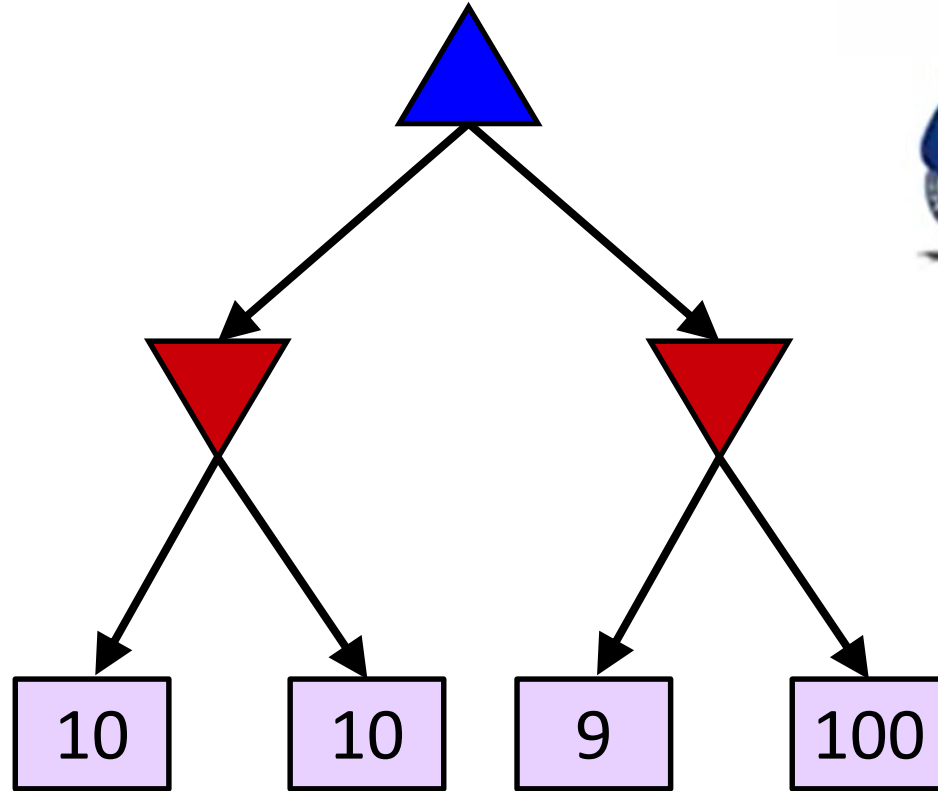
Modeling Assumptions

Know your opponent



Modeling Assumptions

Know your opponent



Modeling Assumptions

Minimax autonomous vehicle?



Image: <https://corporate.ford.com/innovation/autonomous-2021.html>

Minimax Driver?



Clip: How I Met Your Mother, CBS

Modeling Assumptions

Dangerous Pessimism

Assuming the worst case when it's not likely



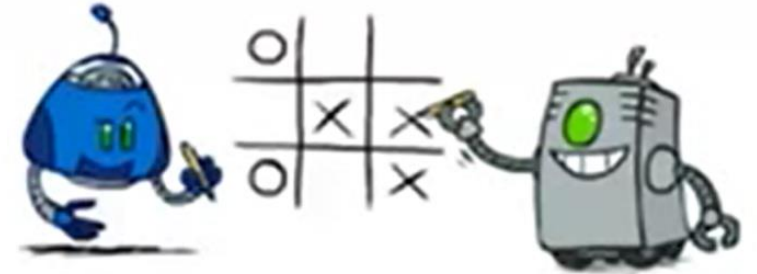
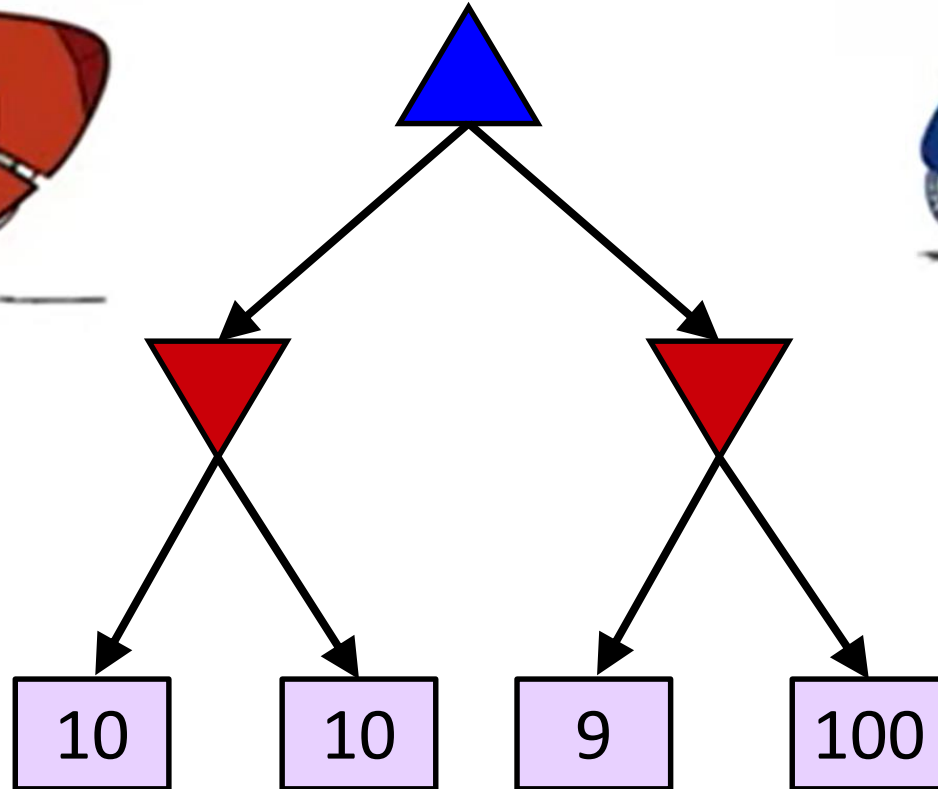
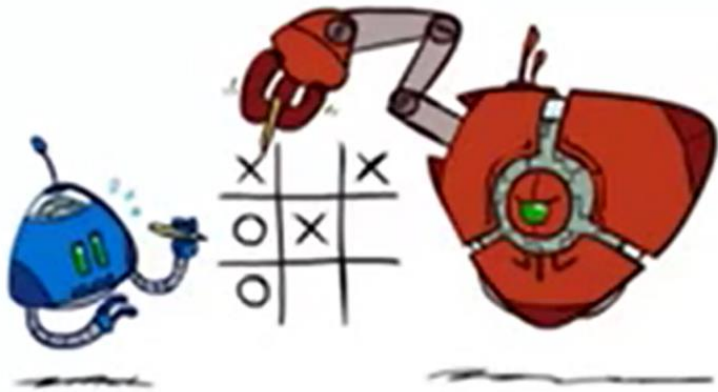
Dangerous Optimism

Assuming chance when the world is adversarial



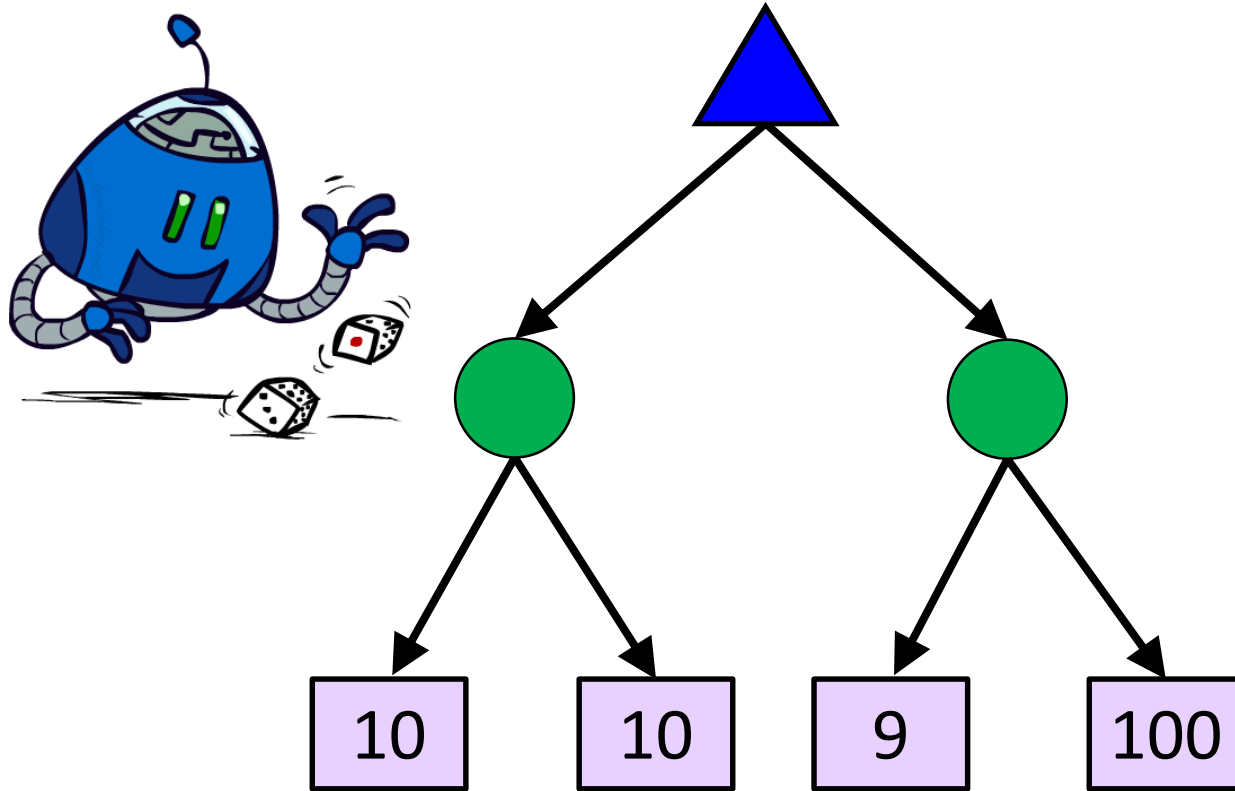
Modeling Assumptions

Know your opponent

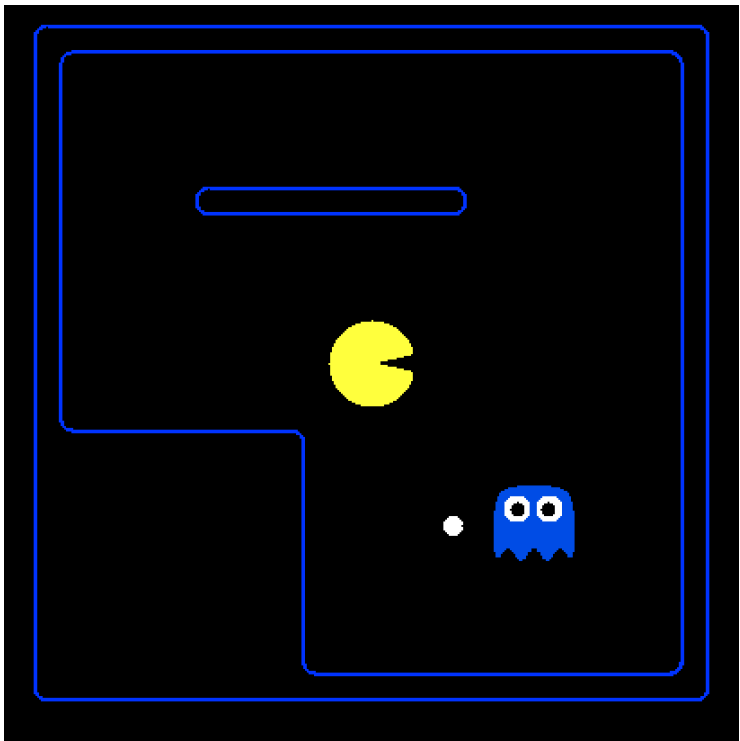


Modeling Assumptions

Chance nodes: Expectimax



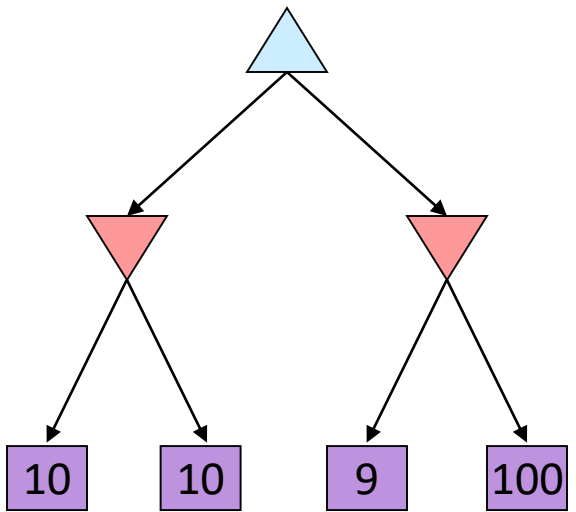
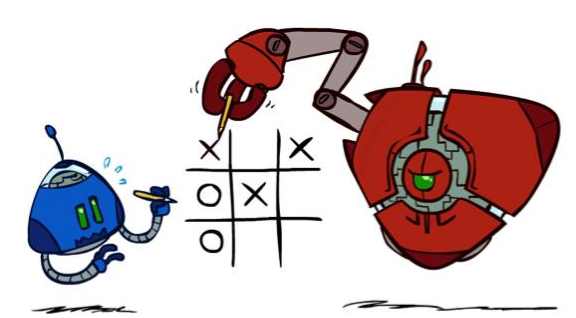
Assumptions vs. Reality



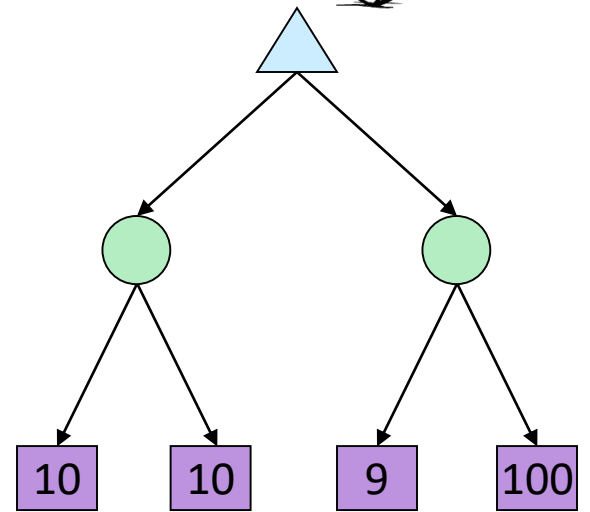
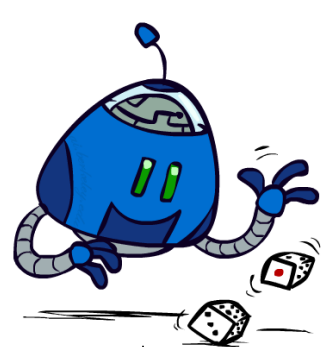
	Minimax Ghost	Random Ghost
Minimax Pacman	Won 5/5 Avg. Score: 493	Won 5/5 Avg. Score: 464
Expectimax Pacman		

Results from playing 5 games

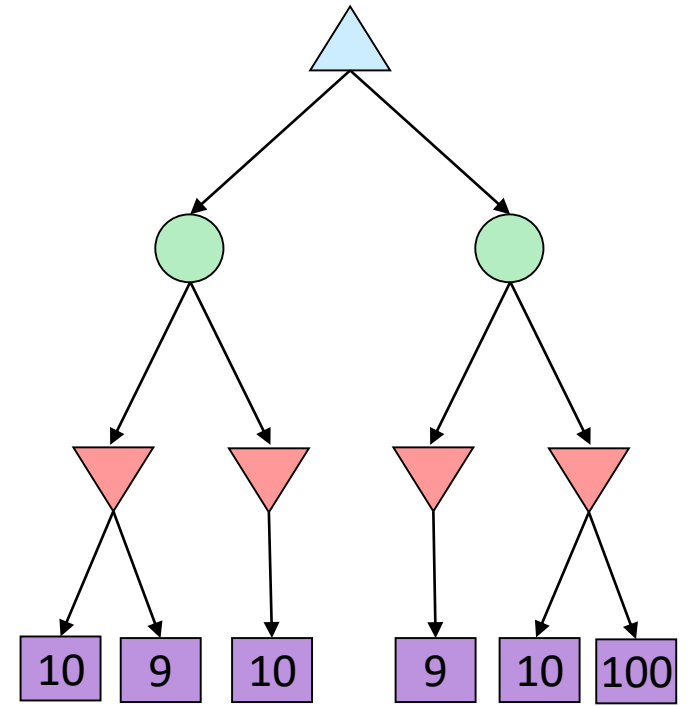
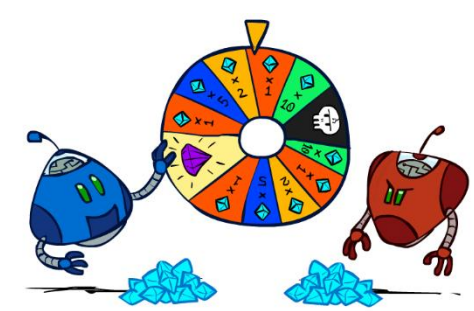
Chance outcomes in trees



Tictactoe, chess
Minimax

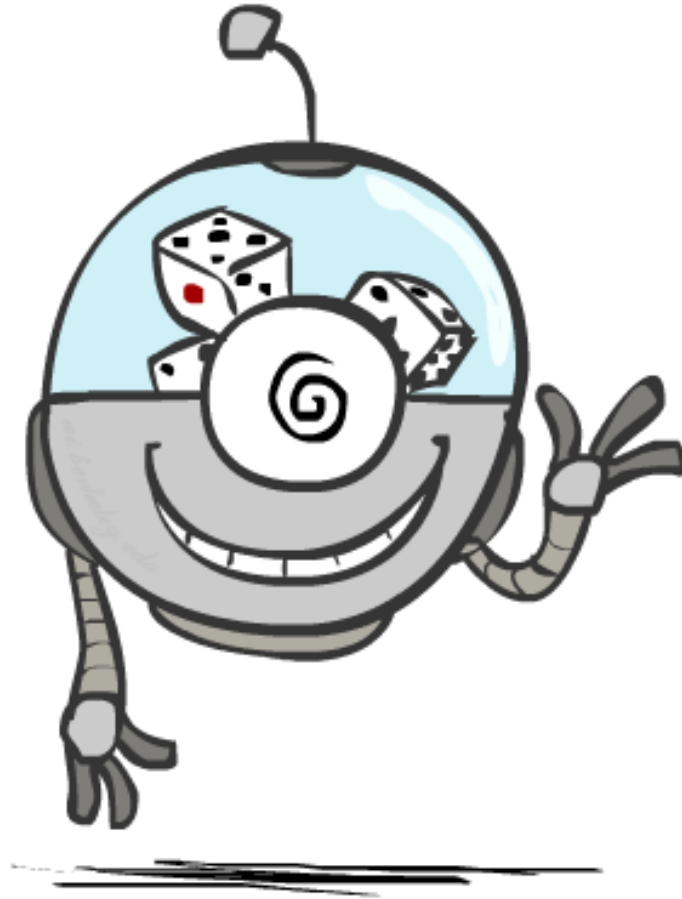


Tetris, investing
Expectimax



Backgammon, Monopoly
Expectiminimax

Probabilities



Probabilities

A **random variable** represents an event whose outcome is unknown

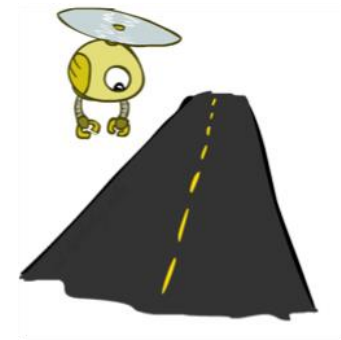
A **probability distribution** is an assignment of weights to outcomes

Example: Traffic on freeway

- Random variable: T = whether there's traffic
- Outcomes: T in {none, light, heavy}
- Distribution:

$$P(T=\text{none}) = 0.25, \quad P(T=\text{light}) = 0.50, \quad P(T=\text{heavy}) = 0.25$$

Probabilities over all possible outcomes sum to one



0.25



0.50



0.25

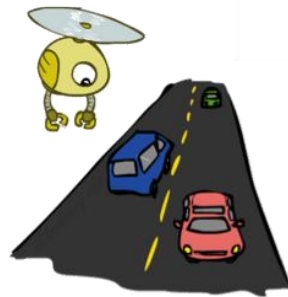
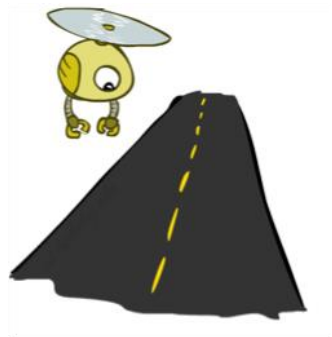
Expected Value

Expected value of a function of a random variable:

Average the **values** of each outcome,
weighted by the **probability** of that outcome

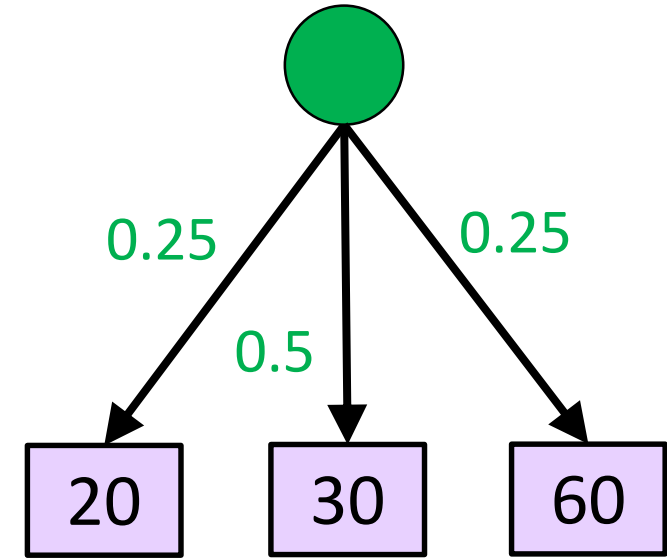
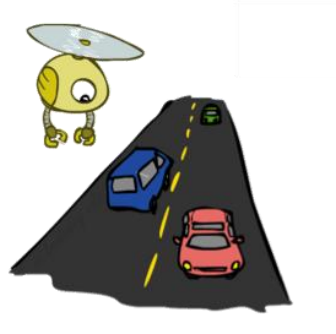
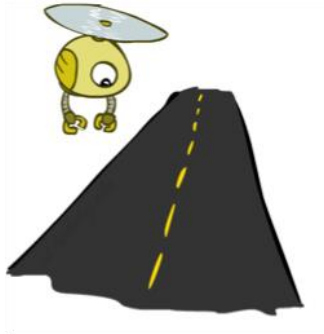
Example: How long to get to the airport?

$$\begin{array}{lclclclclcl} \text{Time:} & 20 \text{ min} & & 30 \text{ min} & & 60 \text{ min} & & & \\ & \times & + & \times & + & \times & & \rightarrow & 35 \text{ min} \\ \text{Probability:} & 0.25 & & 0.50 & & 0.25 & & & \end{array}$$



Expectations

Time: 20 min + 30 min + 60 min
Probability: 0.25 x 0.50 x 0.25



Max node notation

$$V(s) = \max_a V(s'),$$

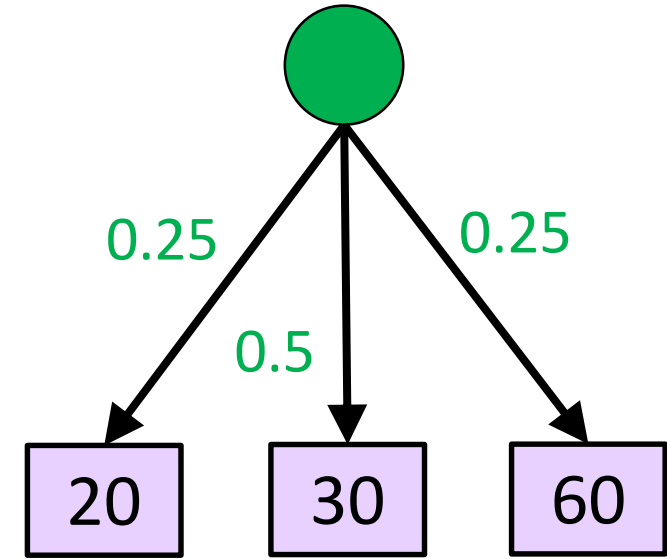
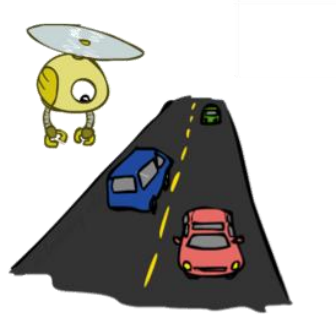
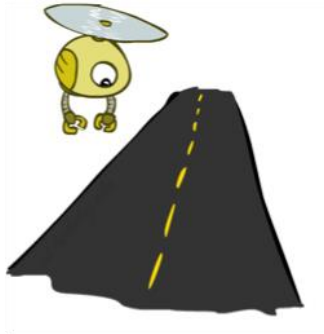
where $s' = result(s, a)$

Chance node notation

$$V(s) =$$

Expectations

Time: 20 min + 30 min + 60 min
Probability: 0.25 x 0.50 x 0.25



Max node notation

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Chance node notation

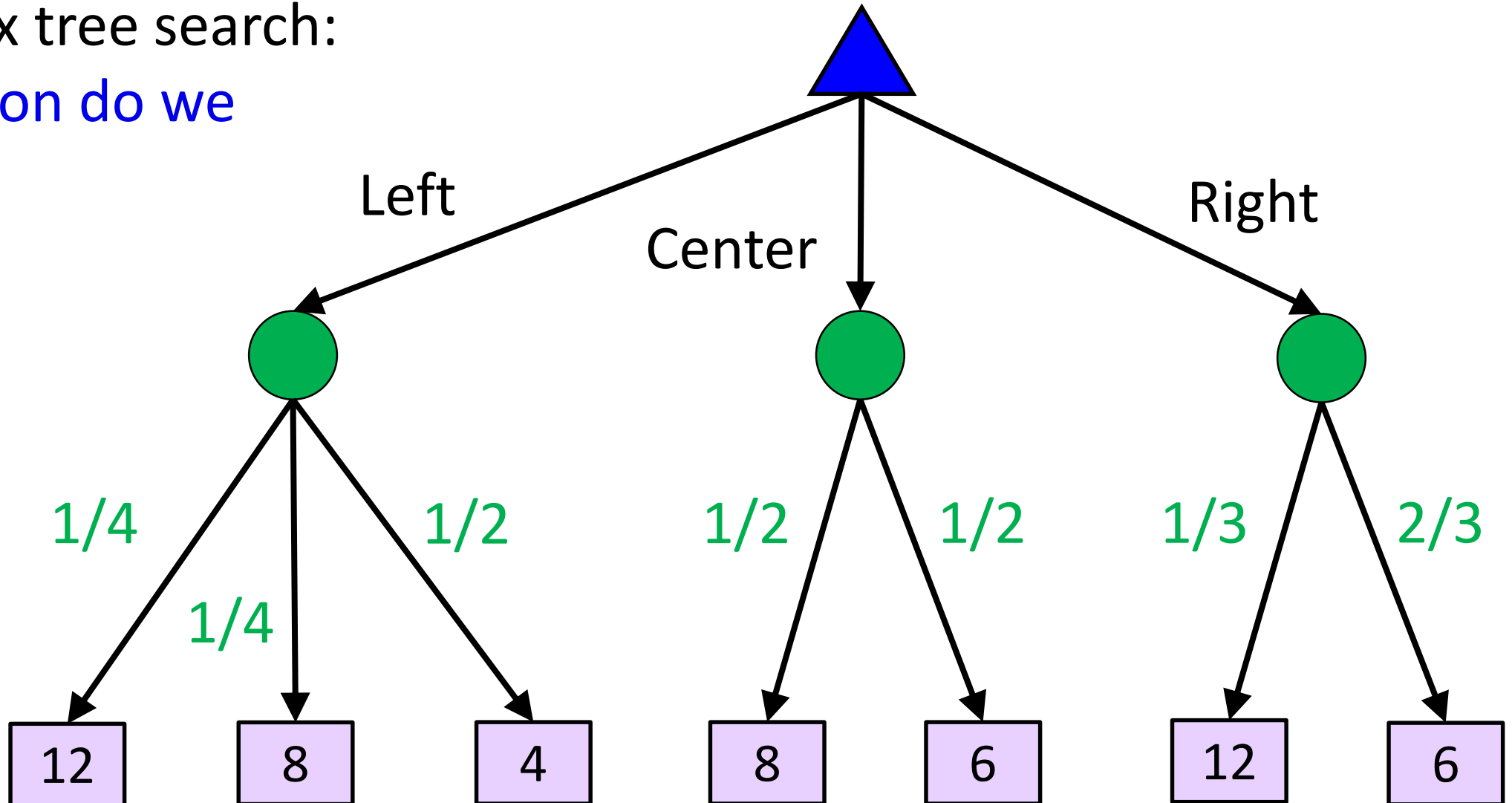
$$V(s) = \sum_{s'} P(s') V(s')$$

Piazza Poll 5

Expectimax tree search:

Which action do we choose?

- A: Left
- B: Center
- C: Right
- D: Eight

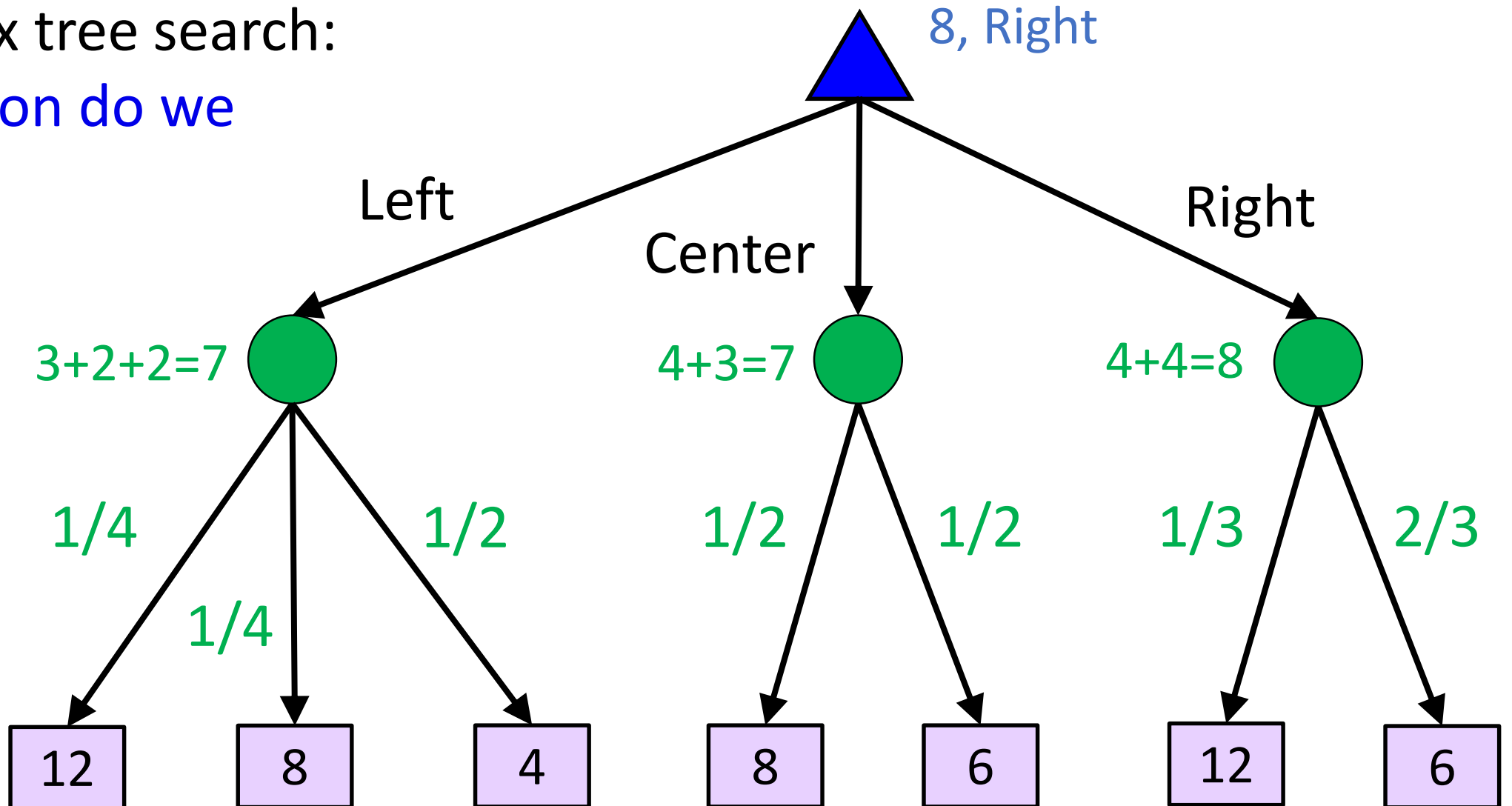


Piazza Poll 5

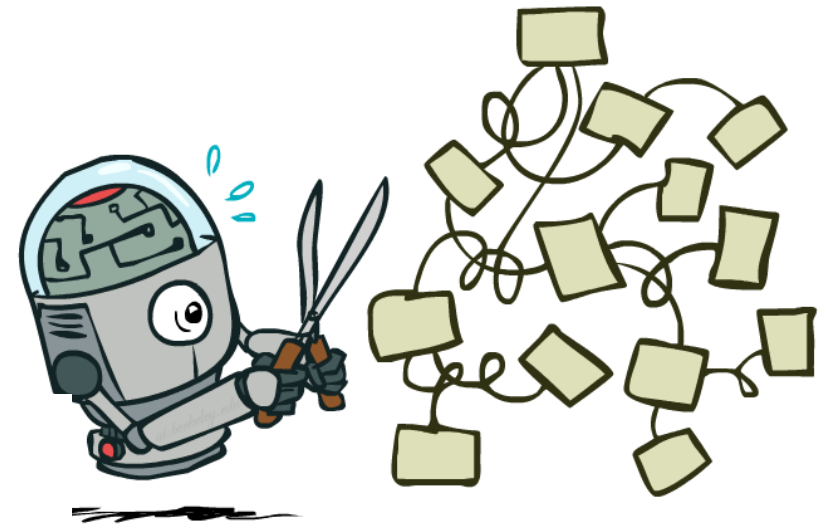
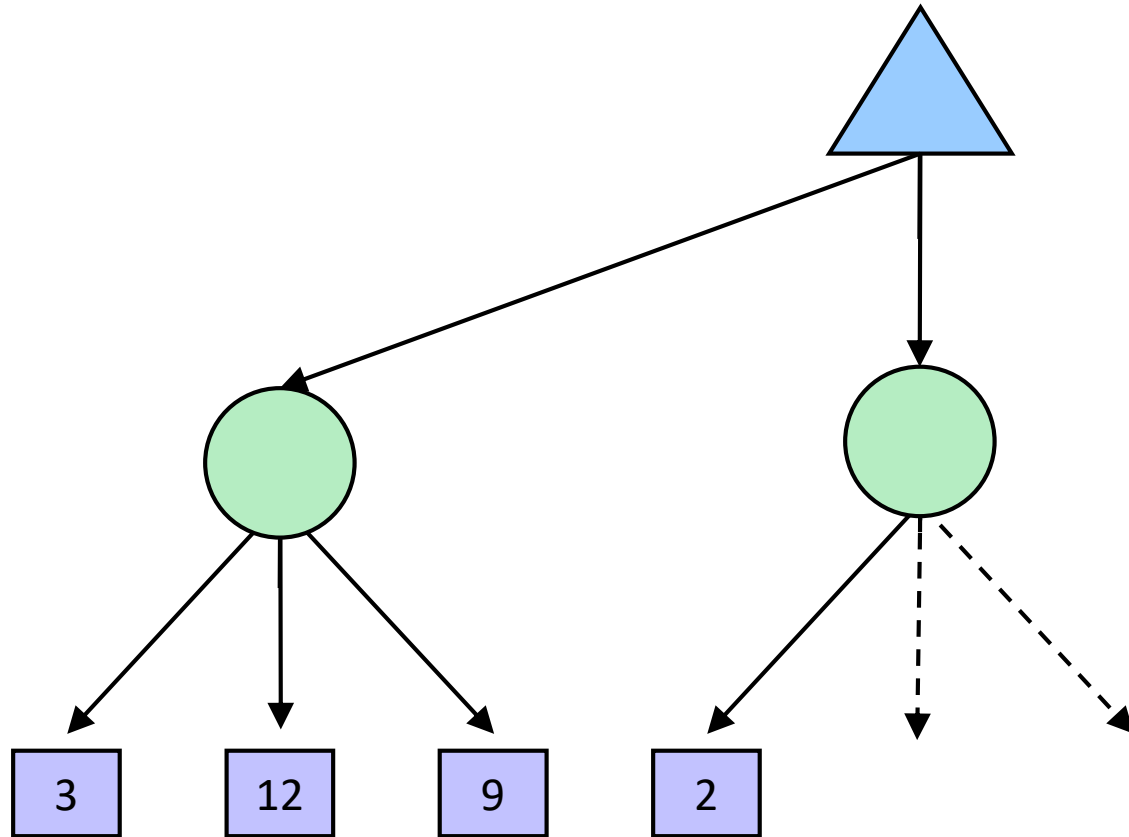
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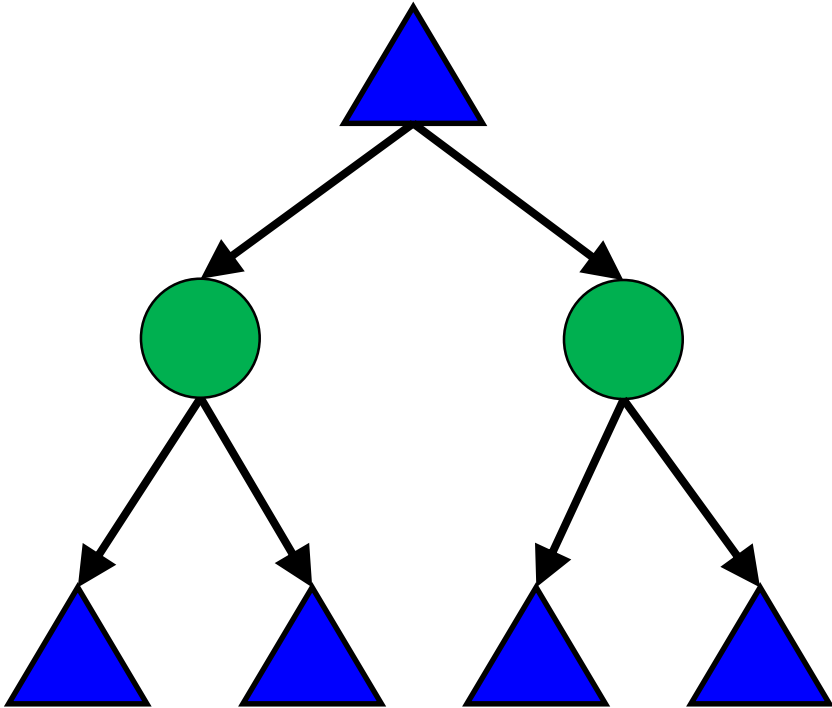
Expectimax Pruning?



Expectimax Code

```
function value( state )  
    if state.is_leaf  
        return state.value  
  
    if state.player is MAX  
        return maxa in state.actions value( state.result(a) )  
  
    if state.player is MIN  
        return mina in state.actions value( state.result(a) )  
  
    if state.player is CHANCE  
        return sums in state.next_states P( s ) * value( s )
```

Preview: MDP/Reinforcement Learning Notation



$$V(s) = \max_a \sum_{s'} P(s') V(s')$$

Preview: MDP/Reinforcement Learning Notation

Standard expectimax:
$$V(s) = \max_a \sum_{s'} P(s'|s, a) V(s')$$

Bellman equations:
$$V(s) = \max_a \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma V(s')]$$

Value iteration:
$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma V_k(s')], \quad \forall s$$

Q-iteration:
$$Q_{k+1}(s, a) = \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma \max_{a'} Q_k(s', a')], \quad \forall s, a$$

Policy extraction:
$$\pi_V(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma V(s')], \quad \forall s$$

Policy evaluation:
$$V_{k+1}^\pi(s) = \sum_{s'} P(s'|s, \pi(s)) [R(s, \pi(s), s') + \gamma V_k^\pi(s')], \quad \forall s$$

Policy improvement:
$$\pi_{new}(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma V^{\pi_{old}}(s')], \quad \forall s$$

Preview: MDP/Reinforcement Learning Notation

Standard expectimax: $V(s) = \max_a \sum_{s'} P(s'|s, a) V(s')$

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Policy extraction: $\pi_V(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s, a) [R(s, a, s') + \gamma V(s')], \quad \forall s$

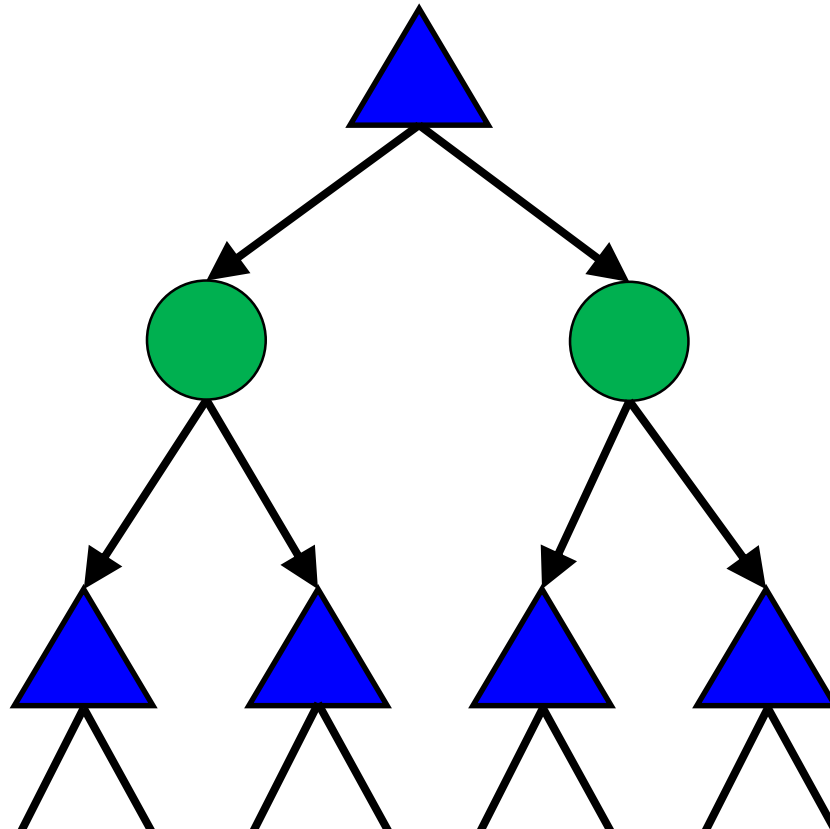
Policy evaluation: $V_{k+1}^\pi(s) = \sum_{s'} P(s'|s, \pi(s)) [R(s, \pi(s), s') + \gamma V_k^\pi(s')], \quad \forall s$

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Why Expectimax?

Pretty great model for an agent in the world

Choose the action that has the: highest expected value



Bonus Question

Let's say you know that your opponent is actually running a depth 1 minimax, using the result 80% of the time, and moving randomly otherwise

Question: What tree search should you use?

A: Minimax

B: Expectimax

C: Something completely different

Summary

Games require decisions when optimality is impossible

- Bounded-depth search and approximate evaluation functions

Games force efficient use of computation

- Alpha-beta pruning

Game playing has produced important research ideas

- Reinforcement learning (checkers)
- Iterative deepening (chess)
- Rational metareasoning (Othello)
- Monte Carlo tree search (Go)
- Solution methods for partial-information games in economics (poker)

Video games present much greater challenges – lots to do!

- $b = 10^{500}$, $|S| = 10^{4000}$, $m = 10,000$