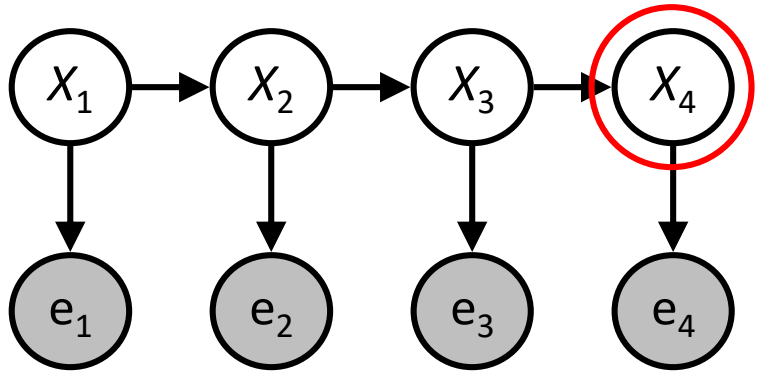
# Warm-up as you walk in

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.

$P(X_4 \mid e_1, e_2, e_3, e_4) =$



# Announcements

## Assignments

- HW10
  - Due **Wed** 11/20
- P5
  - Due **Mon** 11/25

## TA for next semester!

- CSD (15-281): https://www.ugrad.cs.cmu.edu/ta/S20/
- MLD (10-315): https://www.ml.cmu.edu/academics/ta.html

# Sampling Wrap-up

# Likelihood Weighting

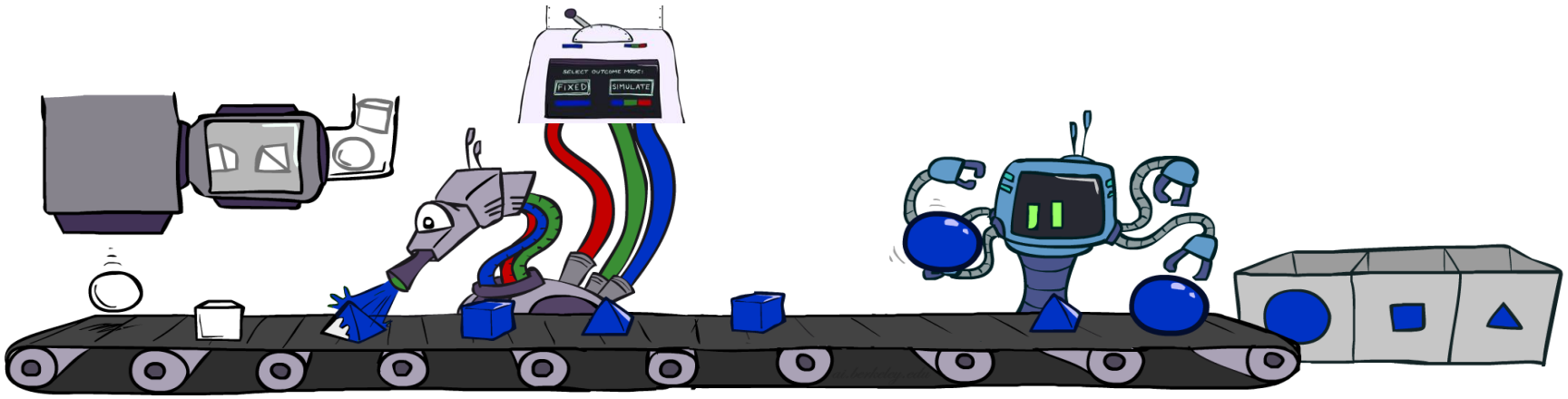
IN: evidence instantiation

w = 1.0

for i=1, 2, …, n

- if $X_i$ is an evidence variable
  - $X_i$ = observation $x_i$ for $X_i$
  - Set w = w * $P(x_i \mid Parents(X_i))$
- else
  - Sample $x_i$ from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_n)$, w



# Likelihood Weighting

## No evidence:
## Prior Sampling

Input: no evidence

for i=1, 2, …, n

- Sample $x_i$ from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_n)$

## Some evidence:
## Likelihood Weighted Sampling

Input: evidence instantiation

w = 1.0

for i=1, 2, …, n

if $X_i$ is an evidence variable

- $X_i$ = observation $x_i$ for $X_i$
- Set $w = w * P(x_i \mid Parents(X_i))$

else

- Sample $x_i$ from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_n)$, w

## All evidence:
## Likelihood Weighted

Input: evidence instantiation

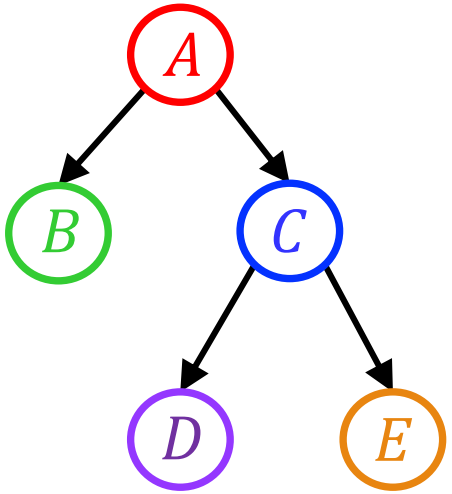
w = 1.0

for i=1, 2, …, n

- Set $w = w * P(x_i \mid Parents(X_i))$

return w

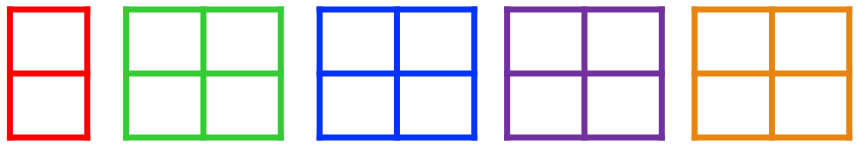
# Likelihood Weighting Distribution



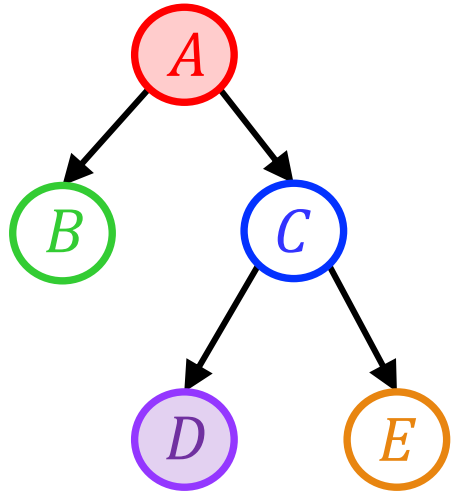

## Consistency of likelihood weighted sampling distribution

## Joint from Bayes nets

$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



# Likelihood Weighting Distribution

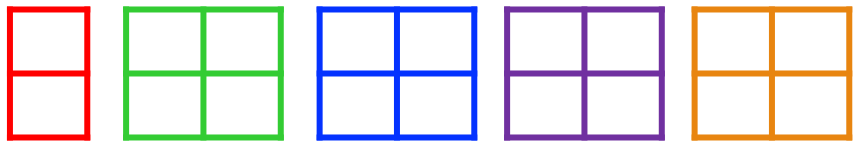



Consistency of likelihood weighted sampling distribution

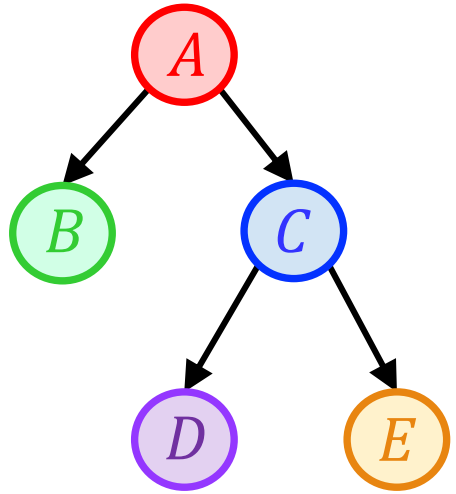
Evidence: $+a$, $-d$

Joint from Bayes nets

$$P(A, B, C, D, E) = P(+a)\ P(B|+a)\ P(C|+a)\ P(-d|C)\ P(E|C)$$



# Likelihood Weighting Distribution

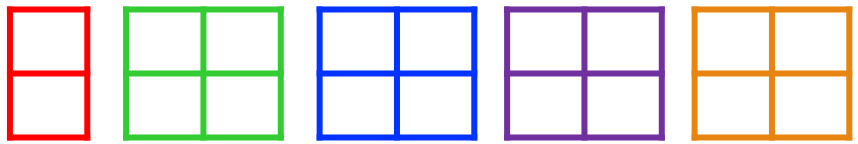



Consistency of likelihood weighted sampling distribution

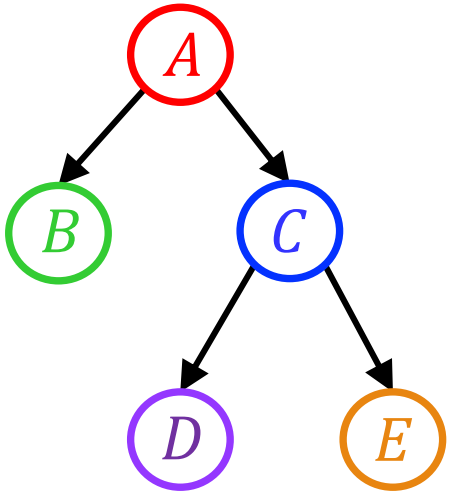
Evidence: $+a$, $+b$, $-c$, $-d$, $+e$

Joint from Bayes nets

$$P(A, B, C, D, E) = P(+a)\, P(+b|+a)\, P(-c|+a)\, P(-d|-c)\, P(+e|-c)$$



# Likelihood Weighting Distribution

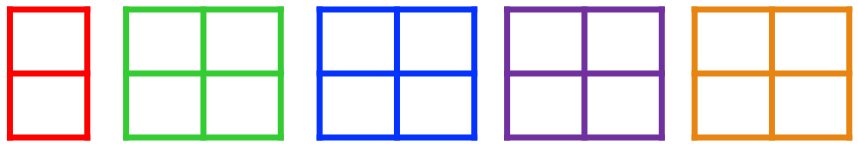



Consistency of likelihood weighted sampling distribution

Evidence: None

Joint from Bayes nets

$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



# Piazza Poll 1

Two identical samples from likelihood weighted sampling will have the same exact weights.

A. True
B. False
C. It depends
D. I don’t know

# Piazza Poll 1

Two identical samples from likelihood weighted sampling will have the same exact weights.

A. True
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# Piazza Poll 2

Given evidence +c, and number of samples $N$,
what does the following likelihood weighted value approximate?

$$\text{weight}_{(+a,-b,+c)} \cdot \frac{N(+a,-b,+c)}{N}$$

A. $P(+a,-b,+c)$
B. $P(+a,-b \mid +c)$
C. I'm not sure

# Piazza Poll 2

Given evidence +c, and number of samples $N$,
what does the following likelihood weighted value approximate?

$$\text{weight}_{(+a,-b,+c)} \cdot \frac{N(+a,-b,+c)}{N}$$

A. $P(+a,-b,+c)$
B. $P(+a,-b \mid +c)$
C. I'm not sure

$$P(AllNodes) = \prod_{e \in EvidenceNodes} P(e \mid Parents(e)) \cdot \prod_{s \in NonEvidenceNodes} P(s \mid Parents(s))$$

# Likelihood Weighting

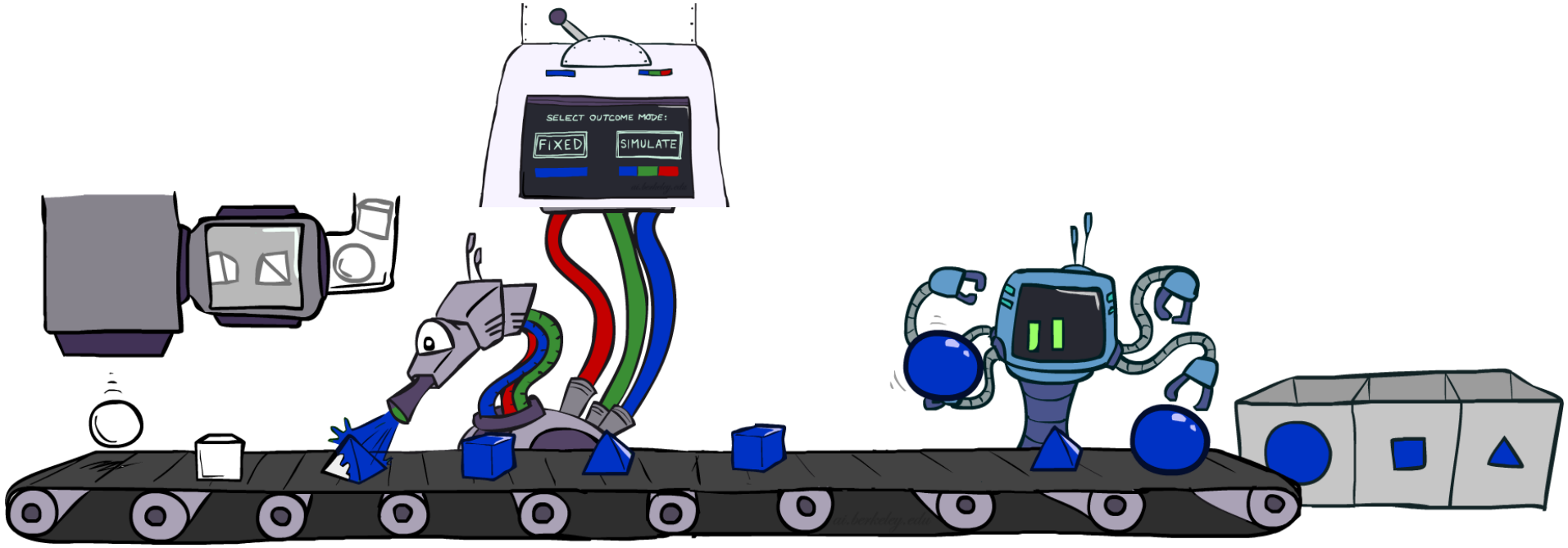
## Likelihood weighting is good

- We have taken evidence into account as we generate the sample
- E.g. here, W's value will get picked based on the evidence values of S, R
- More of our samples will reflect the state of the world suggested by the evidence

## Likelihood weighting doesn't solve all our problems

- Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)

## We would like to consider evidence when we sample every variable



# Likelihood Weighting

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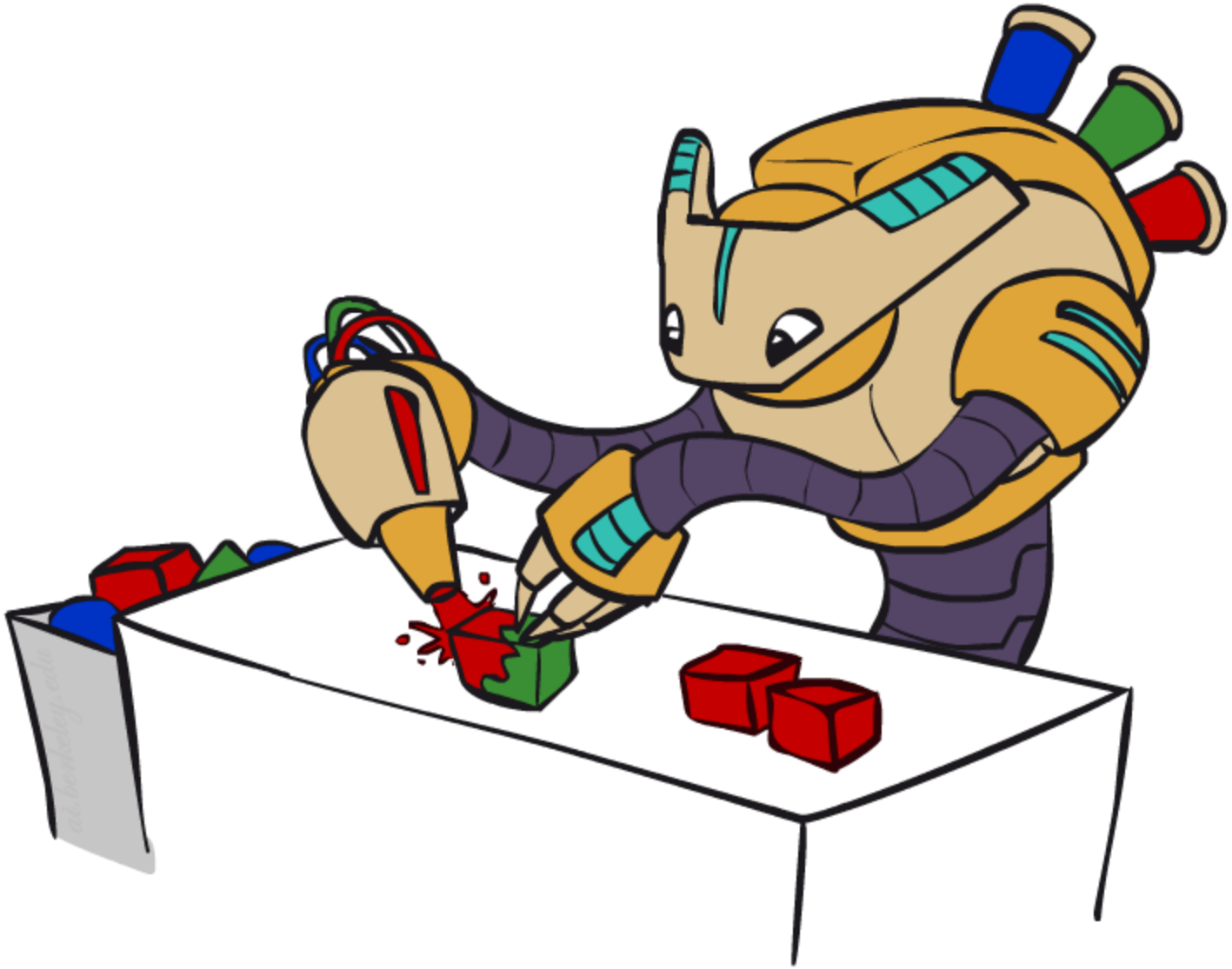
We would like to consider evidence when we sample every variable

→ Gibbs sampling

# Gibbs Sampling



# Gibbs Sampling

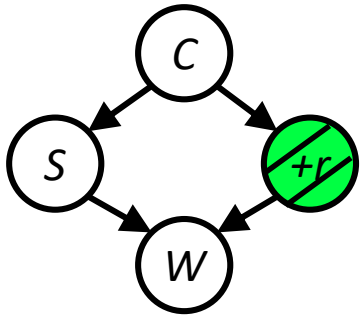
*Procedure:* keep track of a full instantiation $x_1, x_2, \ldots, x_n$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Keep repeating this for a long time.

# Gibbs Sampling Example: P( S | +r)

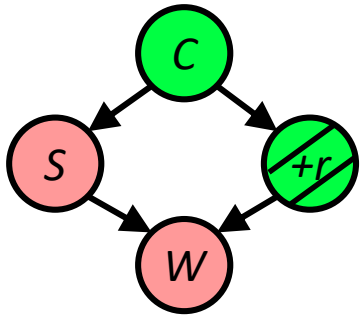
## Step 1: Fix evidence

- R = +r



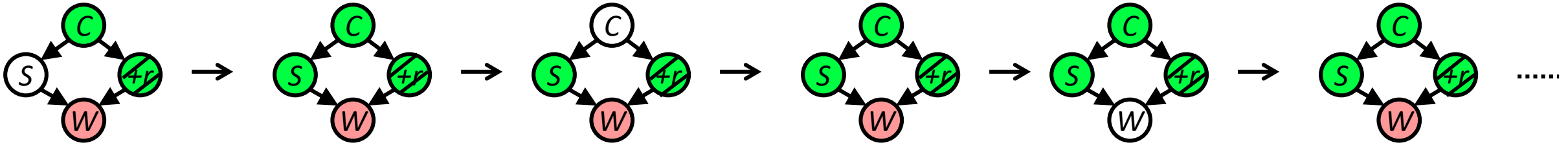

## Step 2: Initialize other variables

- Randomly




## Steps 3: Repeat

- Choose a non-evidence variable X
- Resample X from P( X | all other variables)

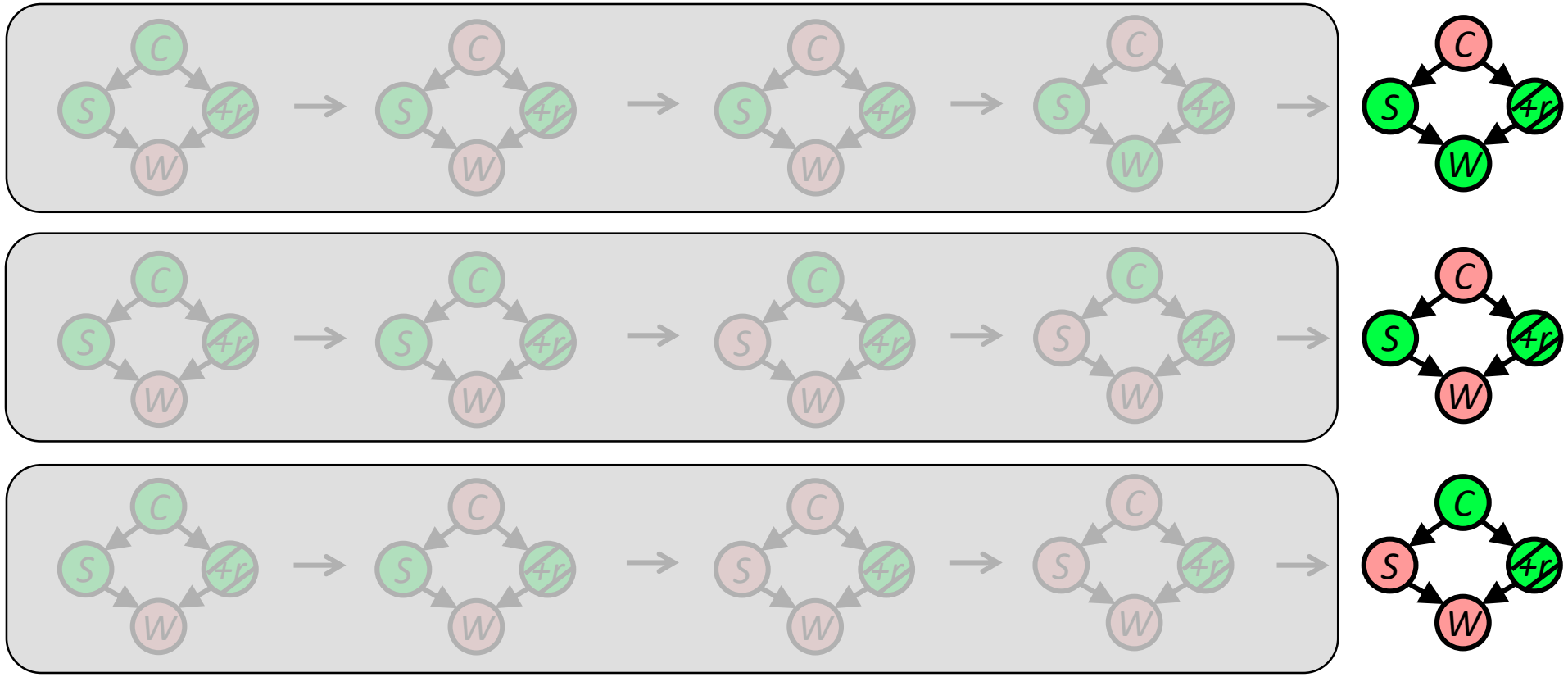



Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

# Gibbs Sampling Example: P( S | +r)

Keep only the last sample from each iteration:

1.

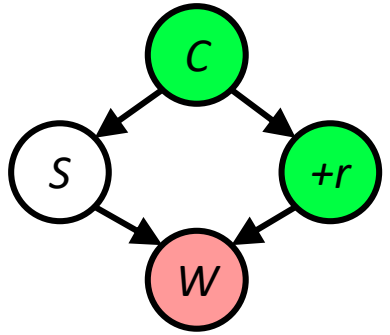



2.

3.

# Efficient Resampling of One Variable

Sample from P(S | +c, +r, -w)




$$
\begin{aligned}
P(S|+c,+r,-w) &= \frac{P(S,+c,+r,-w)}{P(+c,+r,-w)} \\
&= \frac{P(S,+c,+r,-w)}{\sum_s P(s,+c,+r,-w)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{\sum_s P(+c)P(s|+c)P(+r|+c)P(-w|s,+r)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{P(+c)P(+r|+c)\sum_s P(s|+c)P(-w|s,+r)} \\
&= \frac{P(S|+c)P(-w|S,+r)}{\sum_s P(s|+c)P(-w|s,+r)}
\end{aligned}
$$

Many things cancel out – only CPTs with S remain!

More generally: only CPTs that have resampled variable need to be considered, and joined together

# Gibbs Sampling

*Procedure:* keep track of a full instantiation $x_1, x_2, \ldots, x_n$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Keep repeating this for a long time.

*Property:* in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

*Rationale*: both upstream and downstream variables condition on evidence.

In contrast: likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small. Sum of weights over all samples is indicative of how many "effective" samples were obtained, so want high weight.

# Gibbs Sampling

Gibbs sampling produces sample from the query distribution P( Q | e ) in limit of re-sampling infinitely often

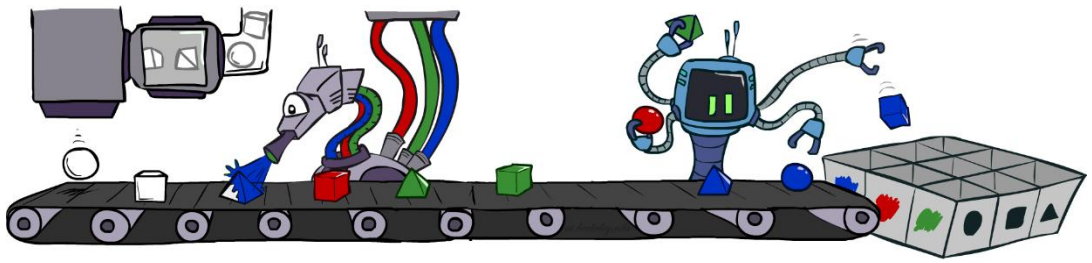
Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods

- Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)

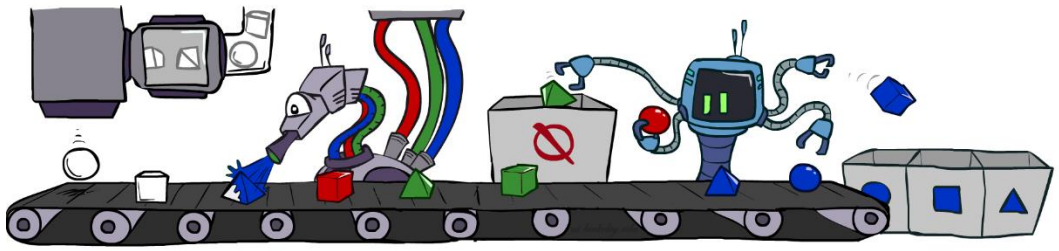
You may read about Monte Carlo methods – they're just sampling

# Bayes' Net Sampling Summary

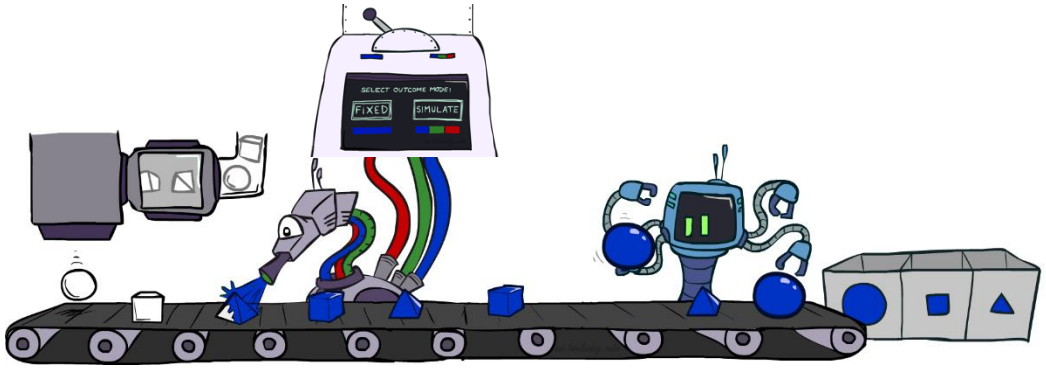
## Prior Sampling P(Q, E)



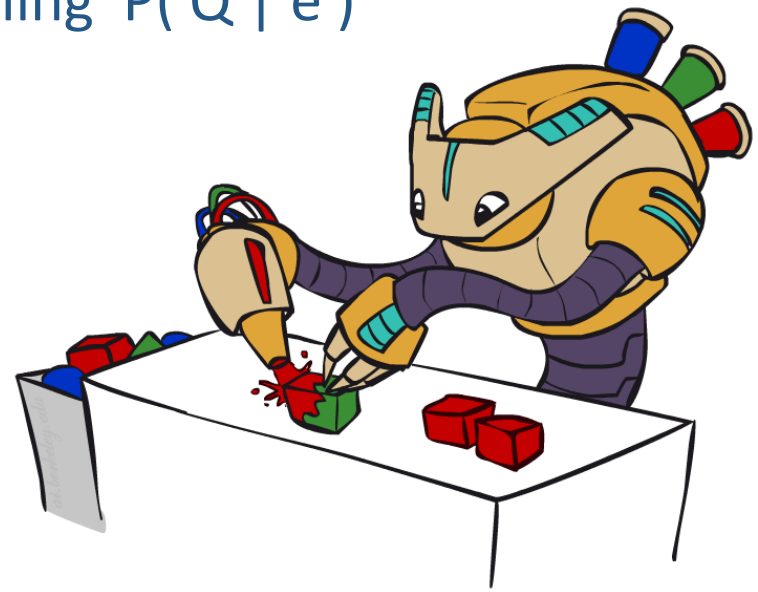
## Rejection Sampling P( Q | e )



## Likelihood Weighting P( Q , e)




## Gibbs Sampling P( Q | e )



# AI: Representation and Problem Solving

## Hidden Markov Models

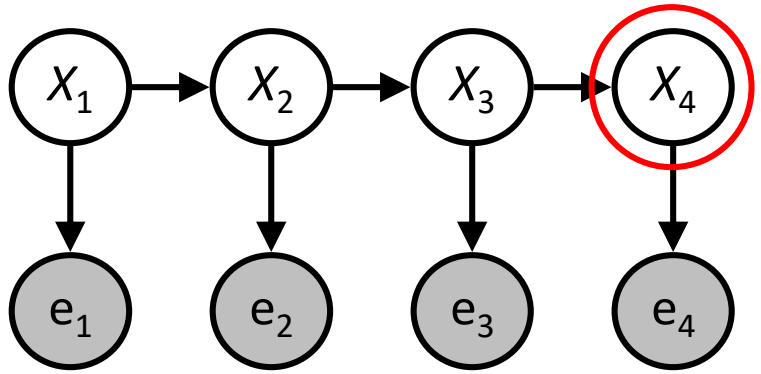

Instructors: Pat Virtue & Fei Fang

Slide credits: CMU AI and http://ai.berkeley.edu

# Warm-up as you walk in

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.

$P(X_4 \mid e_1, e_2, e_3, e_4) =$



# Reasoning over Time or Space

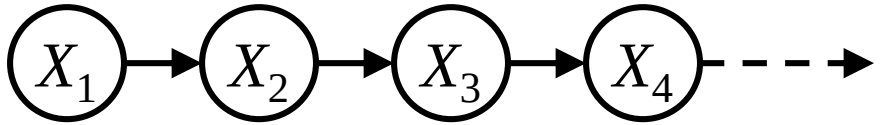
Often, we want to reason about a sequence of observations

- Speech recognition
- Robot localization
- User attention
- Medical monitoring

Need to introduce time (or space) into our models

# Markov Models

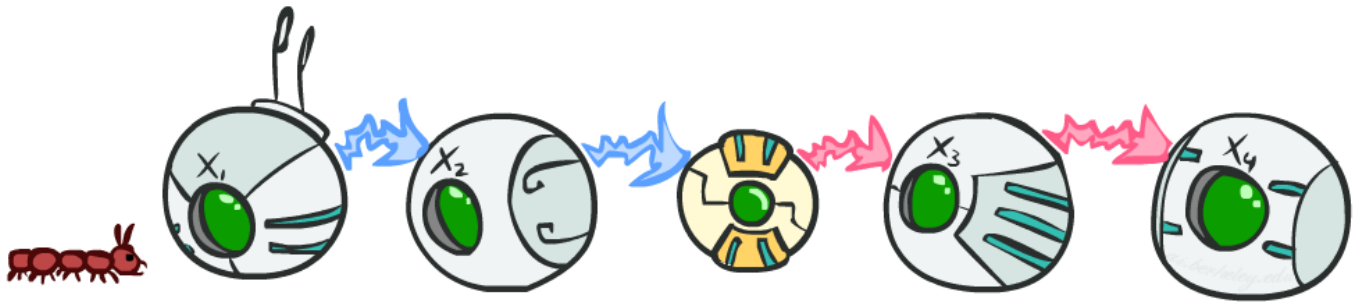
- Value of X at a given time is called the state




$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

# Conditional Independence




## Basic conditional independence:

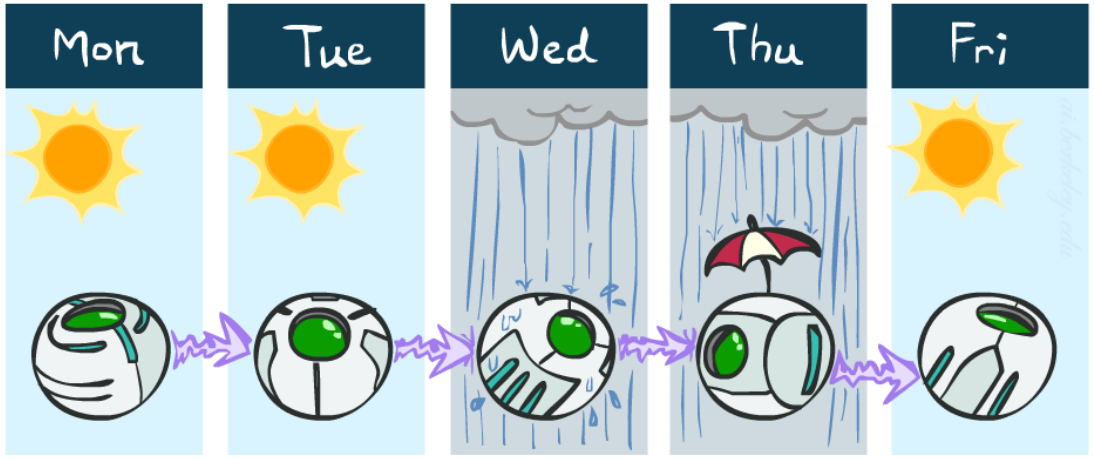
- Past and future independent given the present
- Each time step only depends on the previous
- This is called the (first order) Markov property

## Note that the chain is just a (growable) BN

- We can always use generic BN reasoning on it if we truncate the chain at a fixed length

# Example: Markov Chain Weather

States: X = {rain, sun}

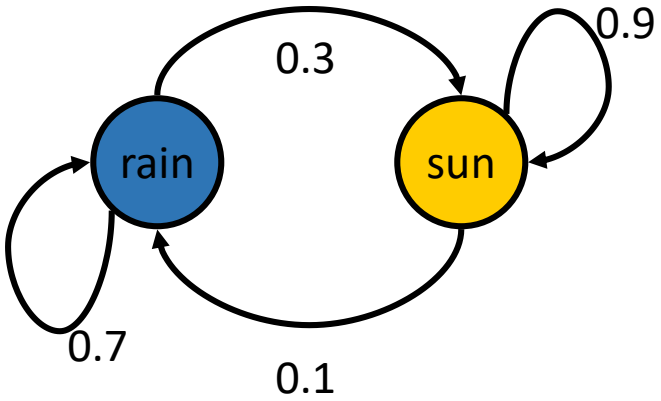


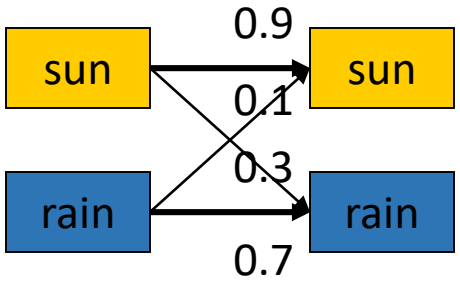

- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

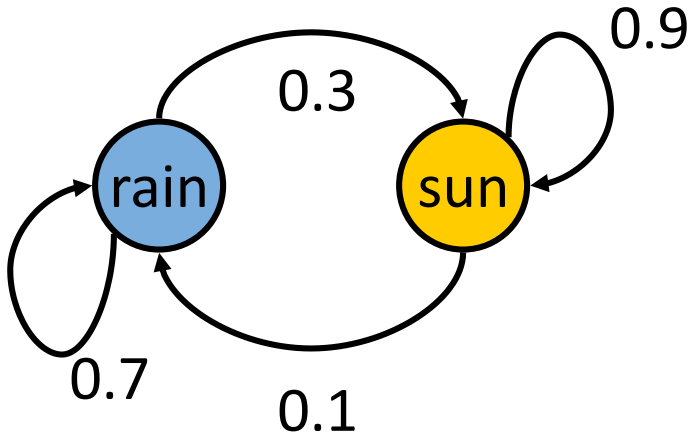
| $X_{t-1}$ | $X_t$ | $P(X_t \mid X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

Two new ways of representing the same CPT



# Example: Markov Chain Weather

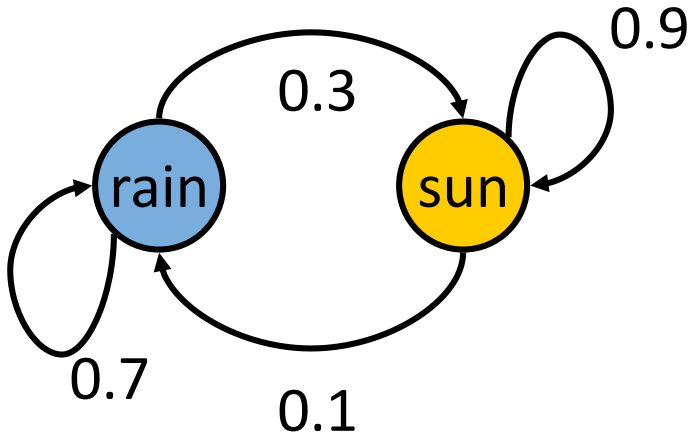



Initial distribution: $P(X_1 = sun) = 1.0$

What is the probability distribution after one step?

$P(X_2 = sun) = ?$

# Example: Markov Chain Weather




Initial distribution: $P(X_1 = sun) = 1.0$

What is the probability distribution after one step?

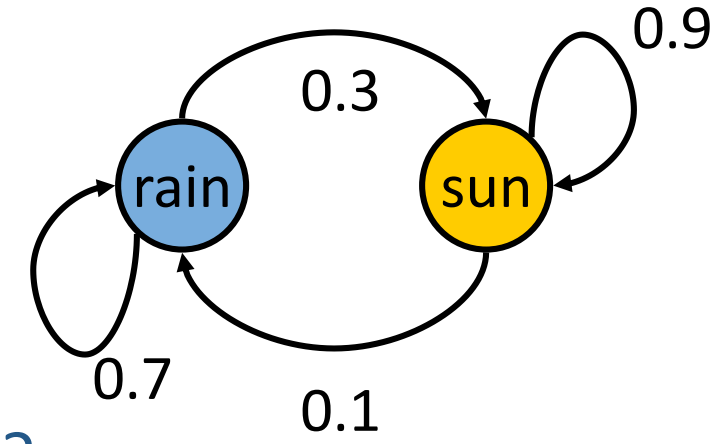
$P(X_2 = sun) = ?$

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Piazza Poll 3

Initial distribution: $P(X_2 = sun) = 0.9$

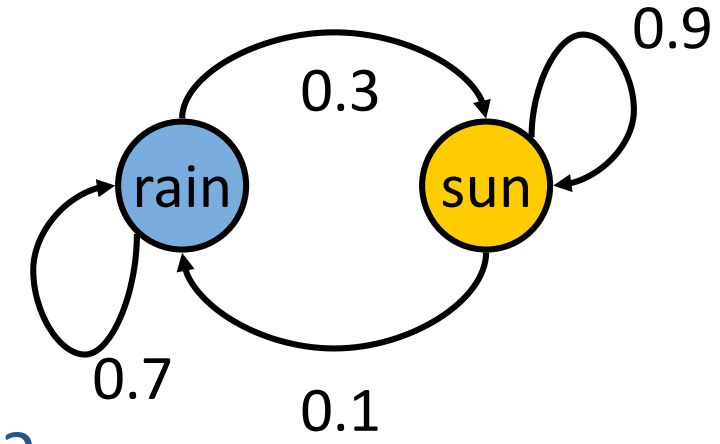



What is the probability distribution after the next step?

$P(X_3 = sun) = ?$

A) 0.81
B) 0.84
C) 0.9
D) 1.0
E) 1.2

# Piazza Poll 3




Initial distribution: $P(X_2 = sun) = 0.9$

What is the probability distribution after the next step?

$P(X_3 = sun) = ?$

A) 0.81
B) 0.84
C) 0.9
D) 1.0
E) 1.2

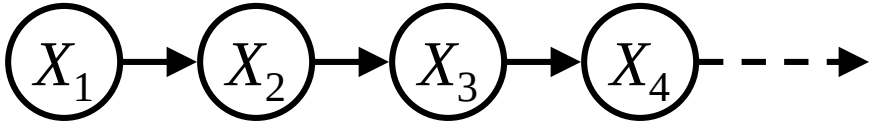
$$P(X_3 = sun) = \sum_{x_2} P(X_3 = sun, X_2 = x_2)$$

$$= \sum_{x_2} P(X_3 = sun \mid X_2 = x_2)\, P(X_2 = x_2)$$

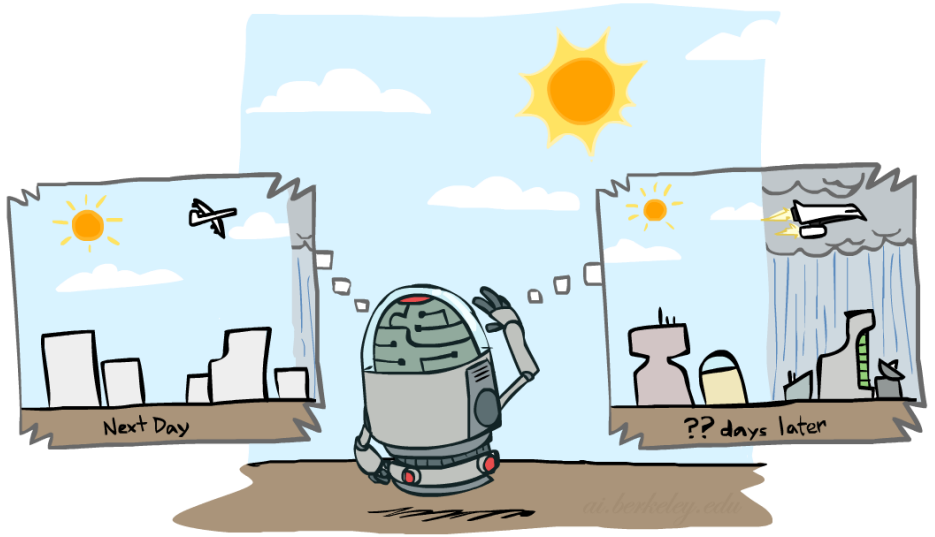
$$= 0.9 \cdot 0.9 + 0.3 \cdot 0.1$$

$$= 0.81 + 0.03 = 0.84$$

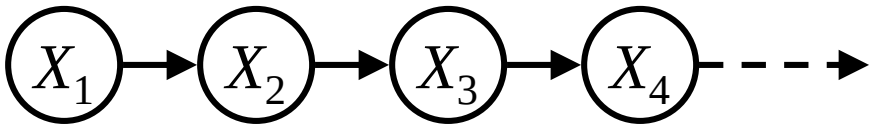
# Markov Chain Inference




If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.



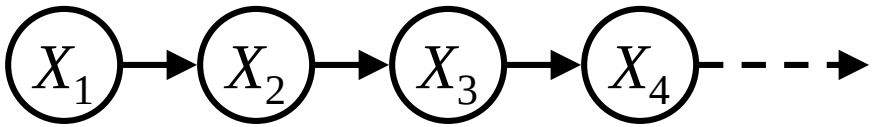
# Markov Chain Inference




If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.

$$P(X_5) = \sum_{x_4} P(x_4, X_5)$$
$$= \sum_{x_4} P(X_5 \mid x_4) P(x_4)$$

# Markov Chain Inference

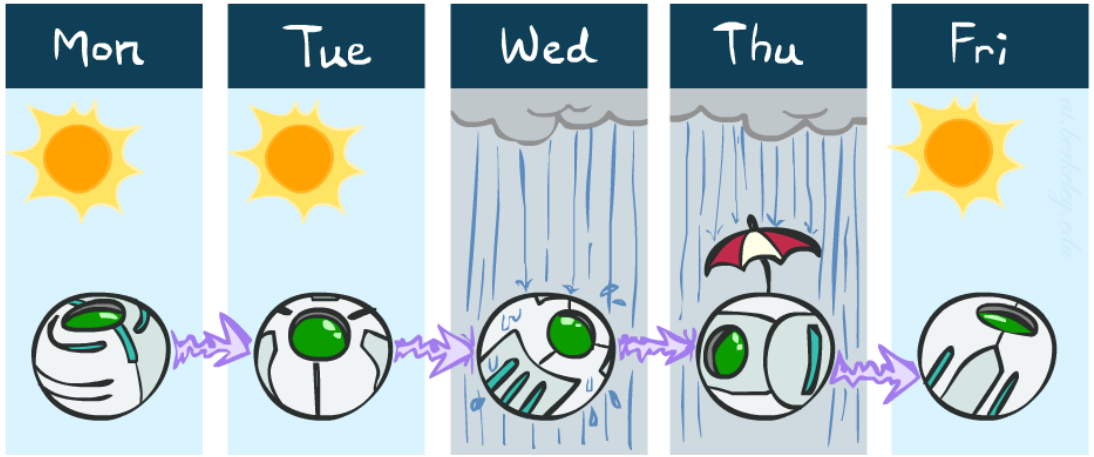



If you know the transition probabilities, $P(X_t \mid X_{t-1})$, and you know $P(X_4)$, write an equation to compute $P(X_5)$.

$$
\begin{aligned}
P(X_5) &= \textstyle\sum_{x_1,x_2,x_3,x_4} P(x_1, x_2, x_3, x_4, X_5) \\
&= \textstyle\sum_{x_1,x_2,x_3,x_4} P(X_5 \mid x_4) P(x_4 \mid x_3) P(x_3 \mid x_2) P(x_2 \mid x_1) P(x_1) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) \sum_{x_1,x_2,x_3} P(x_4 \mid x_3) P(x_3 \mid x_2) P(x_2 \mid x_1) P(x_1) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) \sum_{x_1,x_2,x_3} P(x_1, x_2, x_3, x_4) \\
&= \textstyle\sum_{x_4} P(X_5 \mid x_4) P(x_4)
\end{aligned}
$$

# Weather prediction

States {rain, sun}



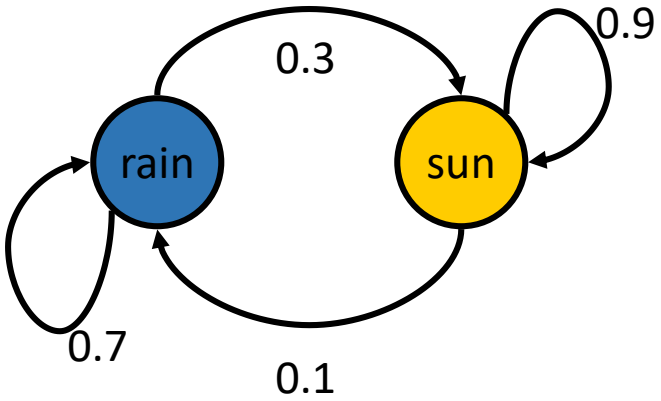

- Initial distribution $P(X_0)$

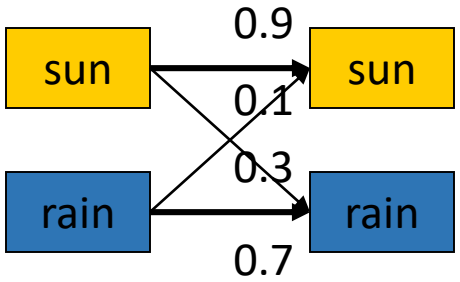
| $P(X_0)$ | |
|---|---|
| sun | rain |
| 0.5 | 0.5 |

- Transition model $P(X_t \mid X_{t-1})$

| $X_{t-1}$ | $P(X_t \mid X_{t-1})$ | |
|---|---|---|
| | sun | rain |
| sun | 0.9 | 0.1 |
| rain | 0.3 | 0.7 |

Two new ways of representing the same CPT

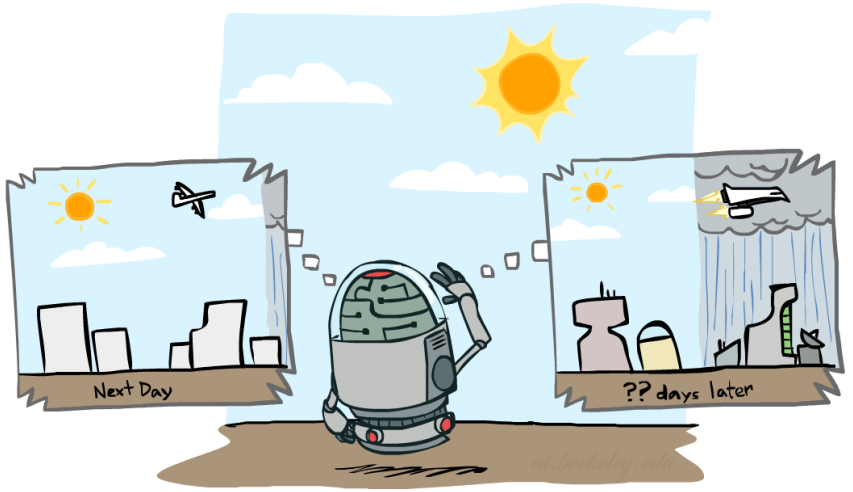


# Weather prediction

Time 0: $P(X_0)$ =<0.5,0.5>

| $X_{t-1}$ | $P(X_t \mid X_{t-1})$ | |
|---|---|---|
| | sun | rain |
| sun | 0.9 | 0.1 |
| rain | 0.3 | 0.7 |



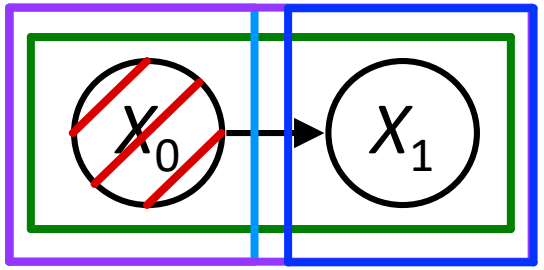

What is the weather like at time 1?

$$P(X_1) = \sum_{x_0} P(X_0=x_0, X_1)$$

$$= \sum_{x_0} P(X_1 \mid X_0=x_0)\, P(X_0=x_0)$$

$$= 0.5\langle 0.9,0.1\rangle + 0.5\langle 0.3,0.7\rangle$$

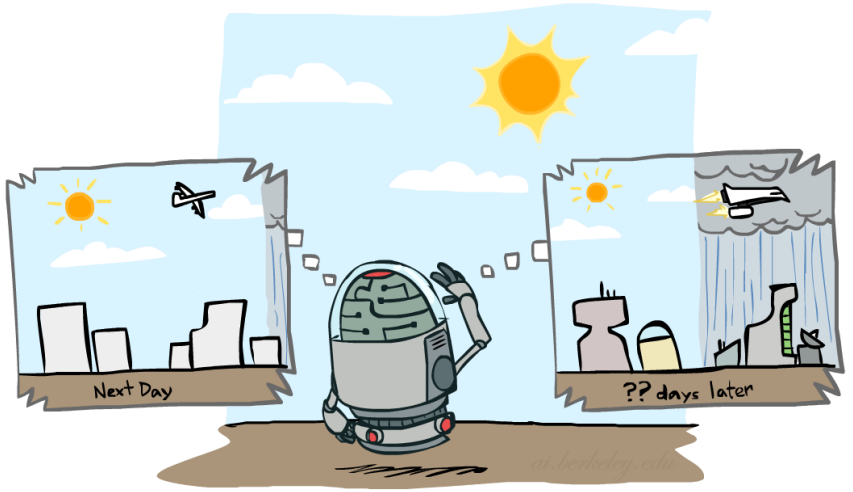
$$= \langle 0.6,0.4\rangle$$



# Weather prediction, contd.

Time 1: $P(X_1)$ =<0.6,0.4>

| $X_{t-1}$ | $P(X_t \mid X_{t-1})$ | |
|---|---|---|
| | sun | rain |
| sun | 0.9 | 0.1 |
| rain | 0.3 | 0.7 |



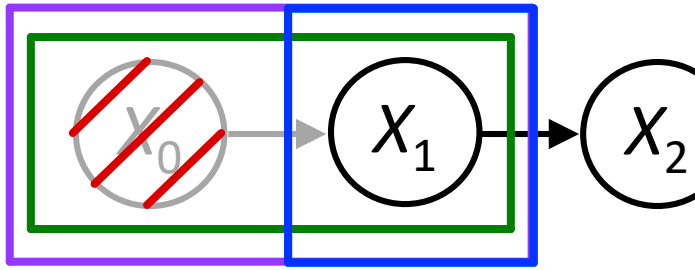

What is the weather like at time 2?

$$P(X_2) = \sum_{x_1} P(X_1=x_1, X_2)$$

$$= \sum_{x_1} P(X_2 \mid X_1=x_1)\, P(X_1=x_1)$$

$$= 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle$$

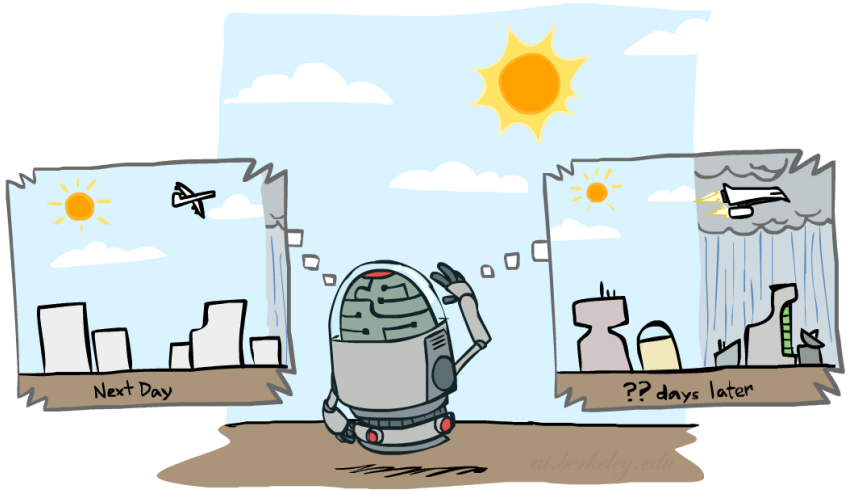
$$= \langle 0.66,0.34\rangle$$



# Weather prediction, contd.

Time 2: $P(X_2)$ =<0.66,0.34>

| $X_{t-1}$ | $P(X_t \mid X_{t-1})$ | |
|---|---|---|
| | sun | rain |
| sun | 0.9 | 0.1 |
| rain | 0.3 | 0.7 |



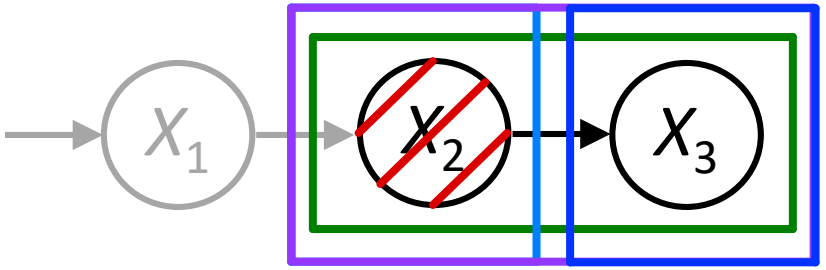

What is the weather like at time 3?

$$P(X_3) = \sum_{x_2} P(X_2=x_2, X_3)$$

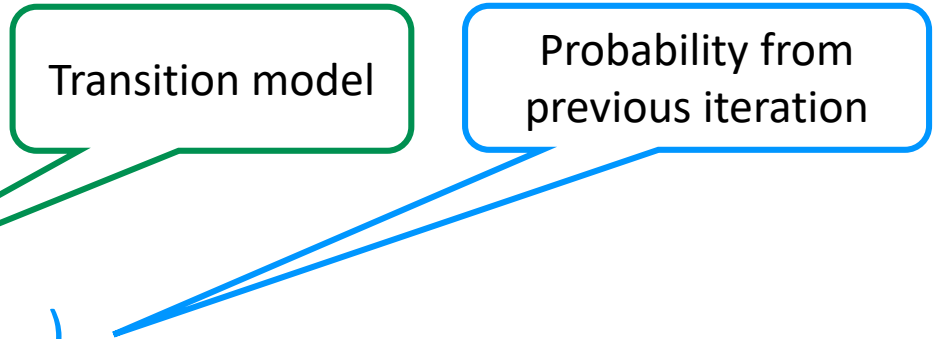
$$= \sum_{x_2} P(X_3 \mid X_2=x_2)\, P(X_2=x_2)$$

$$= 0.66\langle 0.9,0.1\rangle + 0.34\langle 0.3,0.7\rangle$$

$$= \langle 0.696,0.304\rangle$$



# Forward algorithm (simple form)




What is the state at time $t$?

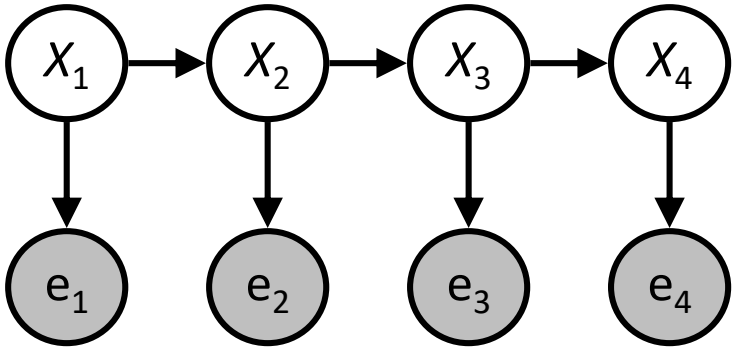
$$P(X_t) = \sum_{x_{t-1}} P(X_{t-1}=x_{t-1}, X_t)$$
$$= \sum_{x_{t-1}} P(X_t \mid X_{t-1}=x_{t-1})\, P(X_{t-1}=x_{t-1})$$

Iterate this update starting at $t$=0

# Hidden Markov Models

# HMM as a Bayes Net Warm-up

- For the following Bayes net, write the query $P(X_4 \mid e_{1:4})$ in terms of the conditional probability tables associated with the Bayes net.



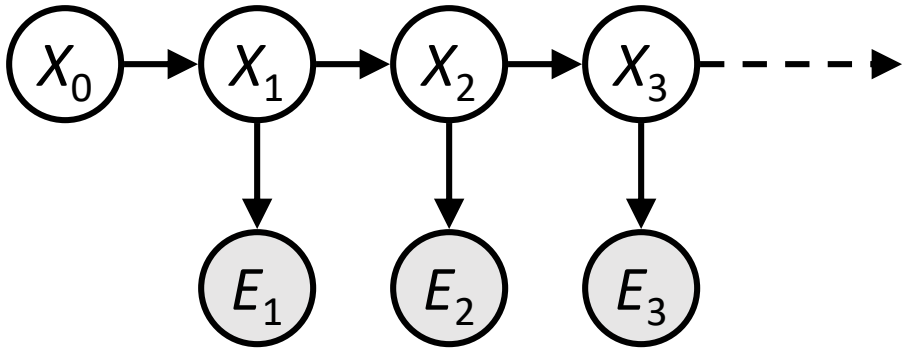

$P(X_4 \mid e_1, e_2, e_3, e_4) =$

# Hidden Markov Models

Usually the true state is not observed directly

## Hidden Markov models (HMMs)

- Underlying Markov chain over states $X$
- You observe evidence $E$ at each time step
- $X_t$ is a single discrete variable; $E_t$ may be continuous and may consist of several variables

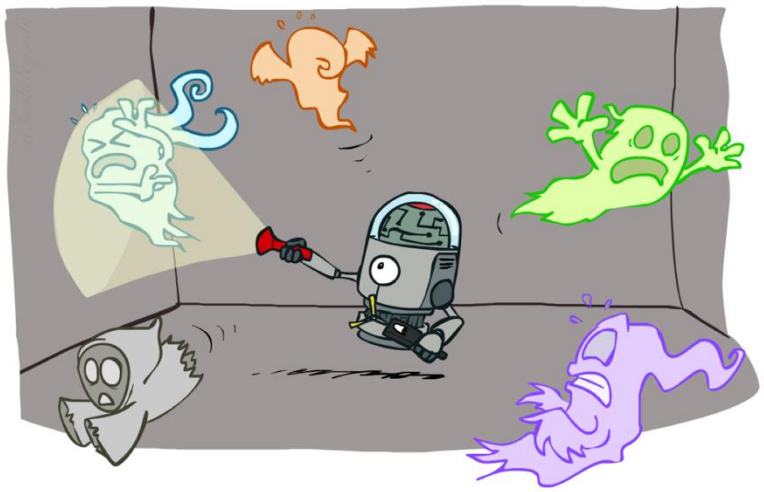


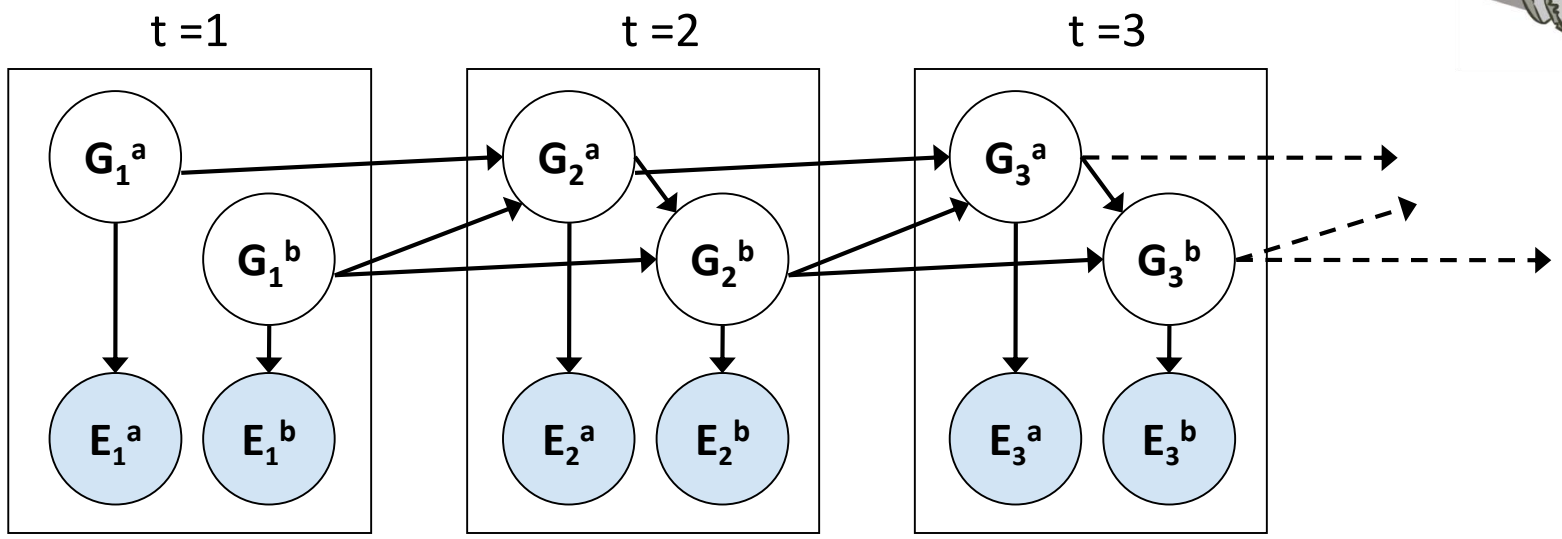

# Dynamic Bayes Nets (DBNs)

We want to track multiple variables over time, using multiple sources of evidence

Idea: Repeat a fixed Bayes net structure at each time

Variables from time *t* can condition on those from *t-1*

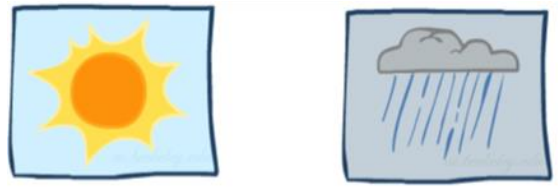




# Example: Weather HMM

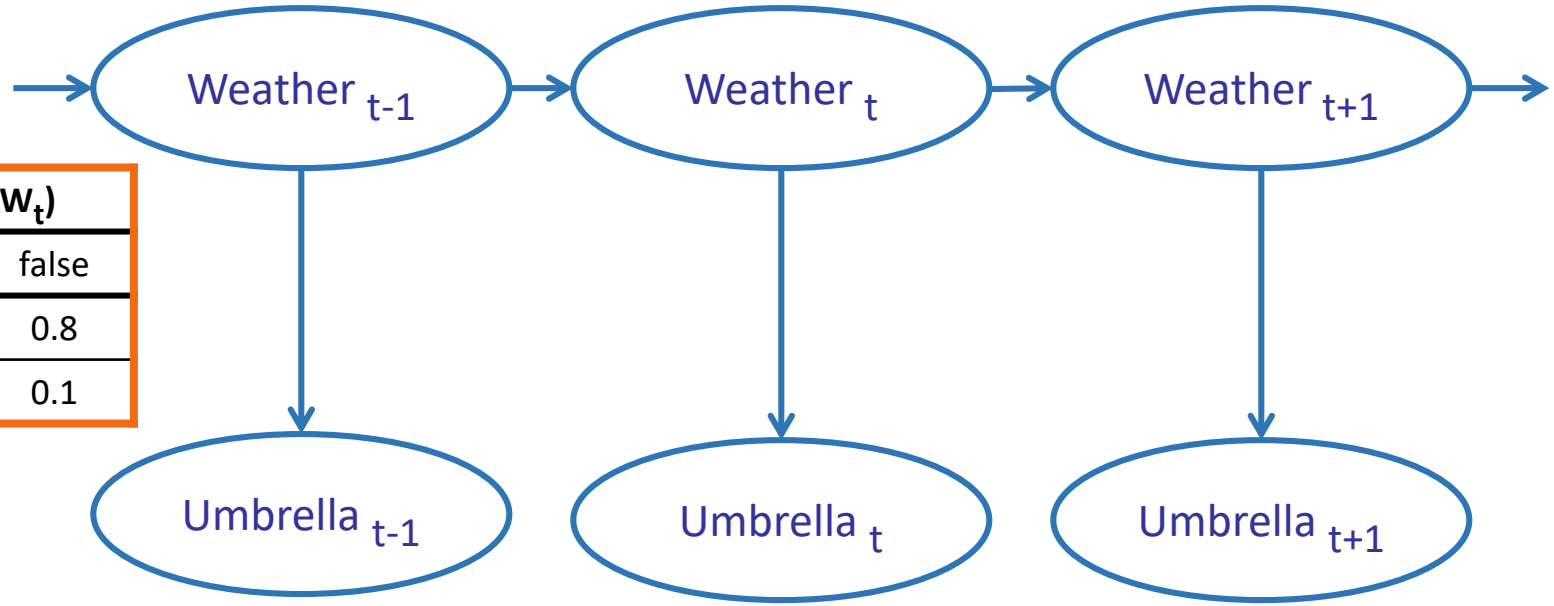
## An HMM is defined by:

- Initial distribution: $P(X_0)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$



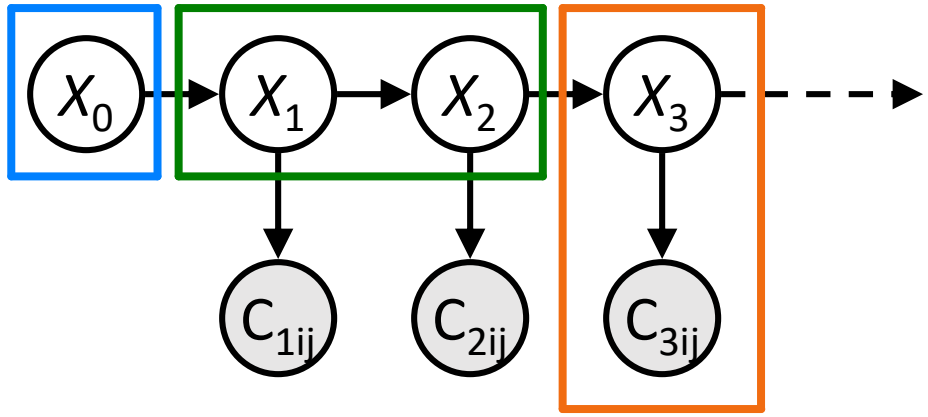
| $W_{t-1}$ | $P(W_t \mid W_{t-1})$ | |
|---|---|---|
| | sun | rain |
| sun | 0.9 | 0.1 |
| rain | 0.3 | 0.7 |

| $W_t$ | $P(U_t \mid W_t)$ | |
|---|---|---|
| | true | false |
| sun | 0.2 | 0.8 |
| rain | 0.9 | 0.1 |



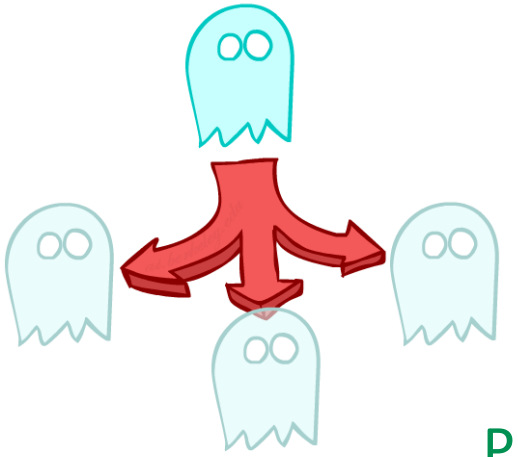
# Example: Ghostbusters HMM

- State: location of moving ghost
- Observations: Color recorded by ghost sensor at clicked squares
- $P(X_0)$ = uniform
- $P(X_t \mid X_{t-1})$ = usually move clockwise, but sometimes move randomly or stay in place
- $P(C_{tij} \mid X_t)$ = same sensor model as before: red means close, green means far away.



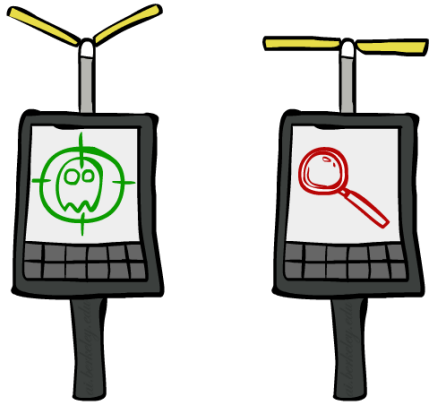

| | | |
|---|---|---|
| 1/9 | 1/9 | 1/9 |
| 1/9 | 1/9 | 1/9 |
| 1/9 | 1/9 | 1/9 |

$P(X_1)$



| | | |
|---|---|---|
| 1/6 | 1/6 | 1/2 |
| 0 | 1/6 | 0 |
| 0 | 0 | 0 |

$P(X_2 \mid X_1=(2,3))$



[Demo: Ghostbusters – Circular Dynamics – HMM (L14D2)]

# HMM as Probability Model

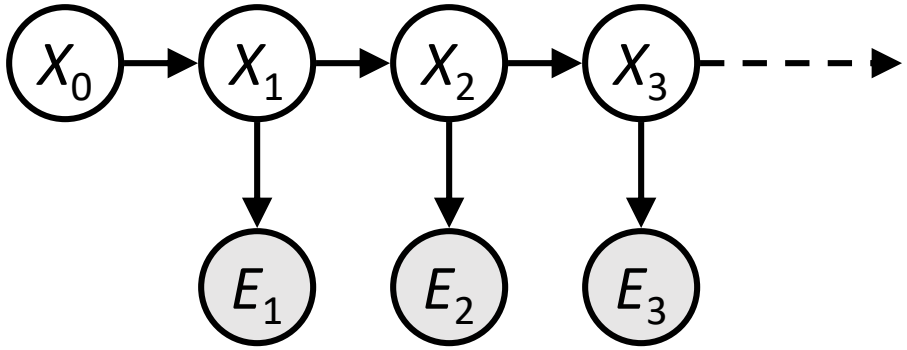
- Joint distribution for Markov model:

$$P(X_0, \ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$$

- Joint distribution for hidden Markov model:

$$P(X_0, X_1, E_1, \ldots, X_T, E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$$

- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?




Useful notation: $X_{a:b} = X_a, X_{a+1}, \ldots, X_b$

For example: $P(X_{1:2} \mid e_{1:3}) = P(X_1, X_2, \mid e_1, e_2, e_3)$

# Real HMM Examples

## Speech recognition HMMs:

- Observations are acoustic signals (continuous valued)
- States are specific positions in specific words (so, tens of thousands)

## Machine translation HMMs:

- Observations are words (tens of thousands)
- States are translation options

## Robot tracking:

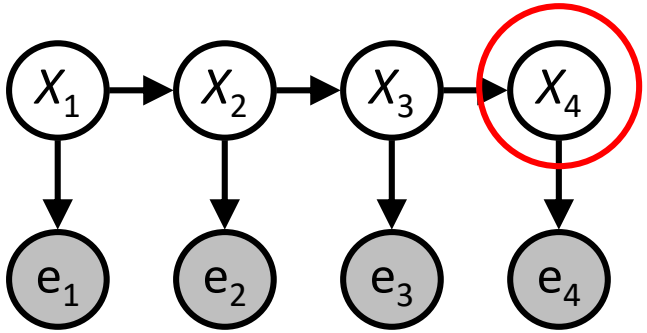
- Observations are range readings (continuous)
- States are positions on a map (continuous)

## Molecular biology:

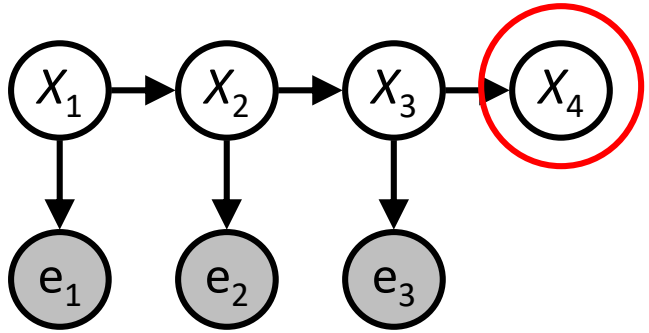
- Observations are nucleotides ACGT
- States are coding/non-coding/start/stop/splice-site etc.

# Other HMM Queries

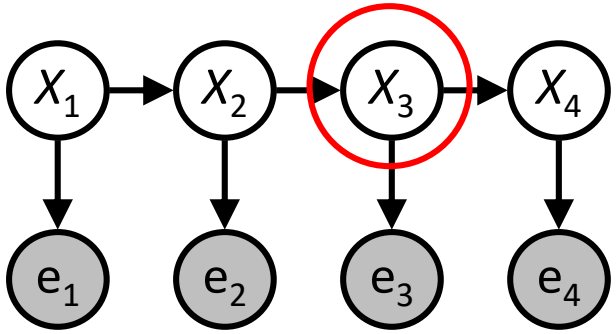
## Filtering: $P(X_t|e_{1:t})$



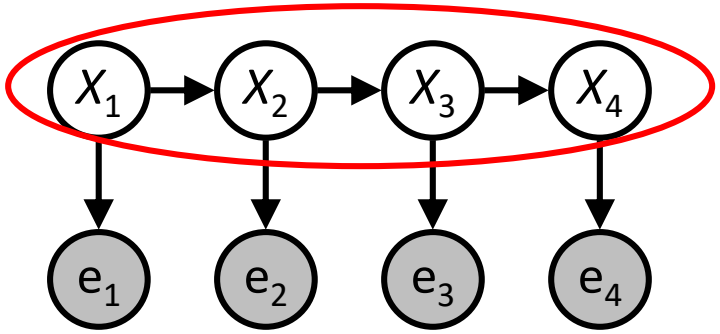

## Prediction: $P(X_{t+k}|e_{1:t})$




## Smoothing: $P(X_k|e_{1:t})$, $k<t$




## Explanation: $P(X_{1:t}|e_{1:t})$



# Inference Tasks

Filtering: $P(X_t | e_{1:t})$

- belief state—input to the decision process of a rational agent

Prediction: $P(X_{t+k} | e_{1:t})$ for $k > 0$

- evaluation of possible action sequences; like filtering without the evidence

Smoothing: $P(X_k | e_{1:t})$ for $0 \leq k < t$

- better estimate of past states, essential for learning

Most likely explanation: $\text{argmax}_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$

- speech recognition, decoding with a noisy channel

# Pacman – Hunting Invisible Ghosts with Sonar


[Demo: Pacman – Sonar – No Beliefs(L14D1)]

# Filtering **Algorithm**

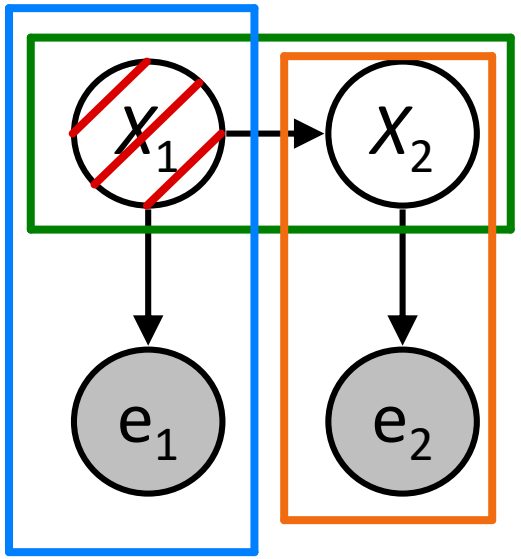
$$P(X_{t+1} \mid e_{1:t+1}) = \alpha \underbrace{P(e_{t+1} \mid X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(X_{t+1} \mid x_t)}_{\text{Normalize}} \underbrace{P(x_t \mid e_{1:t})}_{\text{Predict}}$$

$$\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$$

# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

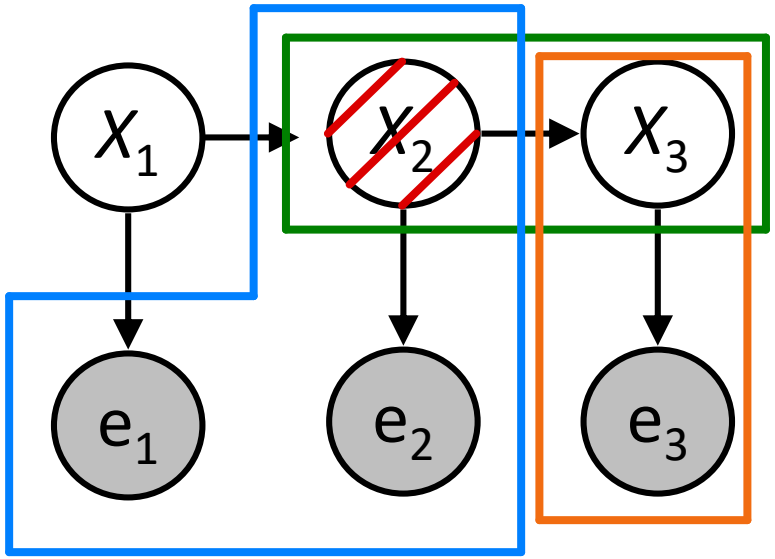
Marching **forward** through the HMM network



# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

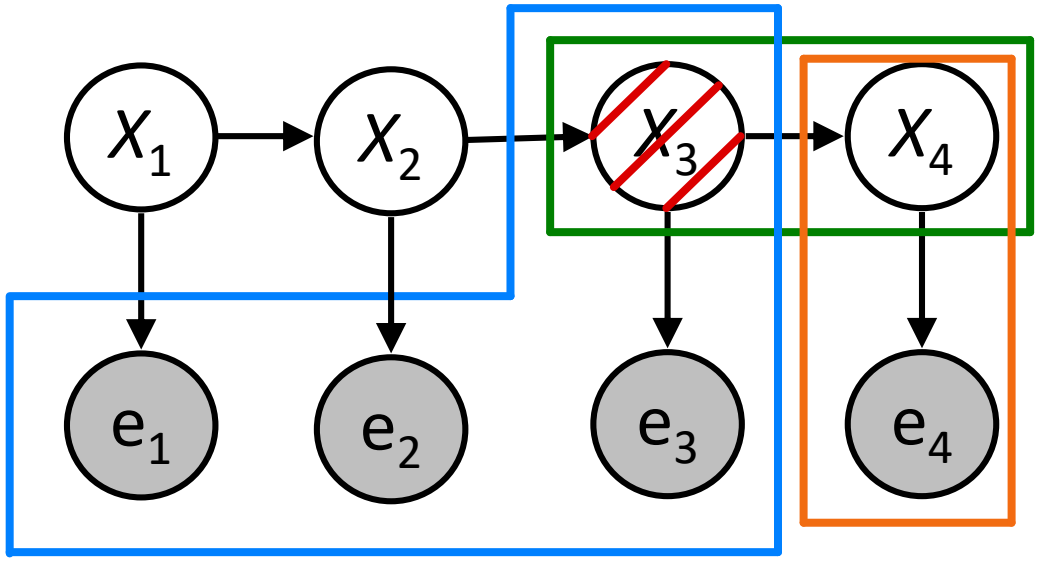
Marching **forward** through the HMM network



# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

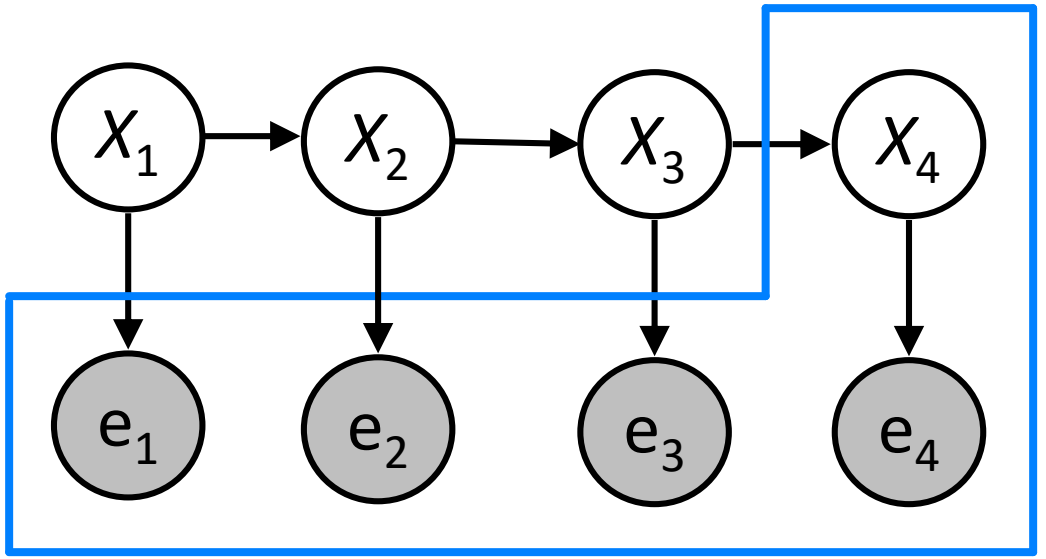
Marching **forward** through the HMM network



# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Marching **forward** through the HMM network



# Example: Prediction step

As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 1.00 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

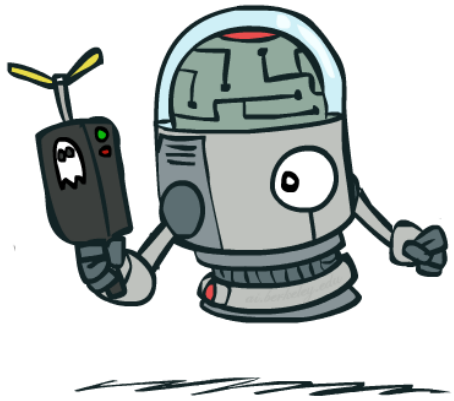
T = 1

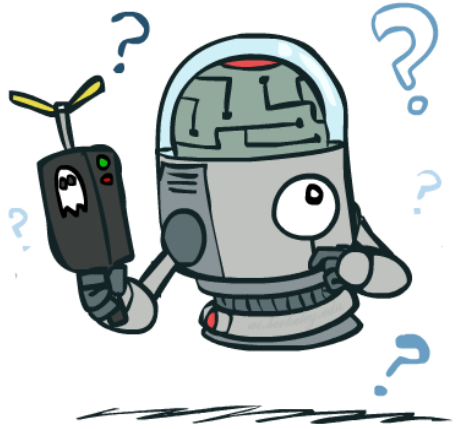
| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |
| <0.01 | 0.76 | 0.06 | 0.06 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |

T = 2

| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

T = 5





# Example: Update step

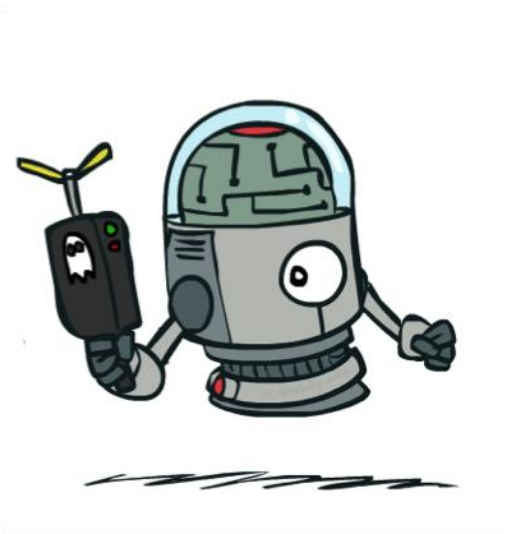
As we get observations, beliefs get reweighted, uncertainty “decreases”

| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

Before observation

| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | 0.02 | <0.01 |
| <0.01 | <0.01 | <0.01 | 0.83 | 0.02 | <0.01 |
| <0.01 | <0.01 | 0.11 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

After observation



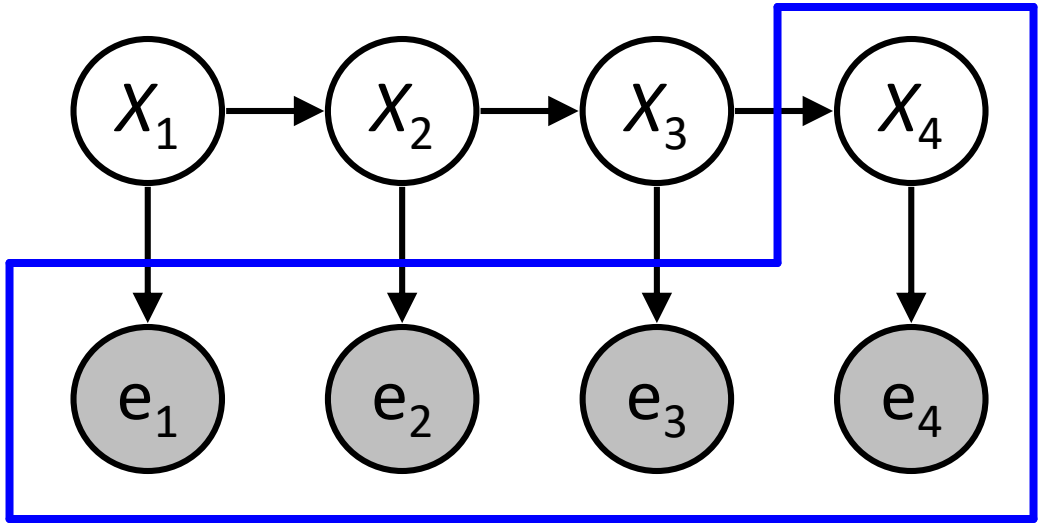
# Demo Ghostbusters – Circular Dynamics -- HMM

# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$
$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$




Def. of cond. probability with $X_t$, $e_t$

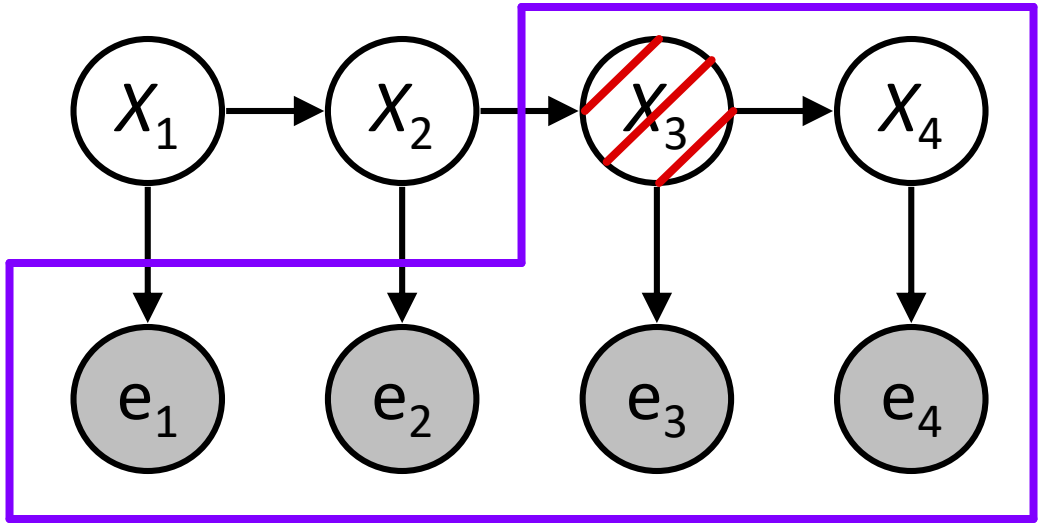
# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$
$$= \alpha\, P(X_t, e_t \mid e_{1:t-1})$$
$$= \alpha \sum_{x_{t-1}} P(x_{t-1}, X_t, e_t \mid e_{1:t-1})$$

Summation over variable $X_{t-1}$



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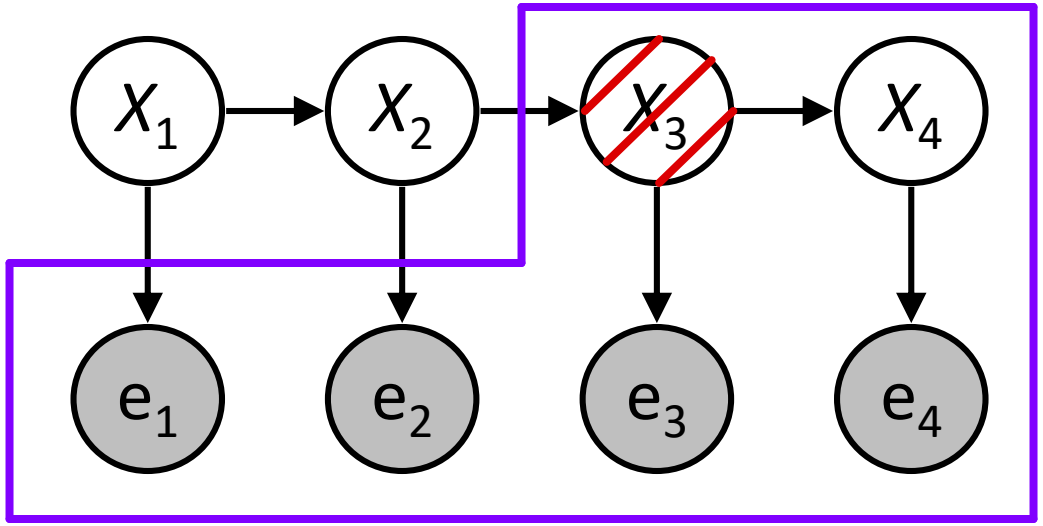


Chain rule with $x_{t-1}$, $X_t$, and $e_t$

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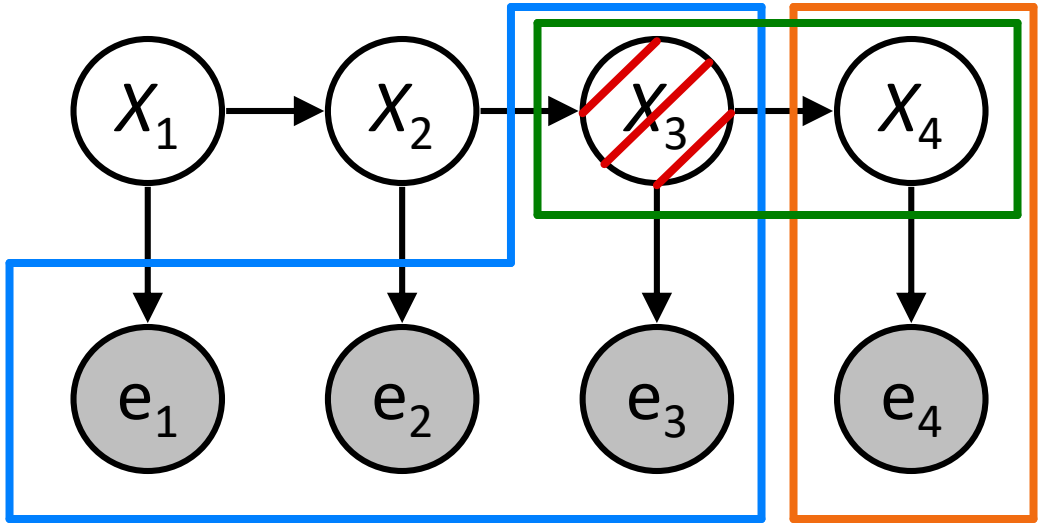
Chain rule with $x_{t-1}$, $X_t$, and $e_t$



# Filtering **Algorithm**

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$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$
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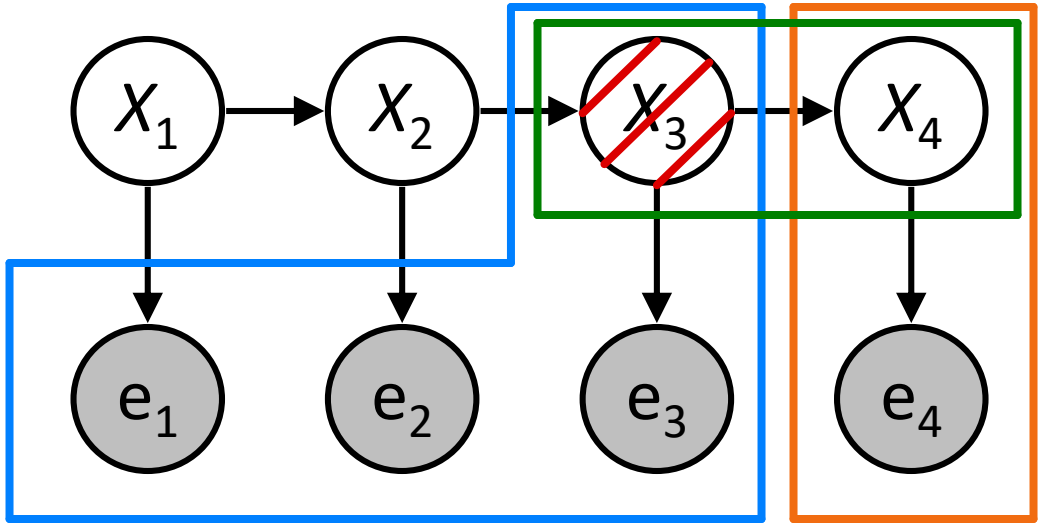


Bayes net conditional independence

# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

$$P(X_t \mid e_{1:t}) = P(X_t \mid e_t, e_{1:t-1})$$

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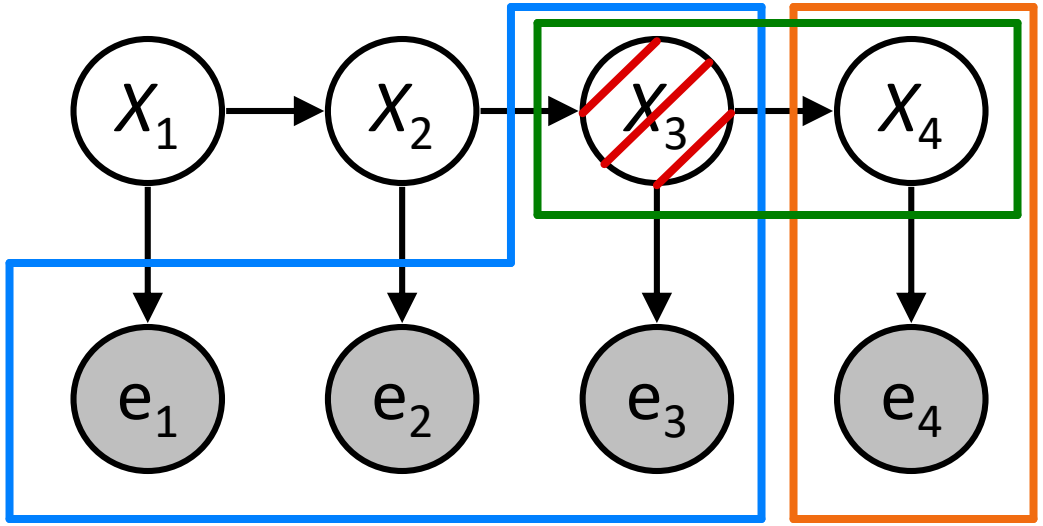
$$= \alpha \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1})\, P(X_t \mid x_{t-1})\, P(e_t \mid X_t)$$

$$= \alpha\, P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1})\, P(x_{t-1} \mid e_{1:t-1})$$

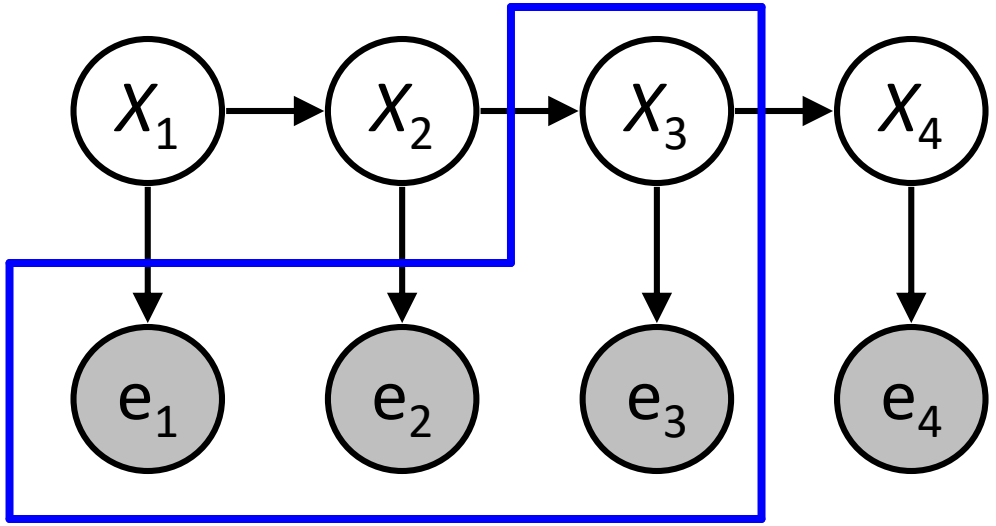



Pulling $P(e_t|X_t)$ out of the summation

# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

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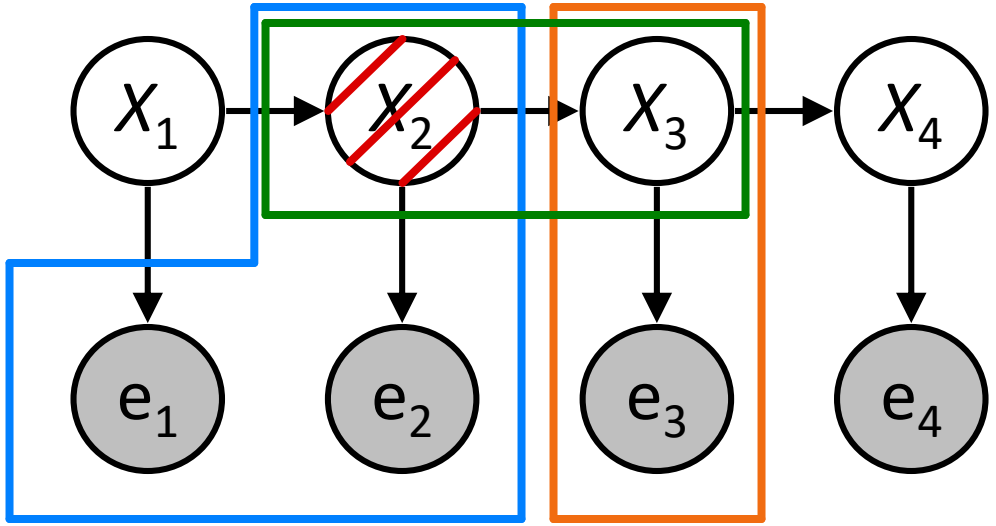
*Recursion!*

# Filtering **Algorithm**

Query: What is the current state, given all of the current and past evidence?

Matching math with Bayes net

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*Recursion!*

# Filtering **Algorithm**

$$P(X_{t+1} | e_{1:t+1}) = \alpha \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(X_{t+1} | x_t)}_{\text{Predict}} \underbrace{P(x_t | e_{1:t})}_{\text{Normalize}}$$

$\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$

Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states

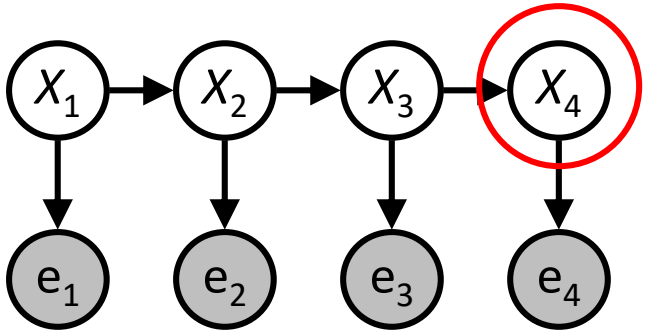
Time and space costs are ***constant***, independent of t

$O(|X|^2)$ is infeasible for models with many state variables

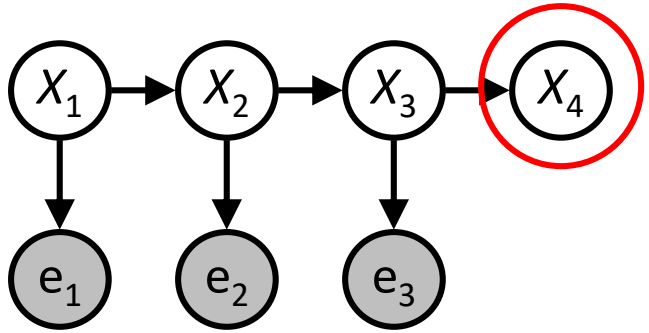
We get to invent really cool approximate filtering algorithms

# Other HMM Queries

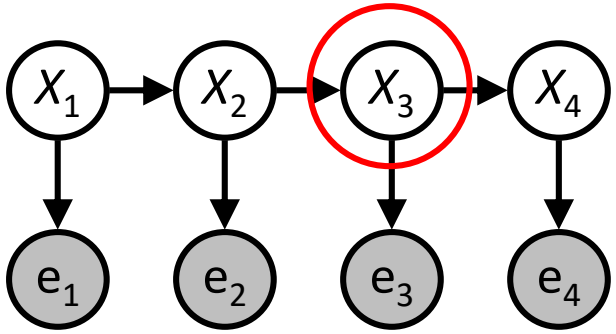
## Filtering: $P(X_t | e_{1:t})$



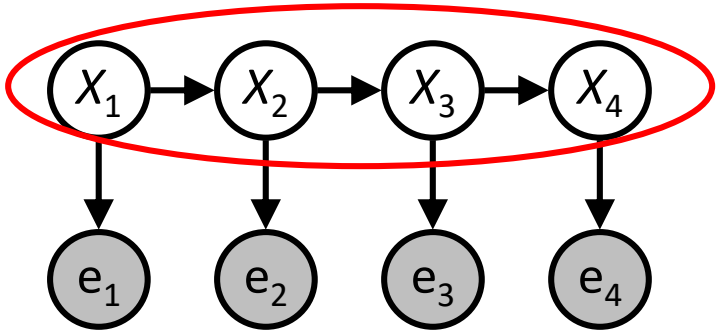

## Prediction: $P(X_t | e_{1:t-1})$




## Smoothing: $P(X_t | e_{1:N})$, $t<N$




## Explanation: $P(X_{1:N} | e_{1:N})$



# Next Time: Particle Filtering and Applications of HMMs