Announcements

Midterm1

- Grade Released

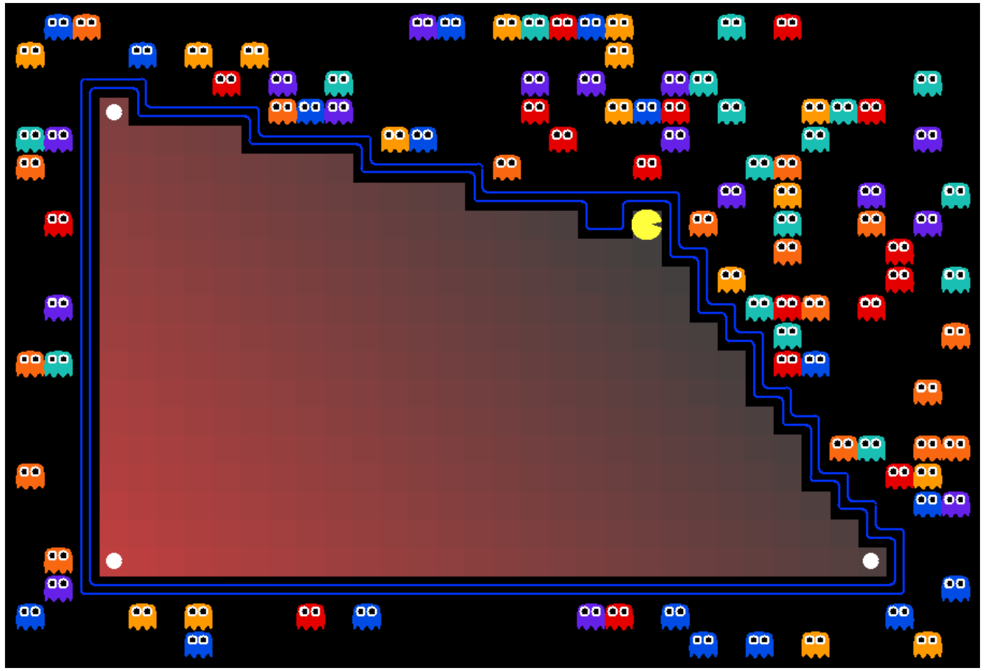
Assignments:

- HW5 (written)
  - Due 10/8 Tue, 10 pm
- HW6 (online)
  - Will be released today, Due 10/15 Tue, 10 pm
- P3: Optimization – expected average completion time < P2
  - Due 10/17 Thu, 10 pm

Final exam: Thursday, Dec 12, 1-4pm, location TBD

AI: Representation and Problem Solving

Integer Programming



Instructors: Fei Fang & Pat Virtue

Slide credits: CMU AI, http://ai.berkeley.edu

Learning Objectives

- Formulate a problem as a Integer (Linear) Program (IP or ILP)
- Write down the Linear Program (LP) relaxation of an IP
- Plot the graphical representation of an IP and find the optimal solution
- Understand the relationship between optimal solution of an IP and the optimal solution of the relaxed LP
- Describe and implement branch-and-bound algorithm

Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

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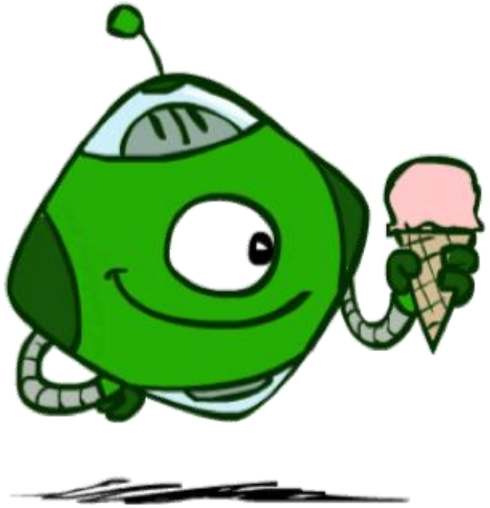
What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation

Formulate Diet Problem with integer constraints as an optimization problem

$$
\begin{aligned}
\min_{x_1, x_2} \quad & 1\,x_1 + 0.5\,x_2 \\
\text{s.t.} \quad & 100\,x_1 + 50\,x_2 \geq 2000 \\
& 100\,x_1 + 50\,x_2 \leq 2500 \\
& 3\,x_1 + 4\,x_2 \leq 100 \\
& 20\,x_1 + 70\,x_2 \geq 700 \\
& x_1, x_2 \in \mathbb{Z}^N
\end{aligned}
$$



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Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned} \qquad\qquad \begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming (BIP)
- A hybrid: Mixed Integer Linear Programming (MIP or MILP)

Notation Alert!

Integer Programming: Graphical Representation

Just add a grid of integer points onto our LP representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$
$$\boldsymbol{x} \in \mathbb{Z}^N$$

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$
$$\text{s.t.} \quad 100\, x_1 + 50\, x_2 \geq 2000$$
$$100\, x_1 + 50\, x_2 \leq 2500$$
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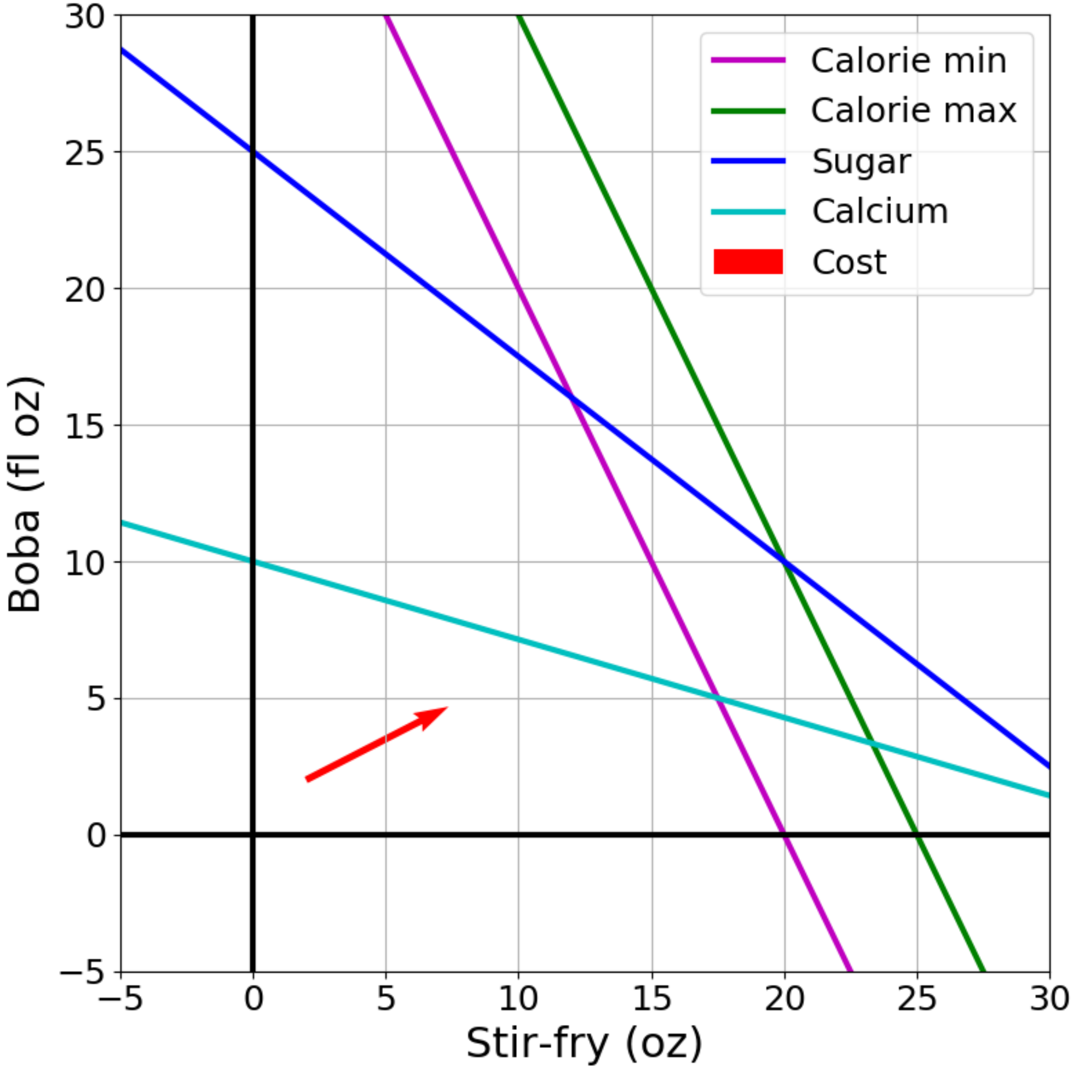
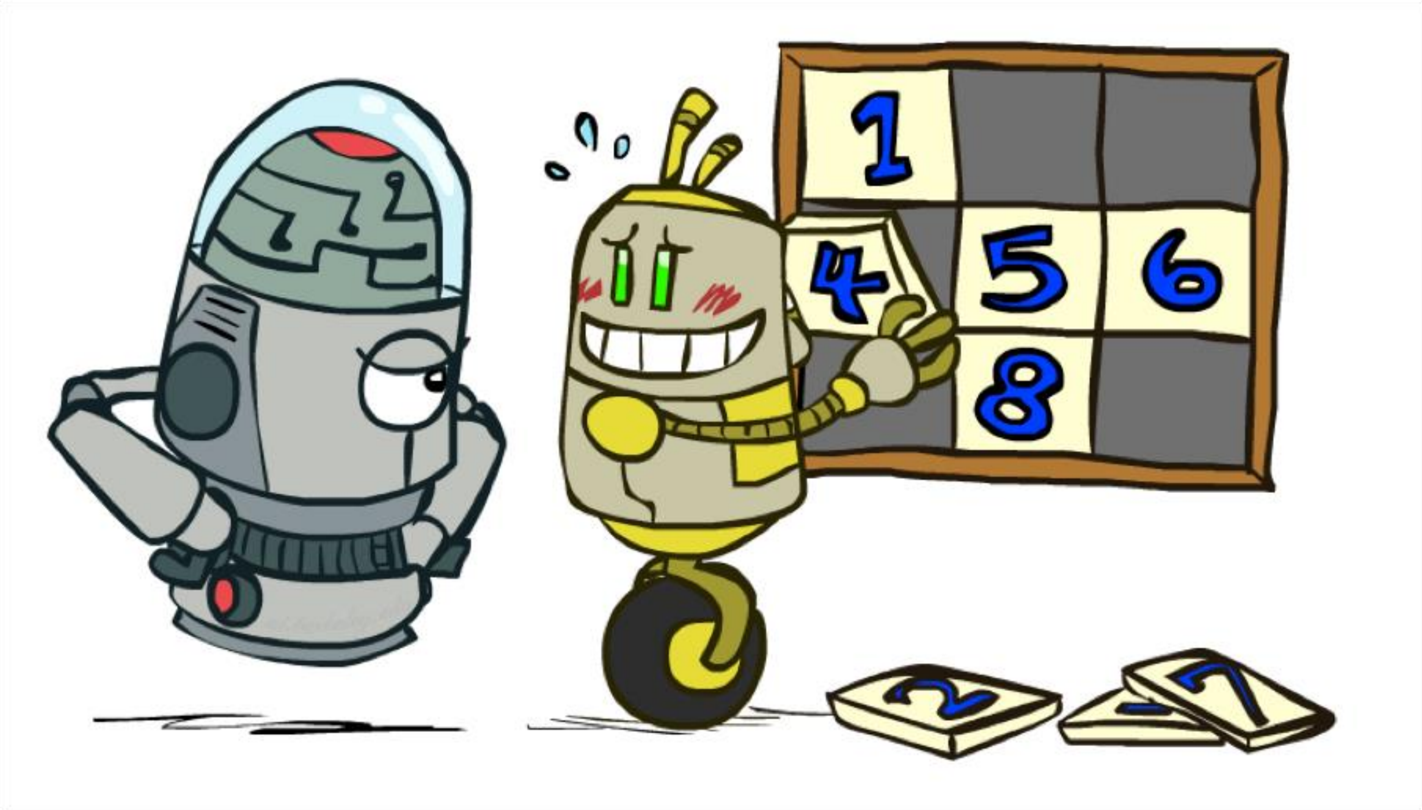


LP Relaxation

Relax IP to LP by dropping integer constraints

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\ & \cancel{\boldsymbol{x} \in \mathbb{Z}^N} \end{aligned}$$

Remember heuristics?



Question

Let y_{IP}^* be the optimal objective of an integer program P.

Let $\boldsymbol{x}_{IP}^*$ be an optimal point of an integer program P.

Let $\boldsymbol{x}_0$ be a feasible point of P.

Let y_0 be the objective value of P at $\boldsymbol{x}_0$

Assume that P is a minimization problem.

Which of the following are true?

A) $\boldsymbol{x}_{IP}^* = \boldsymbol{x}_0$

B) $y_{IP}^* \leq y_0$

C) $y_{IP}^* \geq y_0$

$$
\begin{aligned}
y_{IP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \\
& \boldsymbol{x} \in \mathbb{Z}^N
\end{aligned}
$$

$$
\begin{aligned}
y_{LP}^* = \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\
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\end{aligned}
$$

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$$\begin{aligned} y_{LP}^* = \min_{\boldsymbol{x}}.\quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.}\quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Which of the following are true?

A) $\boldsymbol{x}_{IP}^* = \boldsymbol{x}_0$

B) $y_{IP}^* \le y_0$

C) $y_{IP}^* \ge y_0$

Upper bound

$$y_{LP}^* \le y_{IP}^* \le y_0$$

Lower bound

Solving an IP

Basic Branch and Bound algorithm (essentially a search algorithm)

Assuming a minimization IP

1. Build the root node, which is the original IP. Set $\boldsymbol{x}_{IP}^*$=null, $y_{IP}^* = +\infty$
2. Solve the relaxed LP of the node
3. If relaxed LP is feasible, get solution $\boldsymbol{x}_{LP}^*$ and optimal objective value y_{LP}^*

   (Update) If integer($\boldsymbol{x}_{LP}^*$), update $\boldsymbol{x}_{IP}^*$=best($\boldsymbol{x}_{IP}^*, \boldsymbol{x}_{LP}^*$) and go to step 4

   (Prune) If $y_{LP}^* \geq y_{IP}^*$, go to step 4

   (Branch) Choose a variable x_i that has non-integer value in $\boldsymbol{x}_{LP}^*$, branch and construct two new nodes each representing a more constrained IP:

   New node at left branch: Add constraint $x_i \leq floor(x_i)$

   New node at right branch: Add constraint $x_i \geq ceil(x_i)$
4. (Recursion) Pick an unexplored node and go to step 2. Stop if all nodes explored.
5. Return $\boldsymbol{x}_{IP}^*$

What if it is a maximization IP?

Solving an IP

Basic Branch and Bound algorithm (essentially a search algorithm)

Assuming a minimization IP

(Upper bound of IP)

1. Build the root node, which is the original IP. Set $\boldsymbol{x}_{IP}^{*}$=null, $y_{IP}^{*} = +\infty$
2. Solve the relaxed LP of the node

   (Lower bound for the IP of the node)
3. If relaxed LP is feasible, get solution $\boldsymbol{x}_{LP}^{*}$ and optimal objective value y_{LP}^{*}

   (Update) If integer($\boldsymbol{x}_{LP}^{*}$), update $\boldsymbol{x}_{IP}^{*}$=best($\boldsymbol{x}_{IP}^{*}, \boldsymbol{x}_{LP}^{*}$) and go to step 4

   (Prune) If $y_{LP}^{*} \geq y_{IP}^{*}$, go to step 4 **(Update upper bound)**

   (Branch) Choose a variable x_i that has non-integer value in $\boldsymbol{x}_{LP}^{*}$, branch and construct two new nodes each representing a more constrained IP:

   New node at left branch: Add constraint $x_i \leq floor(x_i)$

   New node at right branch: Add constraint $x_i \geq ceil(x_i)$
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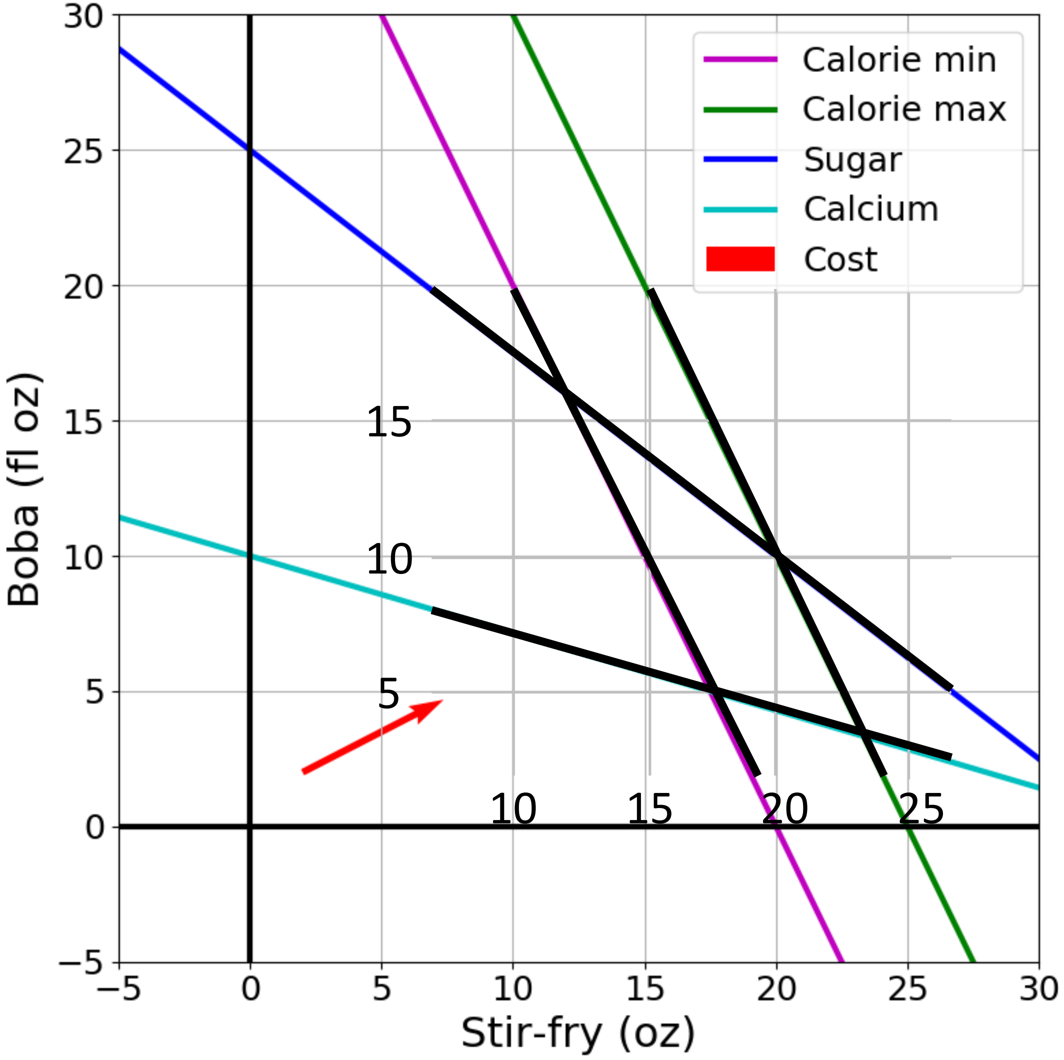
5. Return $\boldsymbol{x}_{IP}^{*}$

What if it is a maximization IP?

Branch and Bound Example



Branch and Bound Example

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\begin{aligned}
\text{s.t.} \quad & 100\,x_1 + 50\,x_2 \geq 2000 \\
& 100\,x_1 + 50\,x_2 \leq 2500 \\
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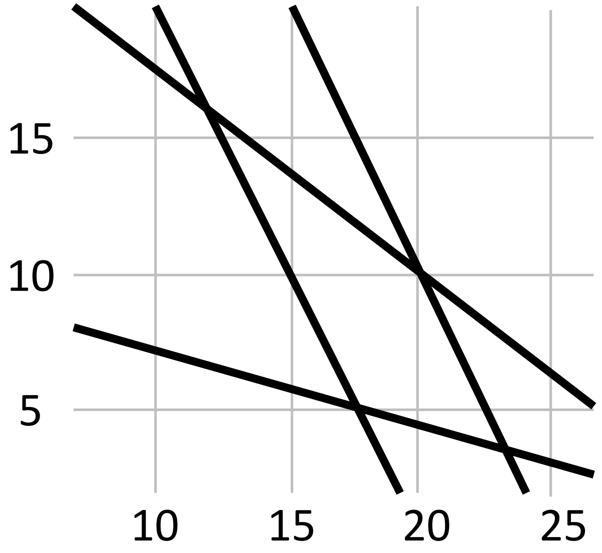
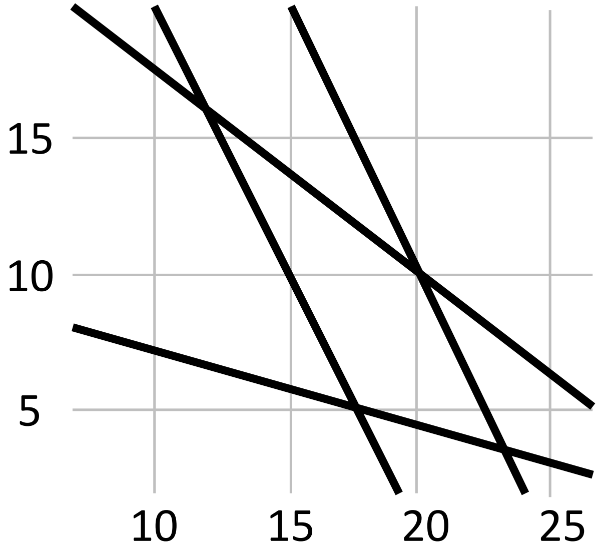
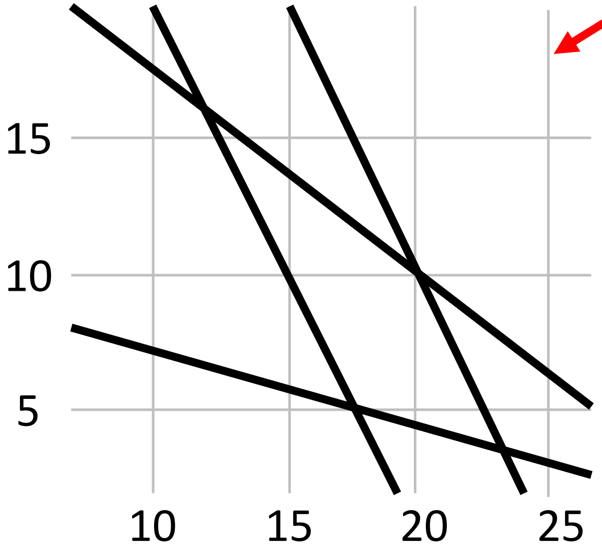




Is it necessary to solve this branch?



Branch and Bound Example

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

s.t.

$$100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
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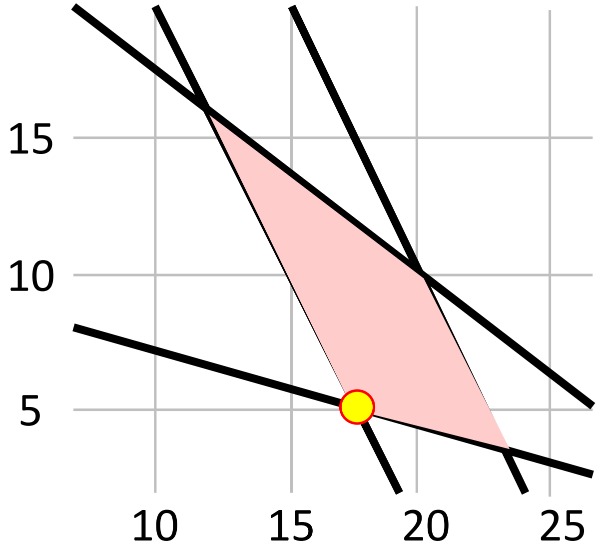
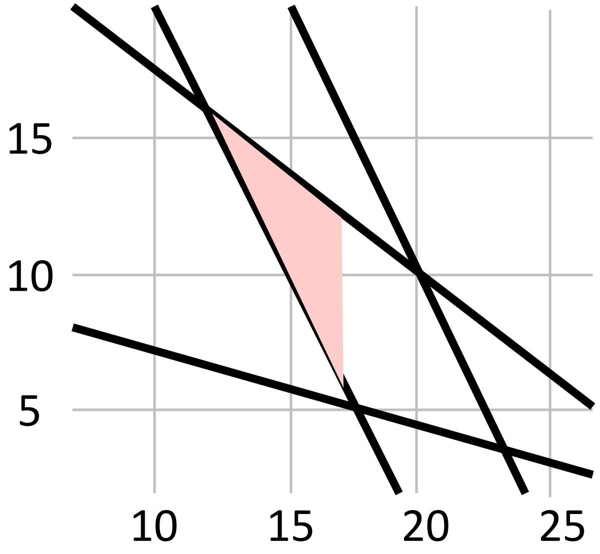
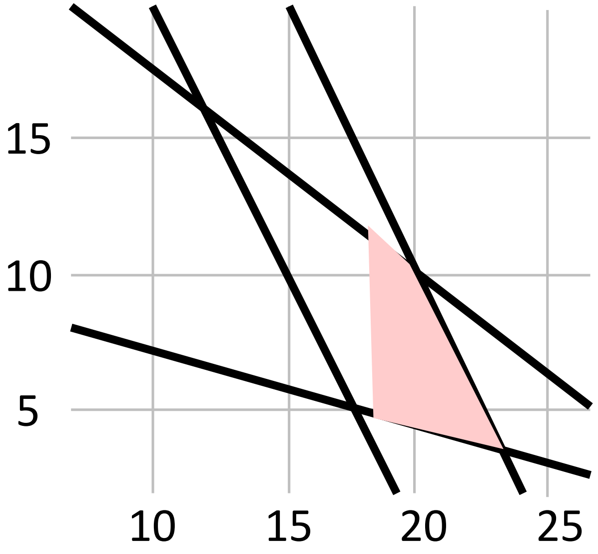


Optimal solution for LP relaxation: (17.5,5)
Optimal value (lower bound for IP): 20



Add constraint $x_1 \leq 17$
Optimal solution for relaxed LP: (17,6) Optimal solution
Optimal value (lower bound for this branch): 20
An integer solution found (update upper bound of IP)
Stop



Add $x_1 \geq 18$

Optimal solution for relaxed LP: (18,4.86)
Optimal value (lower bound for this branch): 20.43
Worse than upper bound
Stop

Solving an IP

Basic Branch and Bound algorithm (essentially a search algorithm)

Assuming a minimization IP

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3. If relaxed LP is feasible, get solution $\boldsymbol{x}_{LP}^*$ and optimal objective value y_{LP}^*

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   New node at right branch: Add constraint $x_i \geq ceil(x_i)$
4. (Recursion) Pick an unexplored node and go to step 2. Stop if all nodes explored.
5. Return $\boldsymbol{x}_{IP}^*$

Anything we can do to make the search more efficient?

Recall: Informed Search

```
function BEST-FIRST-SEARCH (problem, EVAL-FN) returns a solution sequence
   inputs: problem, a problem
                    EVAL-FN, an evaluation function

   Queuing-Fn ← a function that orders nodes by EVAL-FN
   return GENERAL-SEARCH (problem, Queuing-Fn)
```

```
function GREEDY-SEARCH (problem) returns a solution or failure
   return BEST-FIRST-SEARCH (problem, h)
```

Solving an IP

Basic Branch and Bound algorithm (essentially a search algorithm)

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4. (Recursion) Pick an unexplored node and go to step 2. Stop if all nodes explored.
5. Return $\boldsymbol{x}_{IP}^*$

What can be used as an evaluation function?

Optimal value of the LP relaxation!

Solving an IP

Improved Branch and Bound algorithm (essentially a best-first-search algorithm)

Assuming a minimization IP

1. Build the root node, which is the original IP.
2. Solve the relaxed LP of the node. **Return null if infeasible.**
3. Given solution $\boldsymbol{x}_{LP}^*$ and optimal objective value y_{LP}^* of the relaxed LP

   (Update) If integer($\boldsymbol{x}_{LP}^*$), **return $\boldsymbol{x}_{LP}^*$**

   (Branch) Choose a variable x_i that has non-integer value in $\boldsymbol{x}_{LP}^*$, branch and construct two new nodes each representing a more constrained IP:

   New node at left branch: Add constraint $x_i \leq floor(x_i)$

   New node at right branch: Add constraint $x_i \geq ceil(x_i)$
4. (Recursion) Pick an unexplored node **whose LP relaxation is feasible and has the best optimal objective value** and go to step 2. Stop if all nodes explored.
5. Return **null**

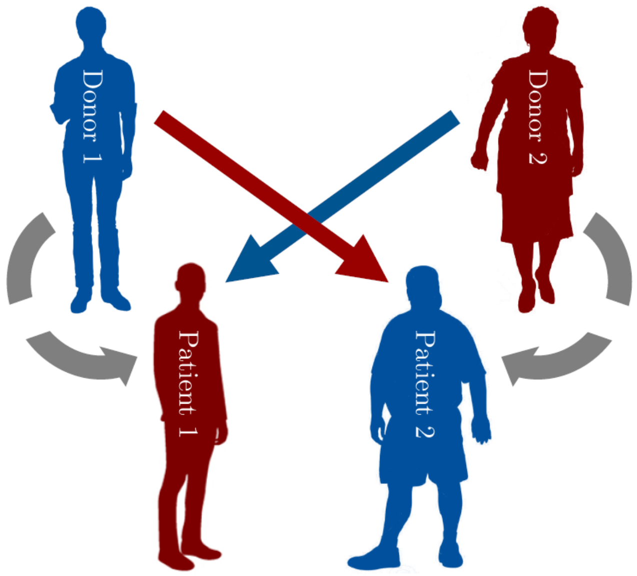
Why still optimal? All the unexplored nodes have no better LP relaxation values!

Formulate a Problem as an IP

Kidney Exchange

UNOS DONATE LIFE®

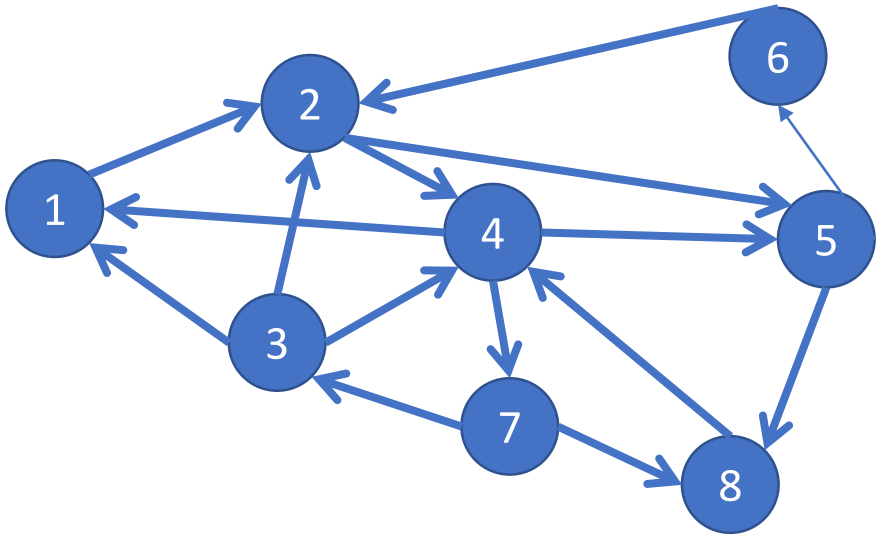
UNITED NETWORK FOR ORGAN SHARING



Formulate a Problem as an IP

Kidney Exchange

- Given directed graph $G = (V, E)$, where each node represent a patient-donor pair, and an edge $\langle u, v \rangle$ means donor of node u can give one kidney to patient of node v
- Find a set of disjoint cycles so as to maximize the number of nodes covered



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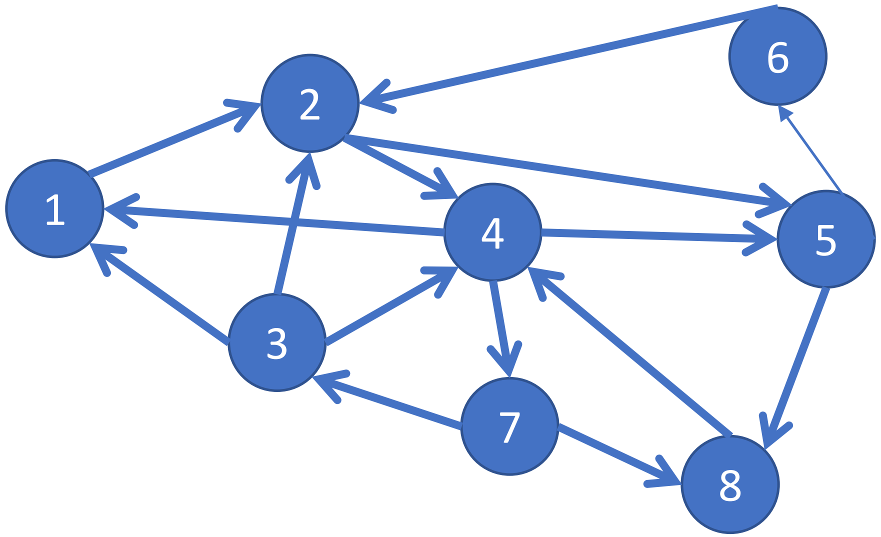
Hint: enumerate all the cycles



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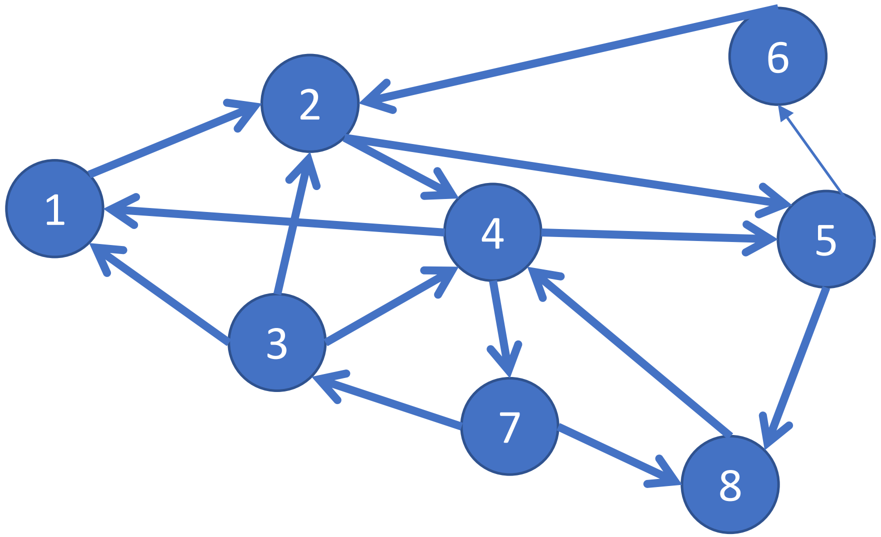


$$\max_x \sum_c x_c l_c$$

$$\text{s.t. } \sum_{c:v\in c} x_c \le 1, \forall v \in V$$

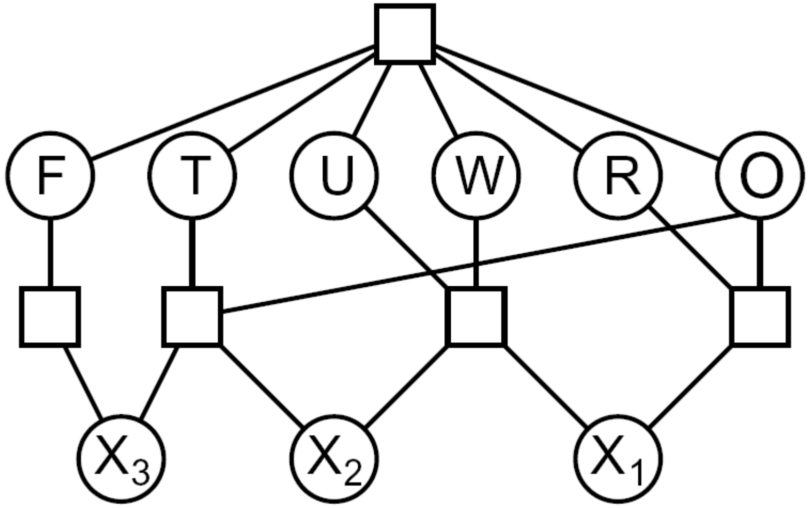
$$x_c \in \{0,1\}, \forall c$$

Formulate a Problem as an IP

Cryptarithmetic

Variables:

IP:



Formulate a Problem as an IP

Cryptarithmetic

Variables: $x_T, x_W, x_O, x_F, x_U, x_R, c_1, c_2, c_3$

IP:

$$\min_{x,c} 1$$

$$\text{s.t. } x_O + x_O = x_R + 10c_1$$
$$x_W + x_W + c_1 = x_U + 10c_2$$
$$x_T + x_T + c_2 = x_O + 10c_3$$
$$0 + c_3 = x_F$$
$$x_T, x_F \in \{1, .., 9\}$$
$$x_W, x_O, x_U, x_R \in \{0, .., 9\}$$
$$c_1, c_2, c_3 \in \{0,1\}$$

Can also write as: $x_T, x_F, x_W, x_O, x_U, x_R \leq 9$,
$x_T, x_F \geq 1$, $x_W, x_O, x_U, x_R \geq 0$,
$c_1, c_2, c_3 \leq 1$, $c_1, c_2, c_3 \geq 0$
$x, c \in \mathbb{Z}^N$

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  T W O
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