

Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of **stir-fry** (ounce) and **boba** (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100 \text{ g}$
- $\text{Calcium} \geq 700 \text{ mg}$

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much **stir-fry** (ounce) and **boba** (fluid ounces) should we buy?

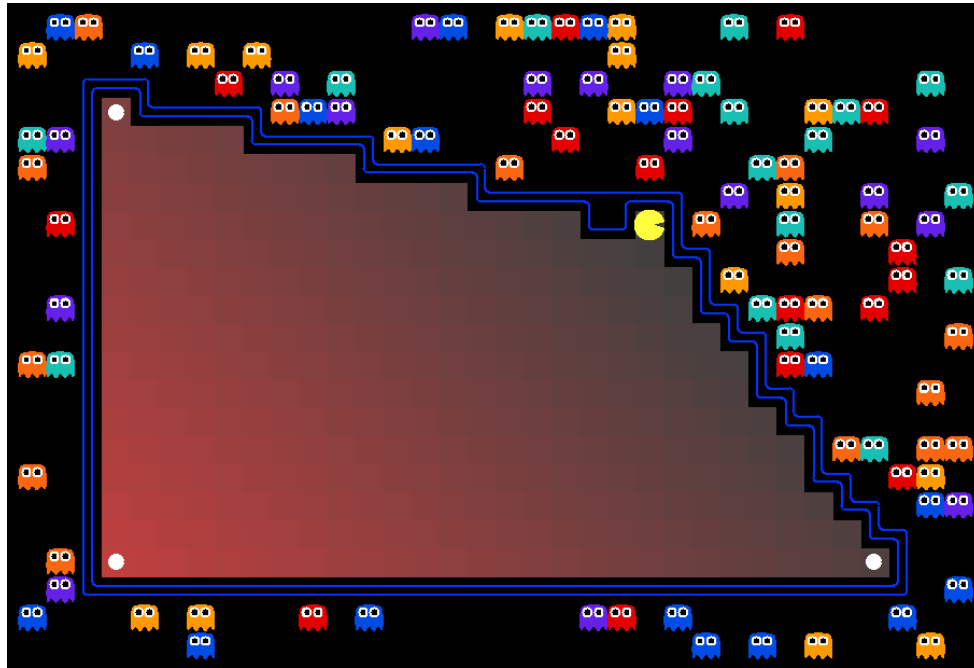
Announcements

Assignments:

- P2 **Sat 10/5, 10 pm**
 - Due ~~Thu 10/3, 10 pm~~
- P3
 - Will be released later today
 - Due Thu 10/17, 10 pm
- HW5 (written)
 - Released Tue 10/1
 - Due Tue **10/8**, 10 pm

AI: Representation and Problem Solving

Optimization & Linear Programming



Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, <http://ai.berkeley.edu>

Learning Objectives

- Formulate a problem as a **Linear Program (LP)**
- Convert a LP into a required **form, e.g., inequality form**
- Plot the **graphical representation** of a linear optimization problem with two variables and find the optimal solution
- Understand the relationship between **optimal solution of an LP** and the intersections of constraints
- Describe and implement a **LP solver based on vertex enumeration**
- Describe the high-level idea of **Simplex algorithm**

Next Lecture

Recap

What have we learned so far?

What do they have in common?

This lecture:

- (1) Continuous space
- (2) General formulation

Recap

What have we learned so far?

- Search: Depth/Breadth-first search, A* search, local search
- Constraint Satisfaction Problem: 8-queen, graph coloring
- Logic and Planning: Propositional Logic, SAT, First-order Logic

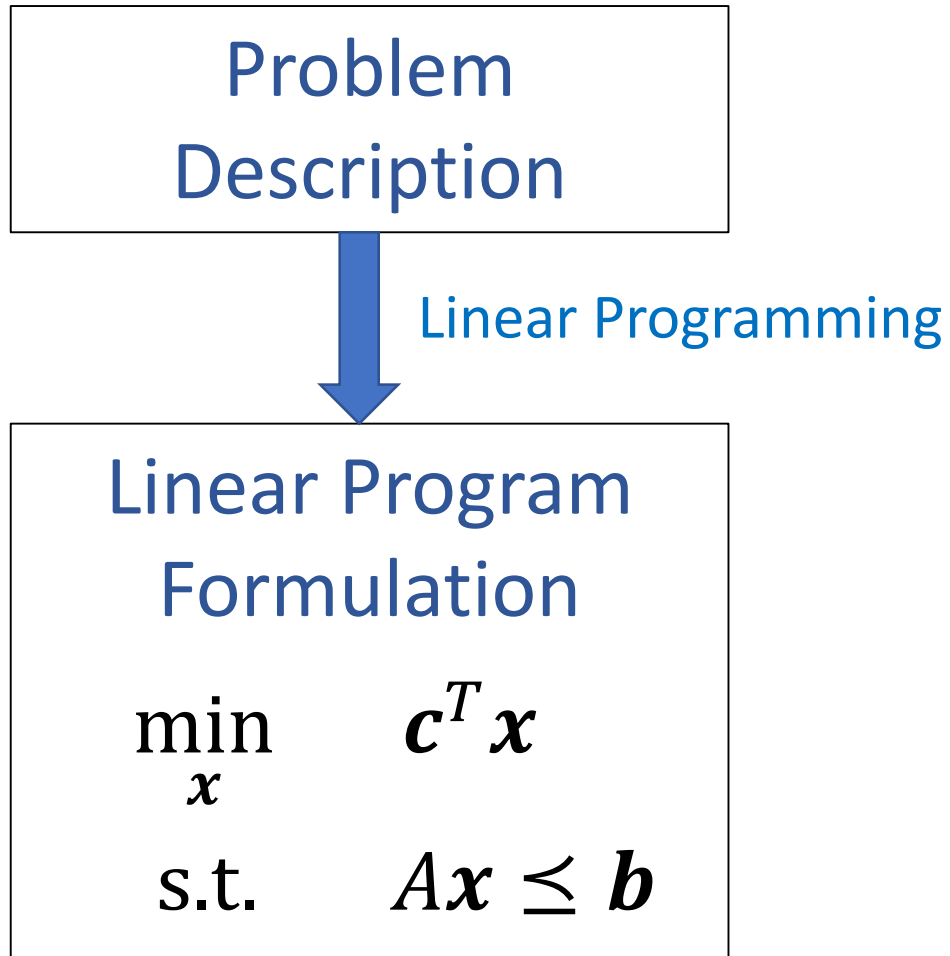
What do they have in common?

- Variables/Symbols, Finite options

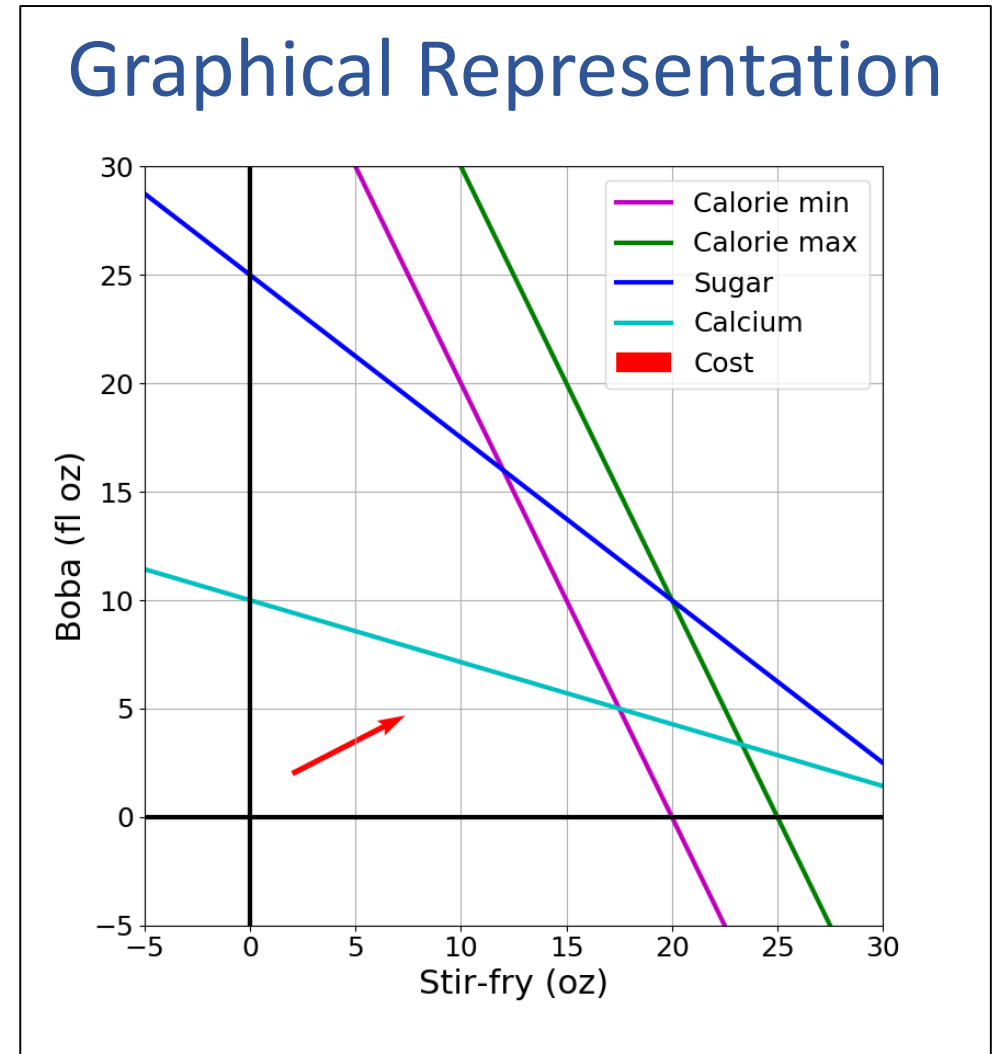
This lecture:

- (1) Move to continuous space (with connections to the discrete space)
- (2) Provide a general formulation that can be used to represent many of the previously seen problems

Focus of Today: (Linear) Optimization Problem



Notation Alert!



Diet Problem: What to eat?

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We can choose the amount of **stir-fry** (ounce) and **boba** (fluid ounces).

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How much **stir-fry** (ounce) and **boba** (fluid ounces) should we buy?

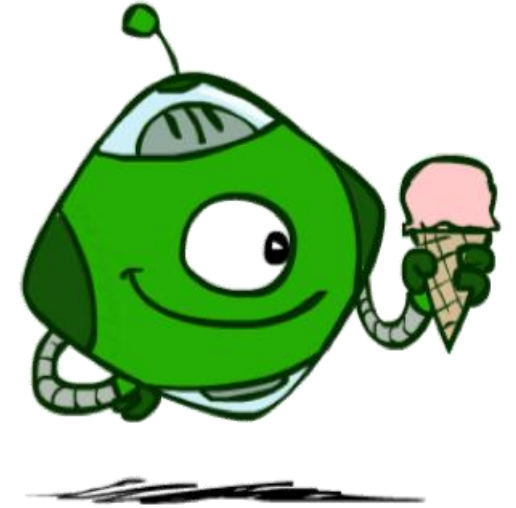
Problem Formulation

Can we formulate it as a Constraint Satisfaction Problem?

Variable:

Domain:

Constraint:



What are the issues with this CSP formulation?

Healthy Squad Goals

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Problem Formulation

Can we formulate it as a Constraint Satisfaction Problem?

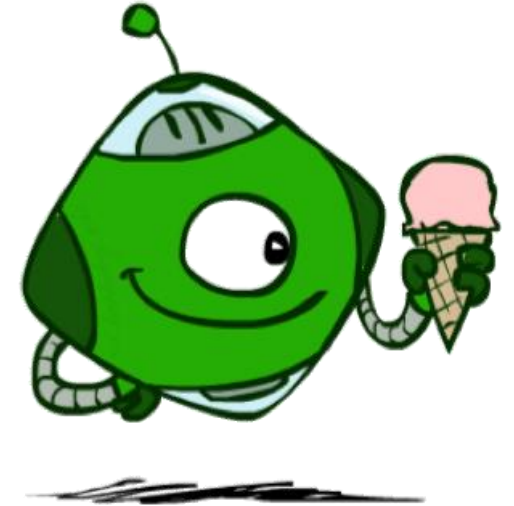
Variable: x_1 (ounces for stir-fry), x_2 (ounces for boba)

Domain: $[0, +\infty)$

Constraint: Implicit: $100 x_1 + 50 x_2 \in [2000, 2500]$

$$3 x_1 + 4 x_2 \leq 100,$$

$$20 x_1 + 70 x_2 \geq 700$$



What are the issues with this CSP formulation?

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Problem Formulation

Optimization problem: Finding the **best** solution from all **feasible** solutions

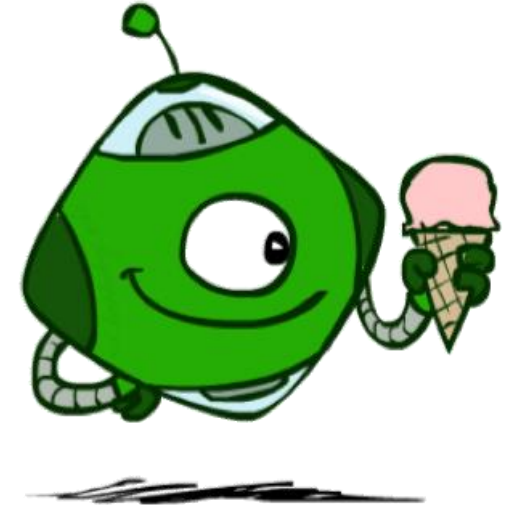
From CSP to Optimization Problem

Variable
Domain
Constraint



$\min_x f(x)$ **Objective**
s.t. x satisfies constraints

Notation Alert!



Healthy Squad Goals

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Problem Formulation

Optimization problem: Finding the **best** solution from all **feasible** solutions

From CSP to Optimization Problem

(Optimization) Variable

~~Domain~~ Can be represented as constraints

Constraint

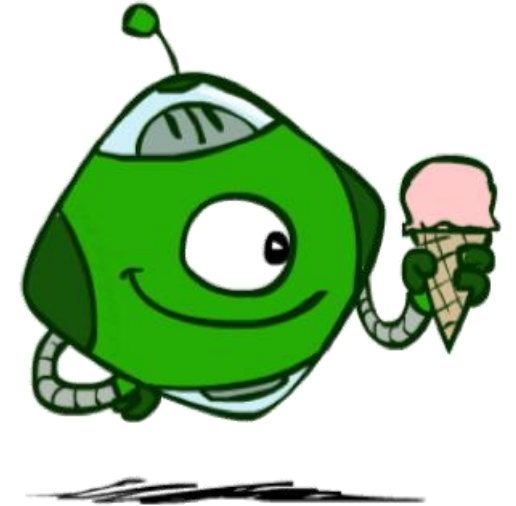
Optimization Objective



$\min_x \quad f(x)$ Objective

s.t. x satisfies constraints

Notation Alert!



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Problem Formulation

Formulate Diet Problem as an optimization problem

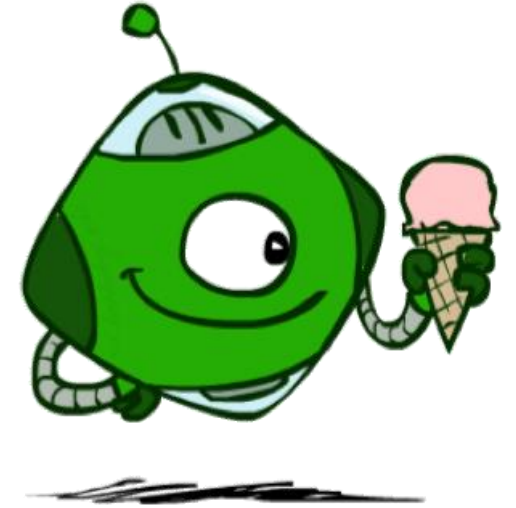
$$\min_{\mathbf{x}} f(\mathbf{x})$$

s.t. $\mathbf{x}$ satisfies constraints

Variable:

Objective:

Constraints:



Healthy Squad Goals

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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$

s.t. $\mathbf{x}$ satisfies constraints

Variable: $\mathbf{x} = [x_1, x_2]^T$. x_1 : ounces for stir-fry, x_2 : ounces for boba

Objective: $\min_{\mathbf{x}} \text{cost}(\mathbf{x})$

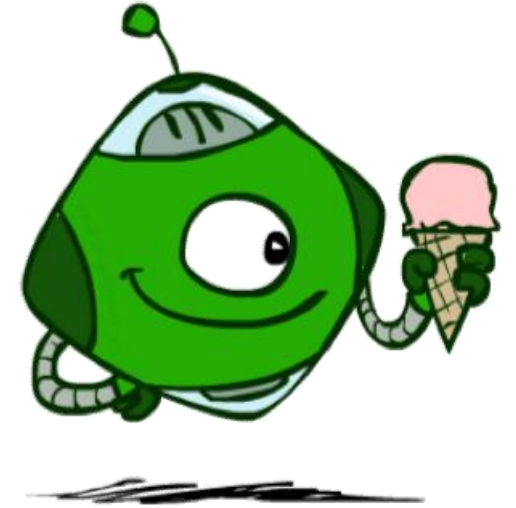
Constraints:

$\text{calories}(\mathbf{x})$ in required range

$\text{sugar}(\mathbf{x}) \leq \text{limit}$

$\text{calcium}(\mathbf{x}) \geq \text{limit}$

Can be ignored in
this problem. Why? ~~$x_1 \geq 0, x_2 \geq 0$~~



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$$\begin{array}{ll}\min_{\mathbf{x}} & \text{cost}(\mathbf{x}) \\ \text{s.t.} & \text{calories}(\mathbf{x}) \text{ in required range} \\ & \text{sugar}(\mathbf{x}) \leq \text{limit} \\ & \text{calcium}(\mathbf{x}) \geq \text{limit}\end{array}$$

$\mathbf{x} = [x_1, x_2]^T$. x_1 : ounces for stir-fry, x_2 : ounces for boba

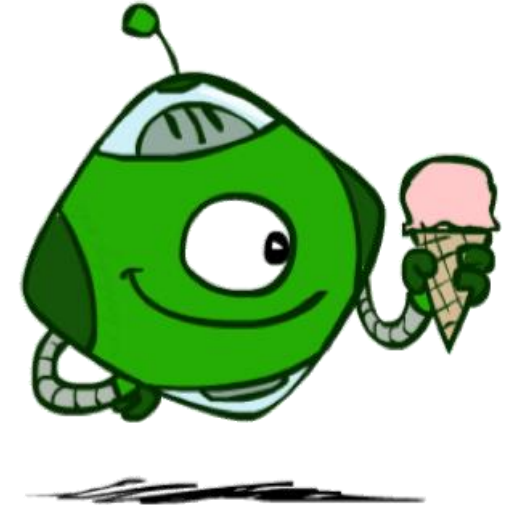
What is the expression of

$\text{cost}(\mathbf{x})$

$\text{calories}(\mathbf{x})$

$\text{sugar}(\mathbf{x})$

$\text{calcium}(\mathbf{x})$



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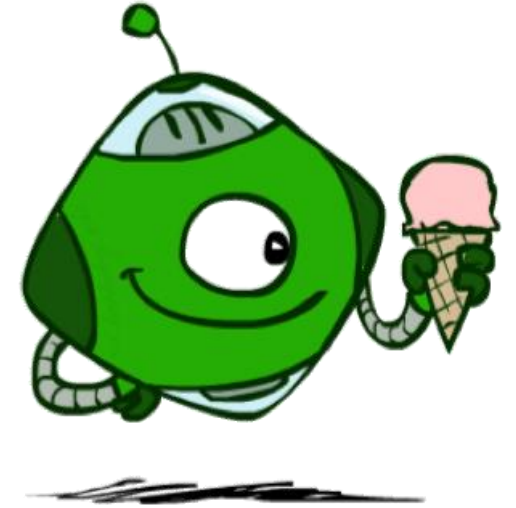
What is the expression of

$$\text{cost}(\mathbf{x}) = x_1 + 0.5x_2$$

$$\text{calories}(\mathbf{x}) = 100x_1 + 50x_2$$

$$\text{sugar}(\mathbf{x}) = 3x_1 + 4x_2$$

$$\text{calcium}(\mathbf{x}) = 20x_1 + 70x_2$$



Healthy Squad Goals

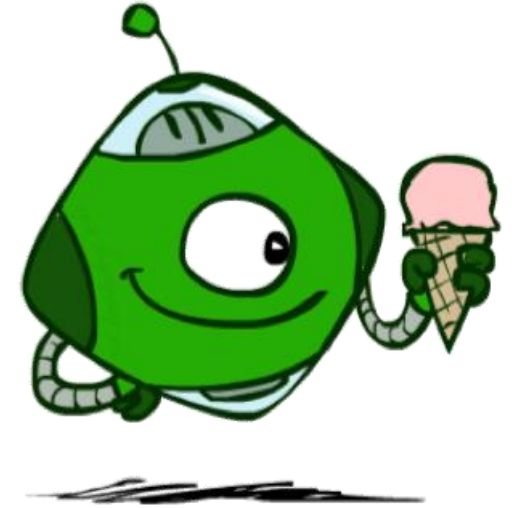
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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\begin{array}{ll}\min_{x_1, x_2} & 1 x_1 + 0.5 x_2 \\ \text{s.t.} & 100 x_1 + 50 x_2 \geq 2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & 20 x_1 + 70 x_2 \geq 700\end{array}$$



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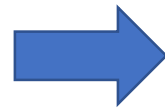
Linear Programming

Linear Programming:

- Technique for the optimization of a **linear objective** function subject to **linear equality** and **linear inequality** constraints

Example Linear program

$$\begin{array}{ll}\min_{x_1, x_2} & 1 x_1 + 0.5 x_2 \\ \text{s.t.} & 100 x_1 + 50 x_2 \geq 2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & 20 x_1 + 70 x_2 \geq 700\end{array}$$



Linear program in Inequality Form

$$\begin{array}{ll}\min. & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b}\end{array}$$

Recap of Linear Algebra

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\mathbf{a} + \mathbf{b}$? $\mathbf{a} - \mathbf{b}$?

What does $\mathbf{a} \leq 0$ mean?

(Note: It is fine if you directly write $\mathbf{a} \leq 0$)

Recap of Linear Algebra

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\mathbf{a} + \mathbf{b}$? $\mathbf{a} - \mathbf{b}$?

$$\mathbf{a} \pm \mathbf{b} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

What does $\mathbf{a} \leq 0$ mean?

$$\forall i, a_i \leq 0$$

(Note: It is fine if you directly write $\mathbf{a} \leq 0$)

Recap of Linear Algebra

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\mathbf{a}^T \mathbf{b}$?

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $A^T \mathbf{b}$?

Recap of Linear Algebra

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\mathbf{a}^T \mathbf{b}$?

$$\begin{aligned} \mathbf{a}^T \mathbf{b} &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &= \sum_{i=1}^n a_i b_i \end{aligned}$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $A^T \mathbf{b}$?

$$A^T \mathbf{b} = \begin{bmatrix} a_{1,1} b_1 + a_{1,2} b_2 + \cdots a_{1,m} b_m \\ \vdots \\ a_{n,1} b_1 + a_{n,2} b_2 + \cdots a_{n,m} b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1,i} b_i \\ \vdots \\ \sum_{i=1}^m a_{n,i} b_i \end{bmatrix}$$

Problem Formulation

$$\begin{array}{ll}\min_{x_1, x_2} & 1 x_1 + 0.5 x_2 \\ \text{s.t.} & 100 x_1 + 50 x_2 \geq 2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & 20 x_1 + 70 x_2 \geq 700\end{array}$$

Convert it into the following inequality form, what should $\mathbf{c}$, A , and $\mathbf{b}$ be?

$$\begin{array}{ll}\min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$

Problem Formulation

$$\begin{array}{ll} \min_{x_1, x_2} & 1 x_1 + 0.5 x_2 \\ \text{s.t.} & 100 x_1 + 50 x_2 \geq 2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & 20 x_1 + 70 x_2 \geq 700 \end{array}$$

$\mathbf{c}^T \mathbf{x}$ with $\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$-100 x_1 - 50 x_2 \leq -2000$

$-20 x_1 - 70 x_2 \leq -700$

Convert it into the following inequality form, what should $\mathbf{c}$, A , and $\mathbf{b}$ be?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \preceq \mathbf{b} \end{array}$$

Problem Formulation

$$\begin{array}{ll}\min_{x_1, x_2} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & -100 x_1 - 50 x_2 \leq -2000 \\ & 100 x_1 + 50 x_2 \leq 2500 \\ & 3 x_1 + 4 x_2 \leq 100 \\ & -20 x_1 - 70 x_2 \leq -700\end{array}$$

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$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Convert it into the following inequality form, what should $\mathbf{c}$, $\mathbf{A}$, and $\mathbf{b}$ be?

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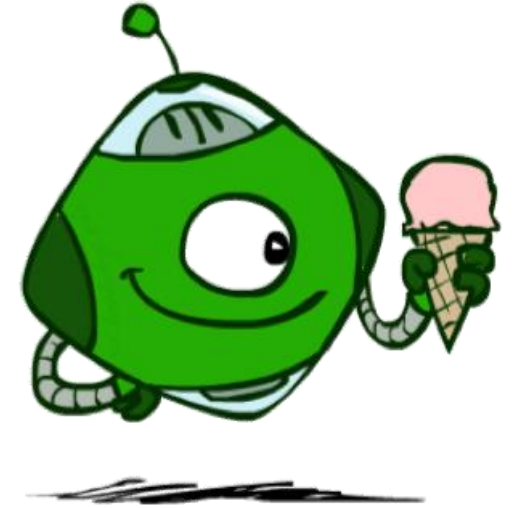
Question

What has to increase to add more menu items?

$$\begin{array}{ll}\min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}\end{array}$$

Select all that apply

- A) length $\mathbf{x}$
- B) length $\mathbf{c}$
- C) height $\mathbf{A}$
- D) width $\mathbf{A}$
- E) length $\mathbf{b}$



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Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
New Item	...	...	...	...

Question

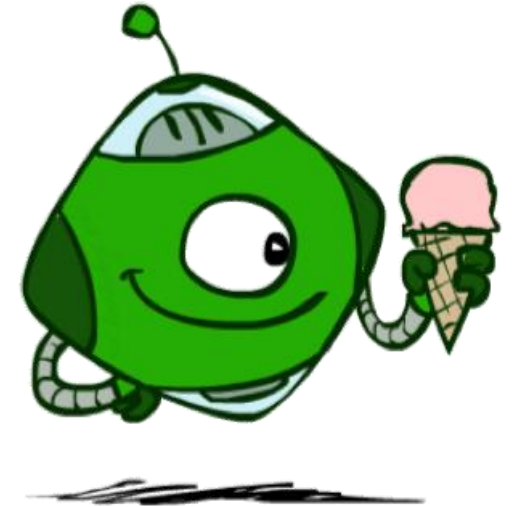
What has to increase to add more menu items?

$$\begin{array}{ll}\min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



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Beef (per oz)	2	80	1	30

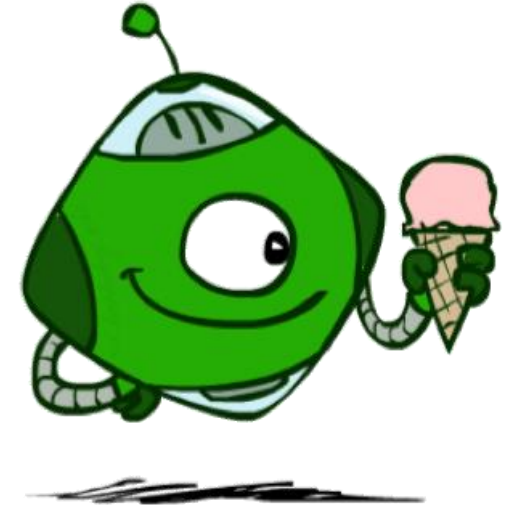
Question

What has to increase to add more menu items?

$$\begin{array}{ll}\min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}\end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 & -80 \\ 100 & 50 & 80 \\ 3 & 4 & 1 \\ -20 & -70 & -30 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



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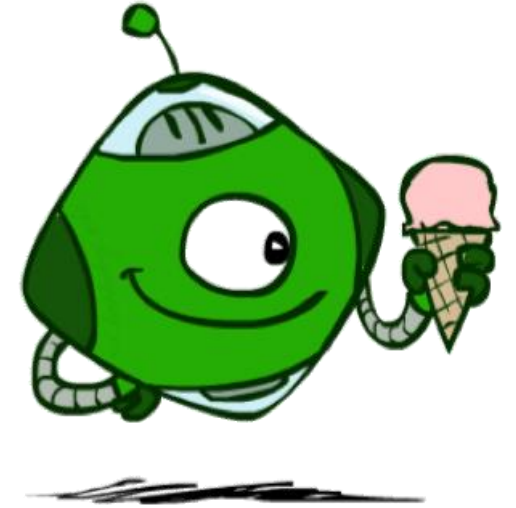
Question

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N ?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

- A) length $\mathbf{x}$
- B) length $\mathbf{c}$
- C) length $\mathbf{b}$



Linear Programming

Different forms

Inequality form

$$\begin{array}{ll}\min. & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b}\end{array}$$

General form

$$\begin{array}{ll}\min. & \mathbf{c}^T \mathbf{x} + \mathbf{d} \\ \text{s.t.} & \mathbf{Gx} \leq \mathbf{h} \\ & \mathbf{Ax} = \mathbf{b}\end{array}$$

Standard form

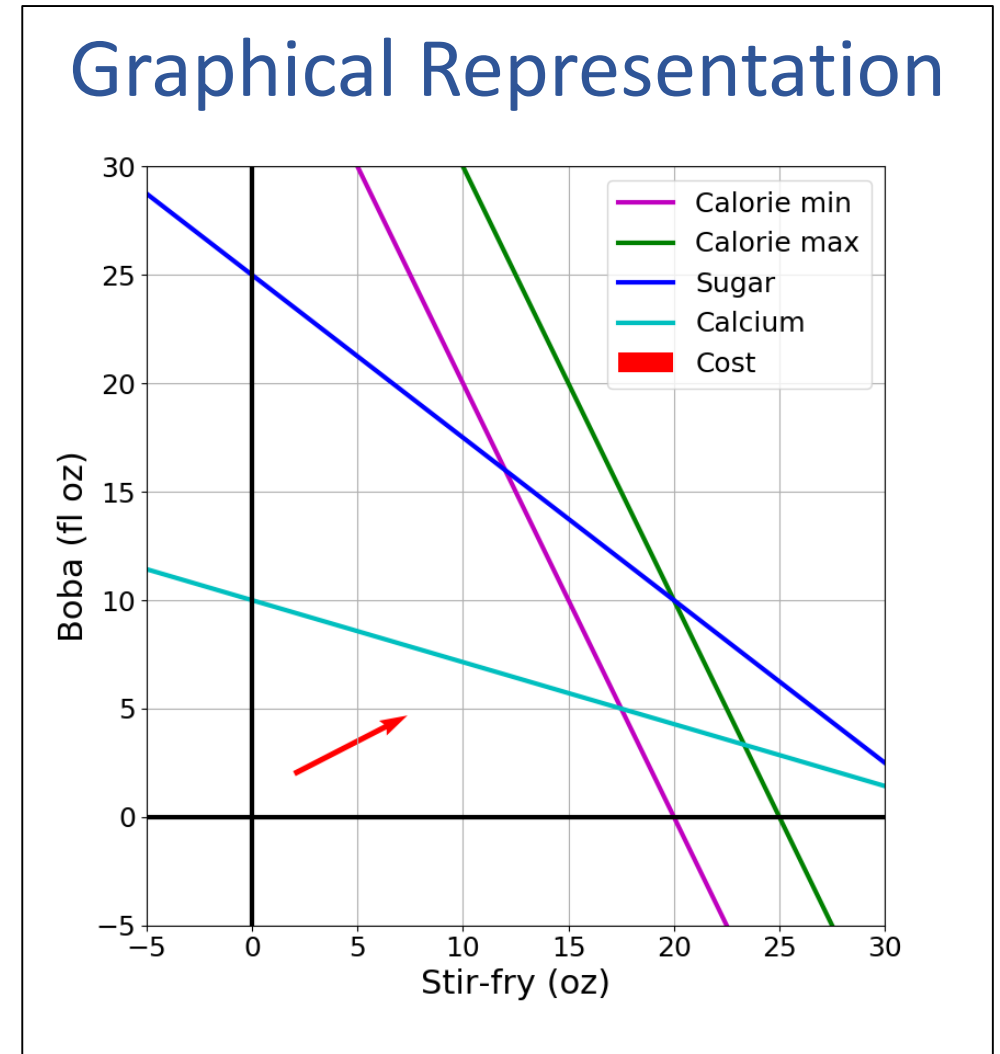
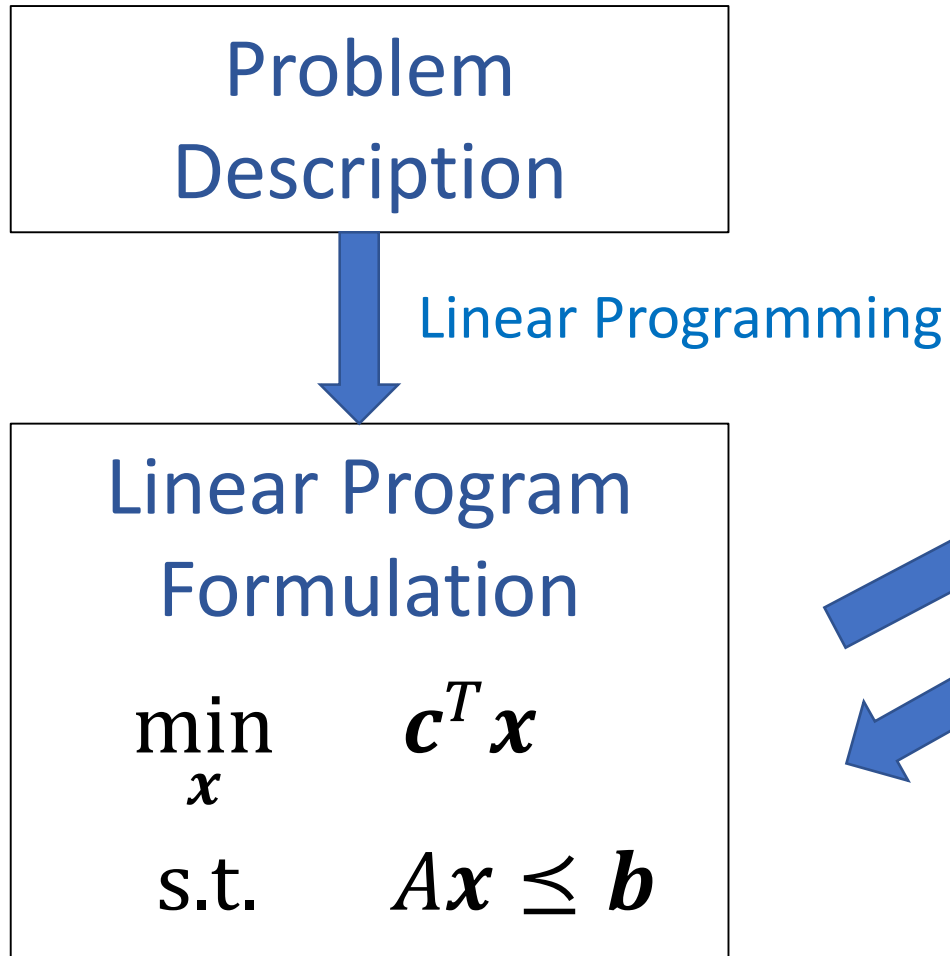
$$\begin{array}{ll}\min. & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0\end{array}$$

Important to pay attention to form!

Can switch between formulations!

Note: Different books have different definitions. We will refer to the forms in this slide (also consistent with B&V book).

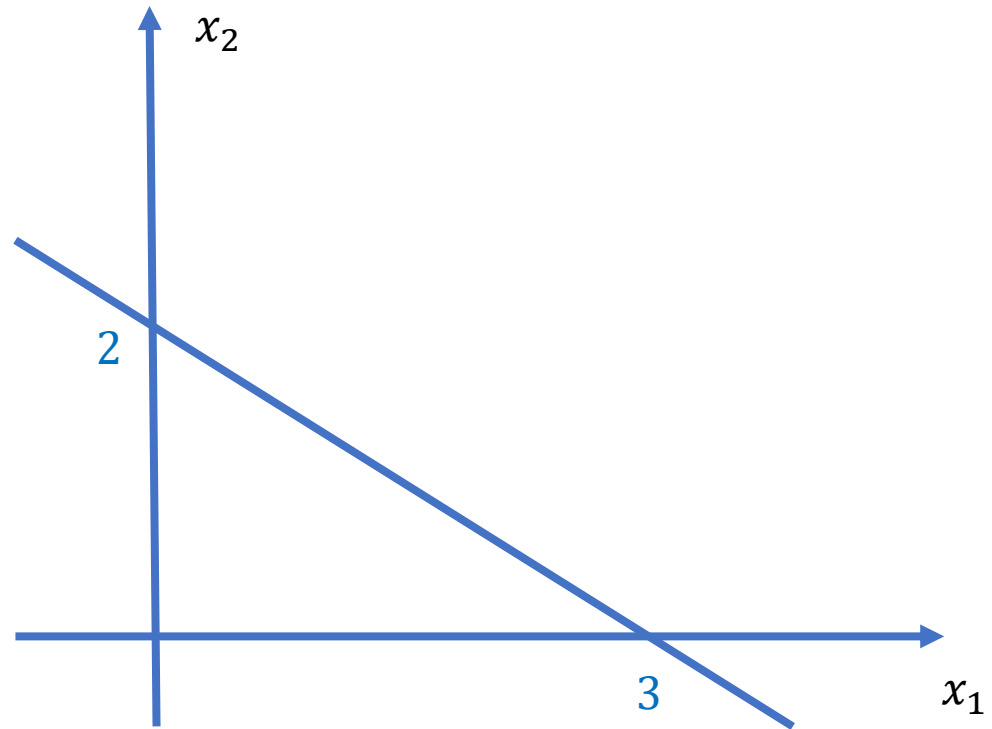
Focus of Today: (Linear) Optimization Problem



Shape of Linear Equality

Geometry / Algebra Question

What is the equality present this line?



Shape of Linear Equality

Geometry / Algebra Question

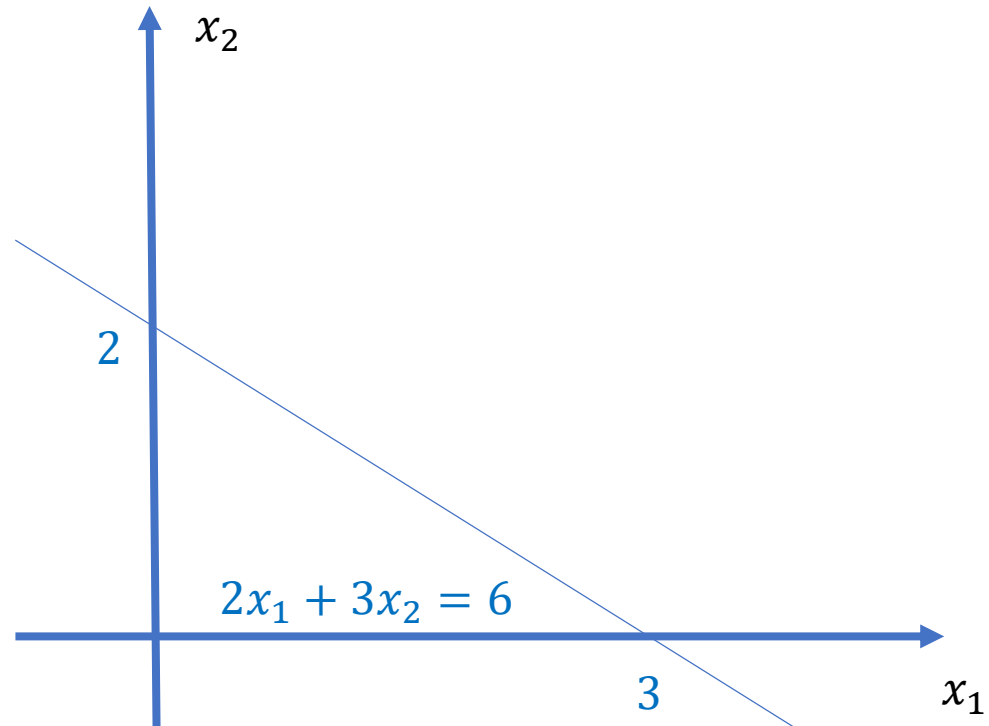
What is the equality present this line?

$$x_2 = -\frac{2}{3}x_1 + 2$$



$$3x_2 + 2x_1 = 6$$

To make sure the line intersects with the axes at (3,0) and (0,2)



Shape of Linear Inequality

Geometry / Algebra Question

What shape does this inequality represent?

$$a_1 x_1 + a_2 x_2 \leq b_1$$

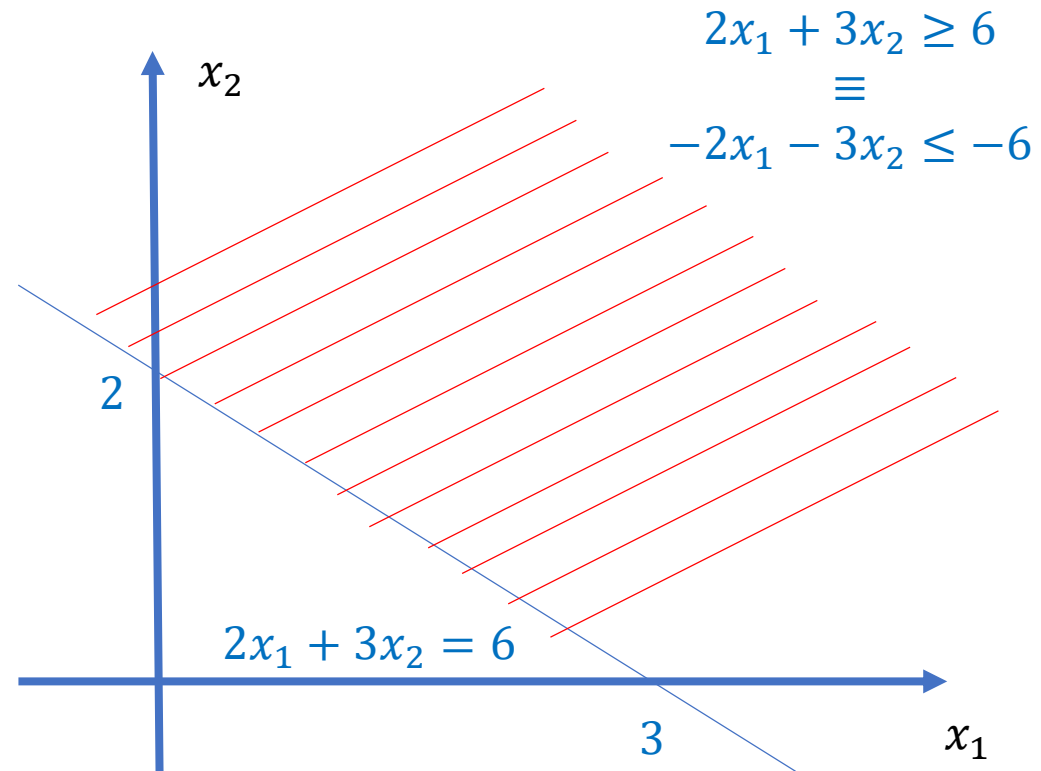
Shape of Linear Inequality

Geometry / Algebra Question

What shape does this inequality represent?

$$a_1 x_1 + a_2 x_2 \leq b_1 \quad \text{Half Plane}$$

$$a_1 x_1 + a_2 x_2 = b_1 \quad \text{Line}$$



Shape of Linear Inequality

Geometry / Algebra Question

What shape does these inequalities jointly represent?

$$a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$$

$$a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$$

$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$

$$a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$$

Shape of Linear Inequality

Geometry / Algebra Question

What shape does these inequalities jointly represent?

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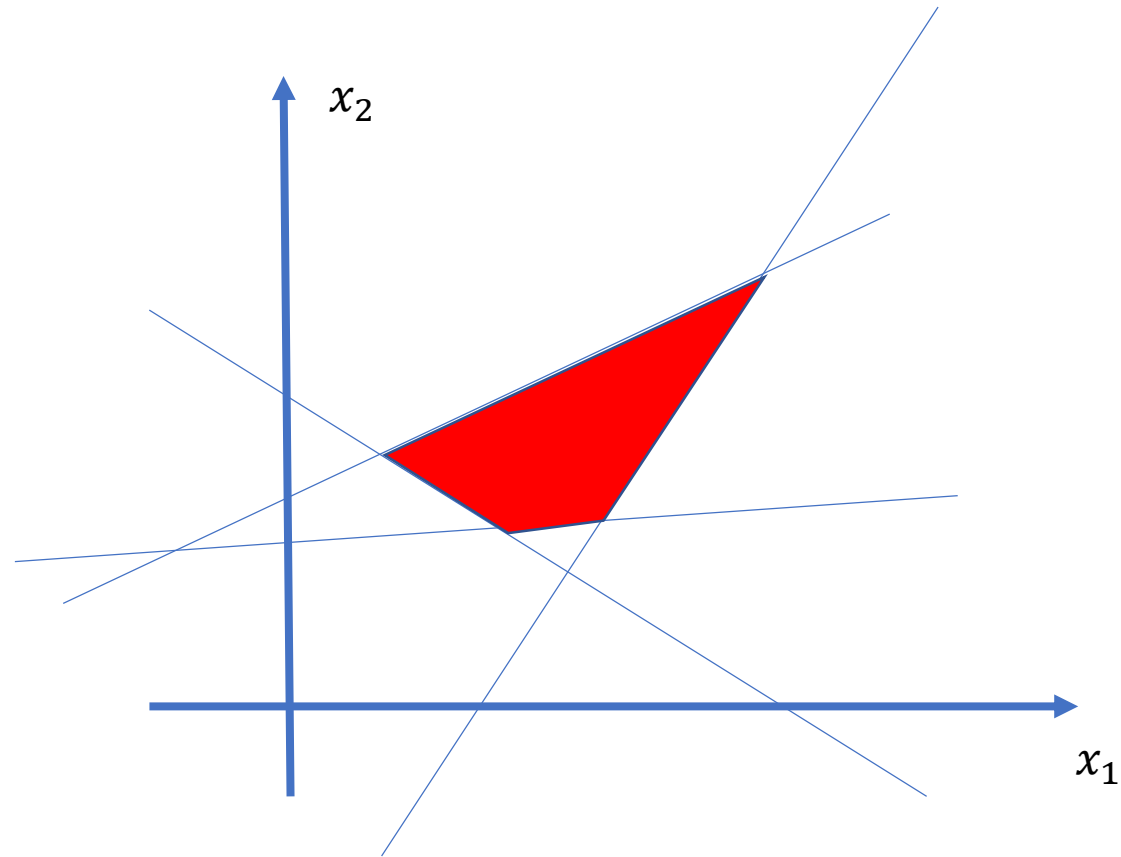
$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$

$$a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$$

Intersection of half planes

Could be polyhedron

Could be empty



Lowering Cost

Given the cost vector $[c_1, c_2]^T$ and initial point $\mathbf{x}^{(0)}$,
Which unit vector step $\Delta \mathbf{x}$ will cause $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \Delta \mathbf{x}$
to have the lowest cost $\mathbf{c}^T \mathbf{x}^{(1)}$?



Lowering Cost

Given the cost vector $[c_1, c_2]^T$ and initial point $\mathbf{x}^{(0)}$,

Which unit vector step $\Delta \mathbf{x}$ will cause $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \Delta \mathbf{x}$ to have the lowest cost $\mathbf{c}^T \mathbf{x}^{(1)}$?

$$\mathbf{c}^T \mathbf{x}^{(1)} = \mathbf{c}^T (\mathbf{x}^{(0)} + \Delta \mathbf{x}) = \mathbf{c}^T \mathbf{x}^{(0)} + \mathbf{c}^T \Delta \mathbf{x}$$

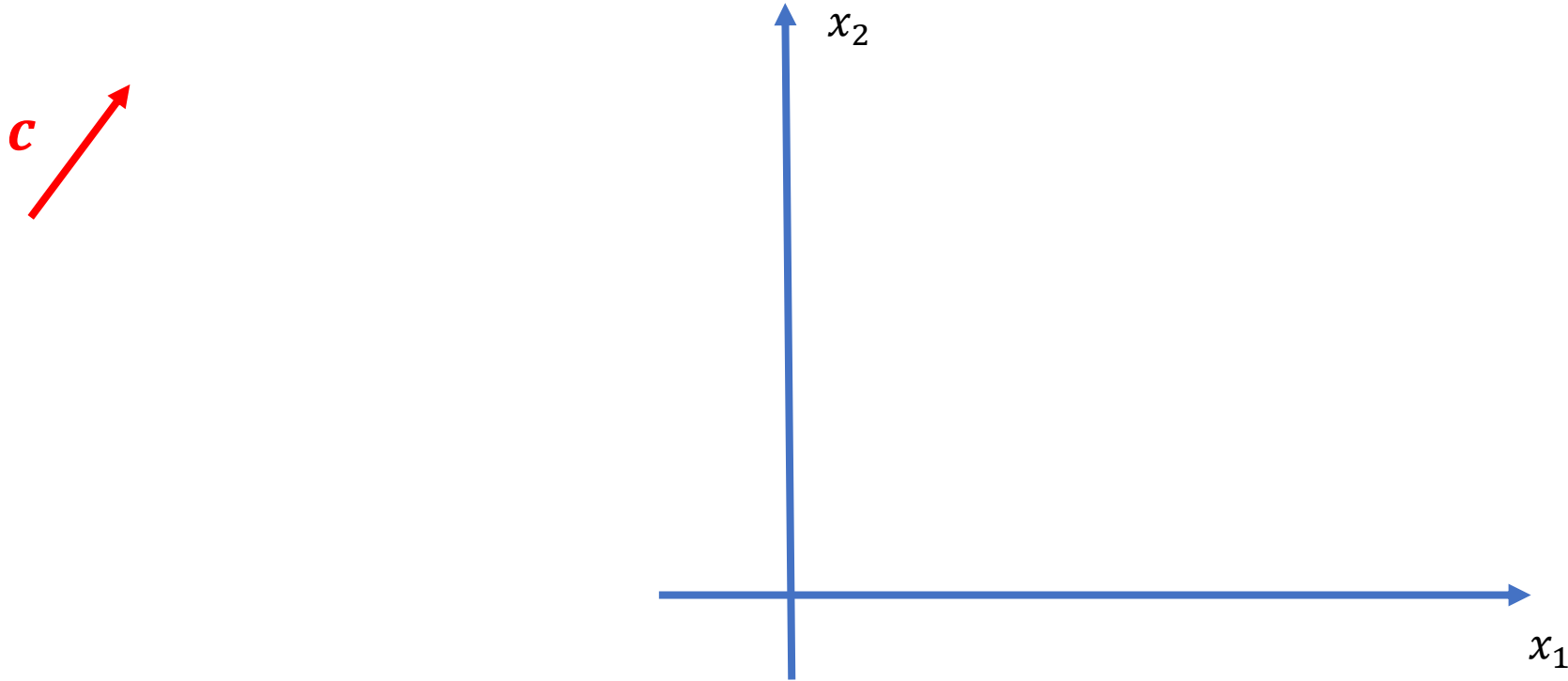


$$\mathbf{c}^T \Delta \mathbf{x} = |\mathbf{c}| |\Delta \mathbf{x}| \cos(\mathbf{c}, \Delta \mathbf{x})$$

$\cos(\mathbf{c}, \Delta \mathbf{x}) = -1$ when $\mathbf{c}$ and $\Delta \mathbf{x}$ have opposite directions

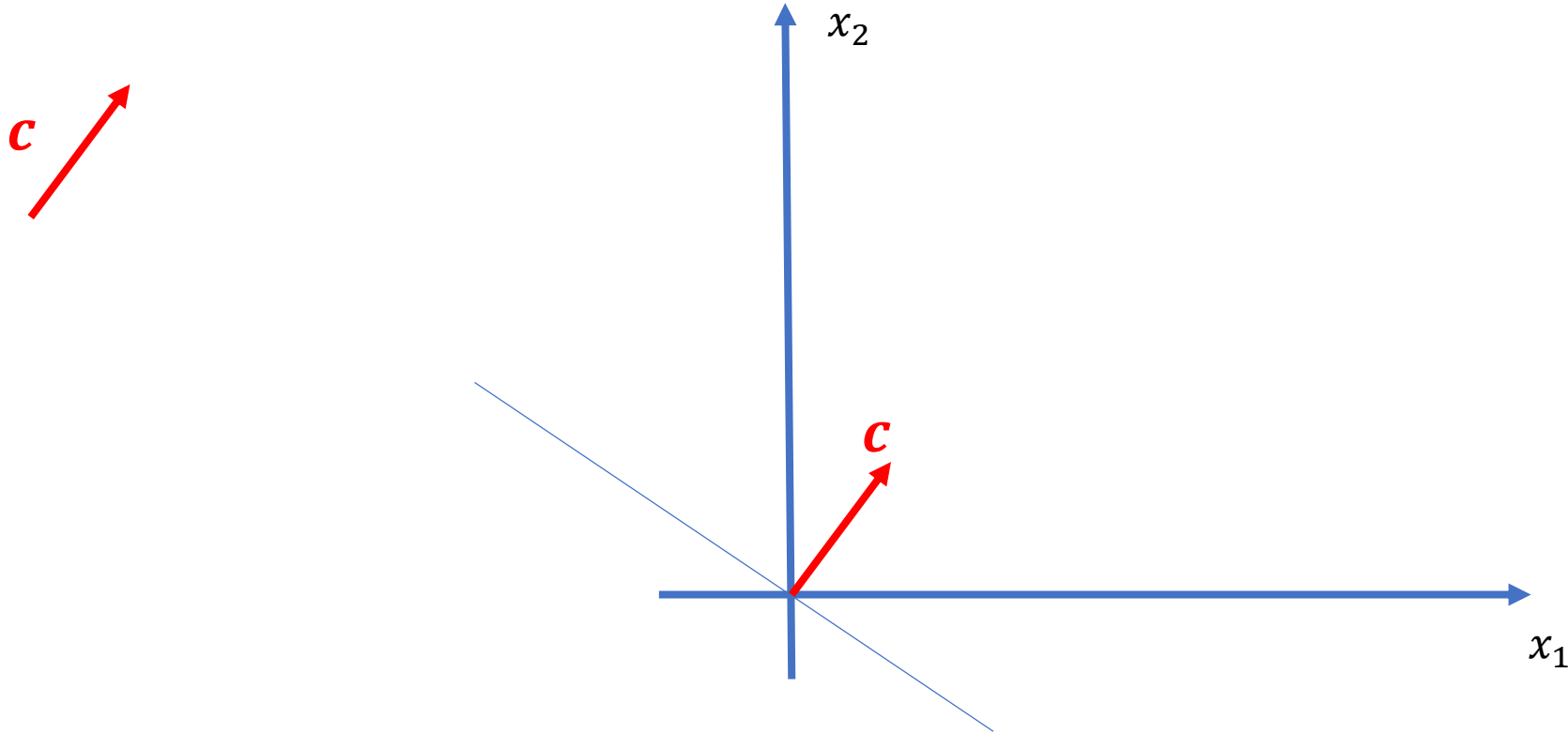
Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will
 $\mathbf{c}^T \mathbf{x} = 0$?



Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will $\mathbf{c}^T \mathbf{x} = 0$?



Cost Contours

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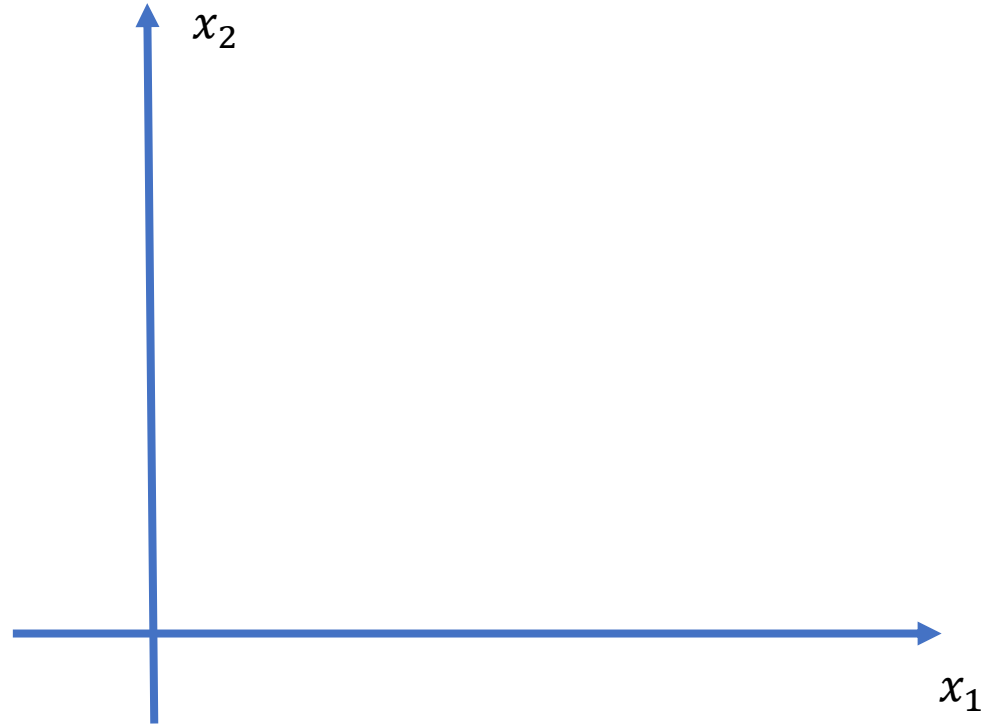
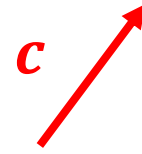
$$\mathbf{c}^T \mathbf{x} = 0 ?$$

$$\mathbf{c}^T \mathbf{x} = 1 ?$$

$$\mathbf{c}^T \mathbf{x} = 2 ?$$

$$\mathbf{c}^T \mathbf{x} = -1 ?$$

$$\mathbf{c}^T \mathbf{x} = -2 ?$$



Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

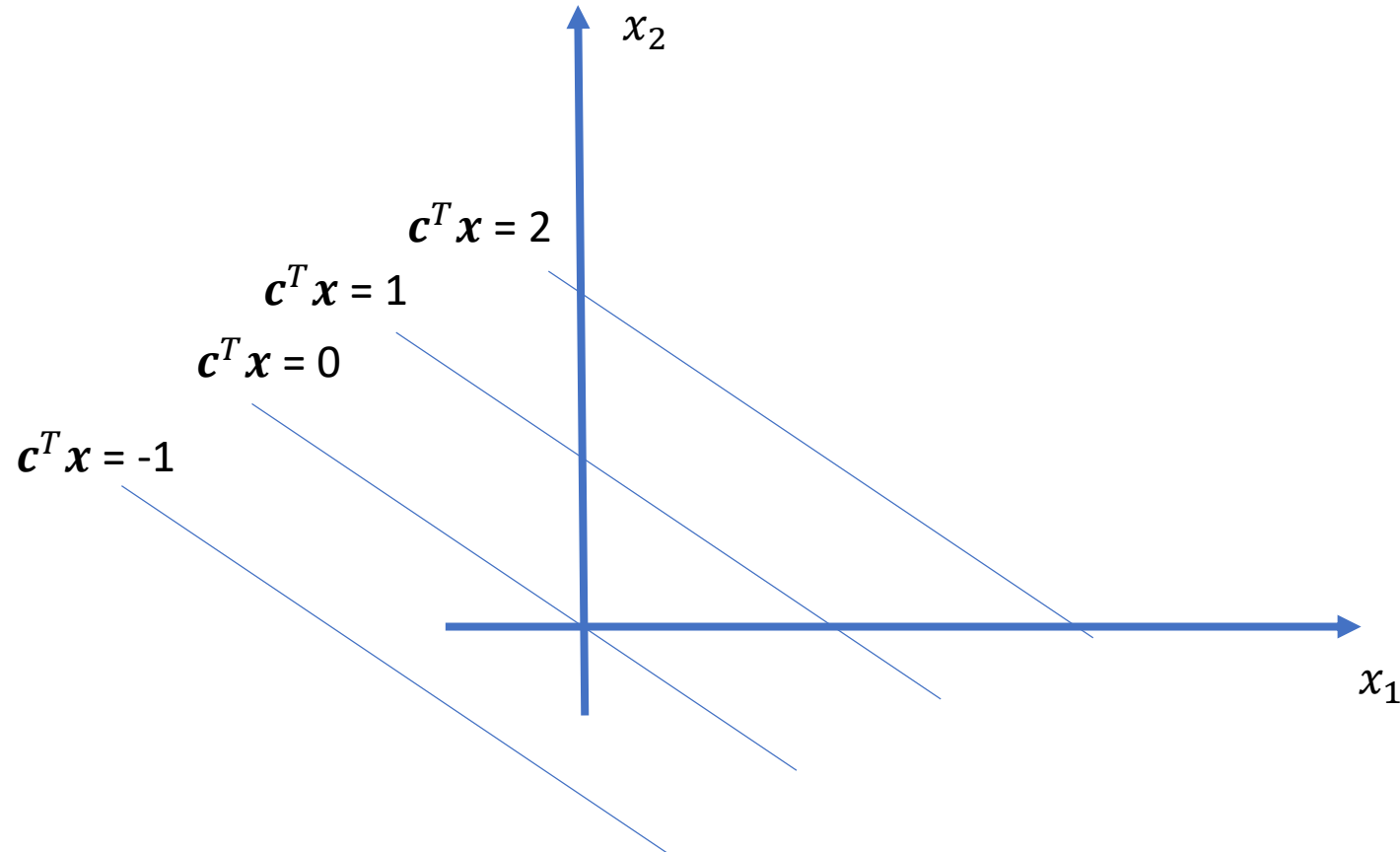
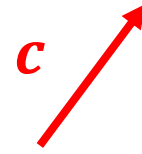
$$\mathbf{c}^T \mathbf{x} = 0 ?$$

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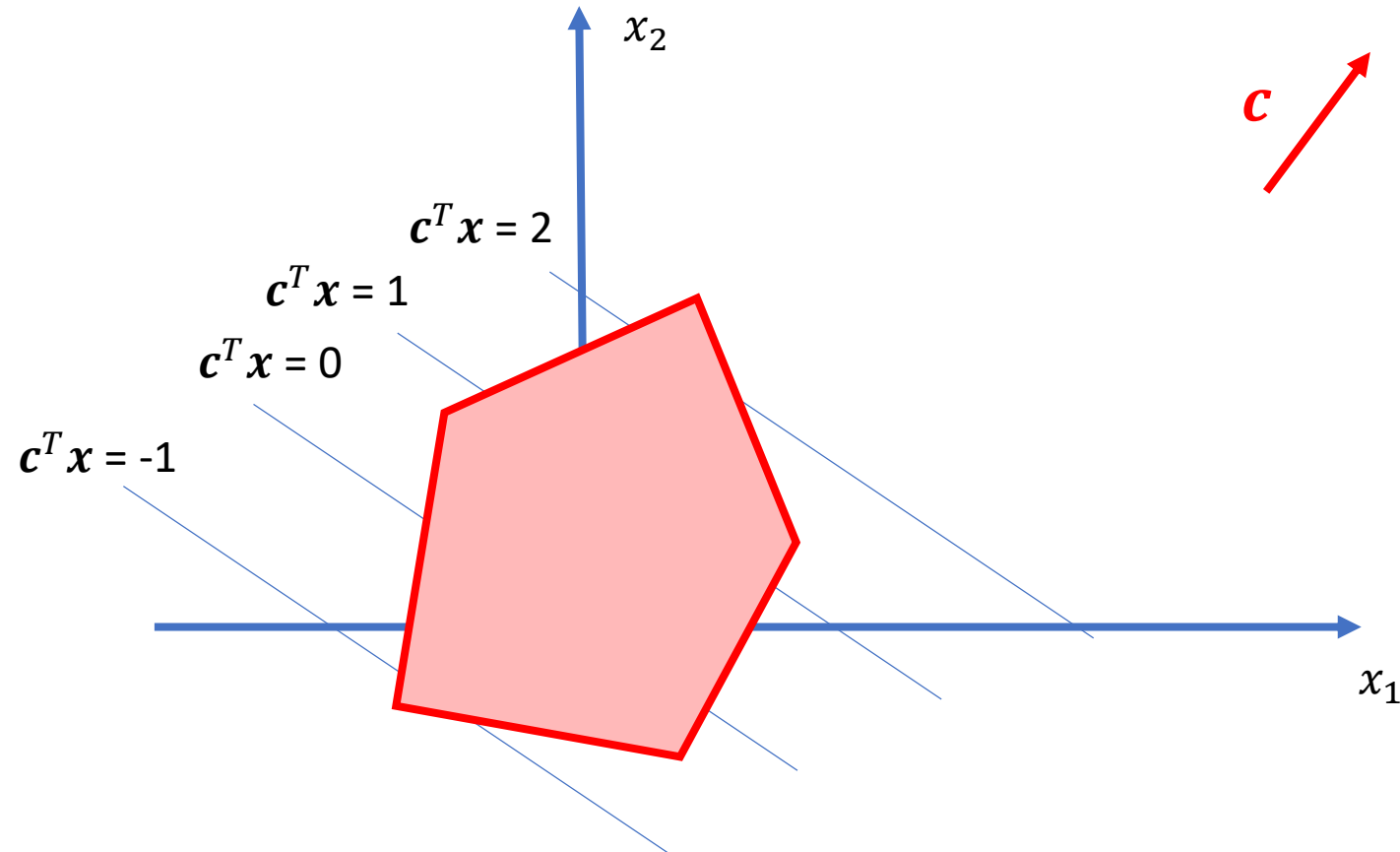
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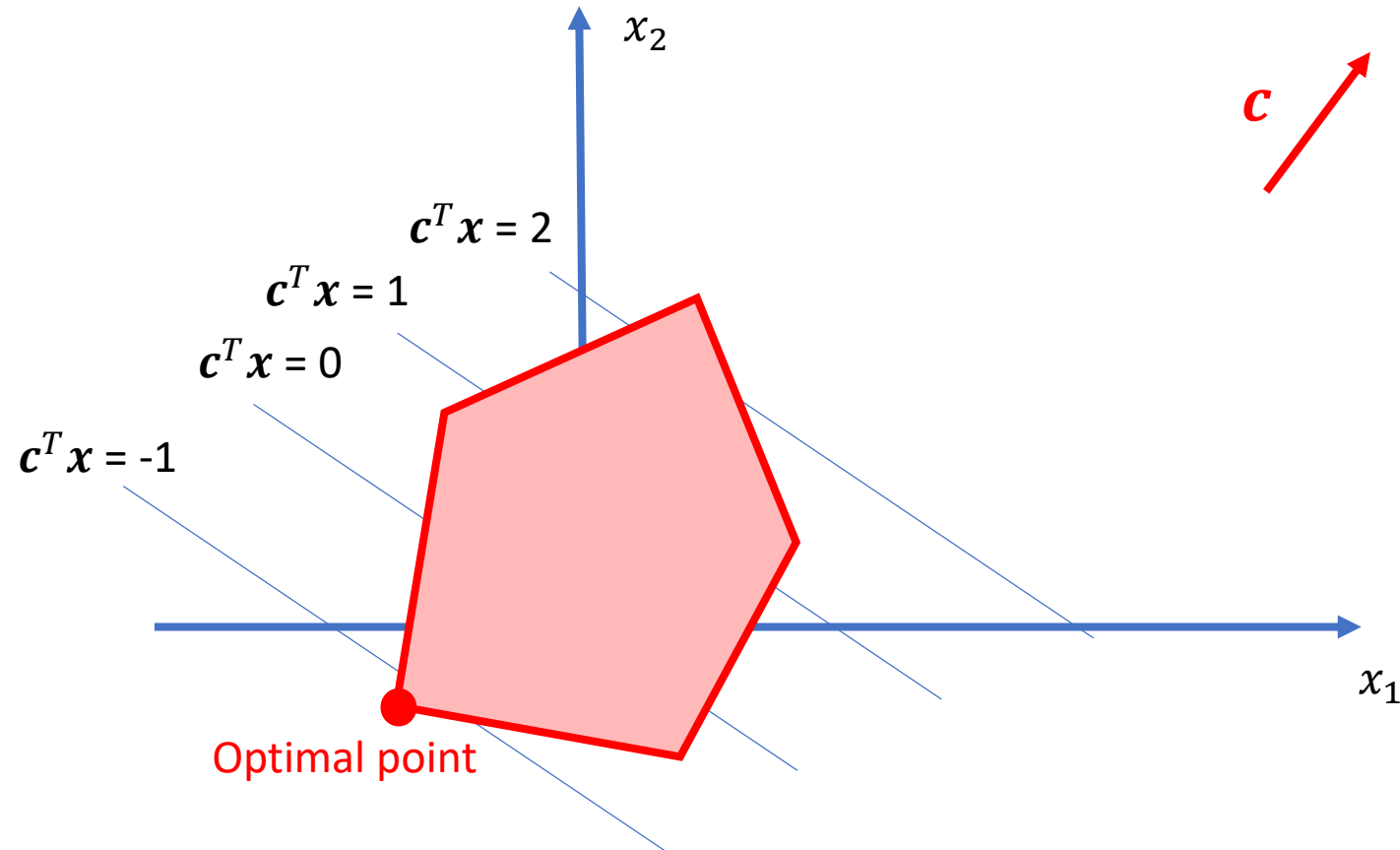
Optimizing Cost

Given the cost vector $[c_1, c_2]^T$, which point in the red polyhedron below can minimize $\mathbf{c}^T \mathbf{x}$?



Optimizing Cost

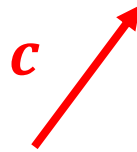
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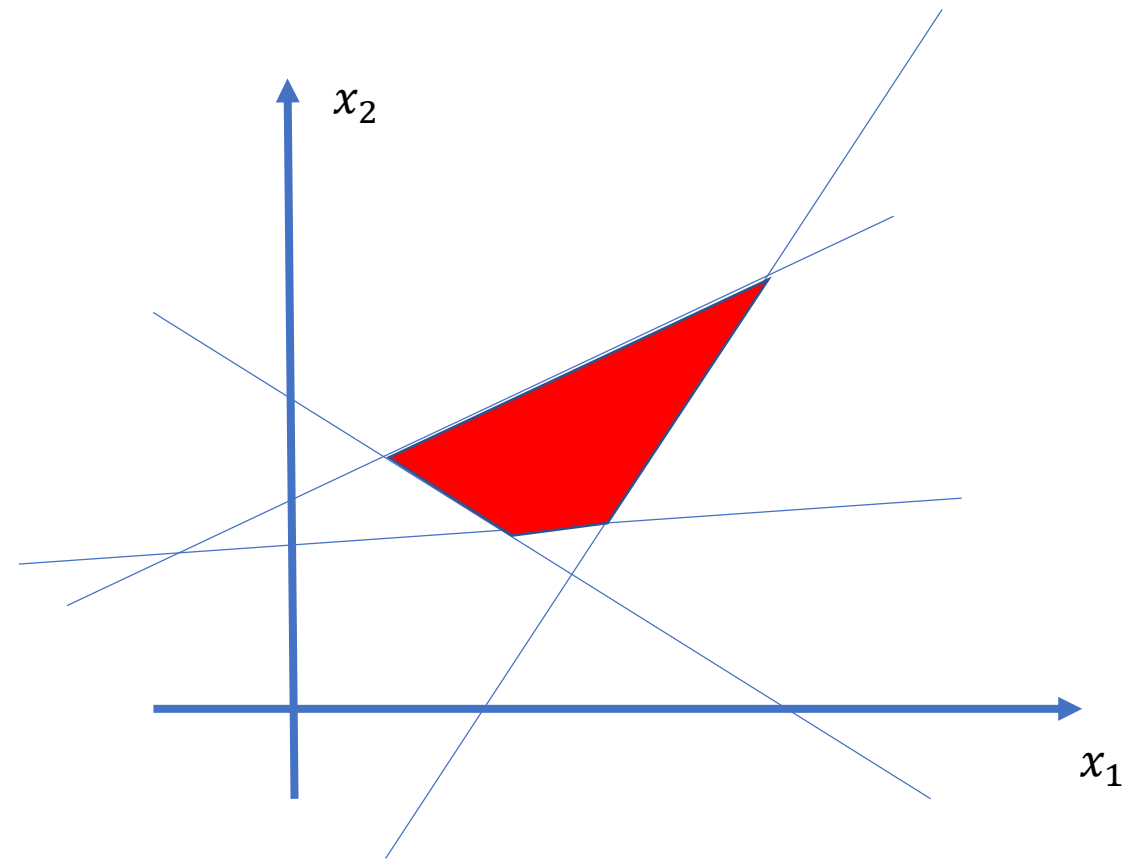
LP Graphical Representation

Inequality form

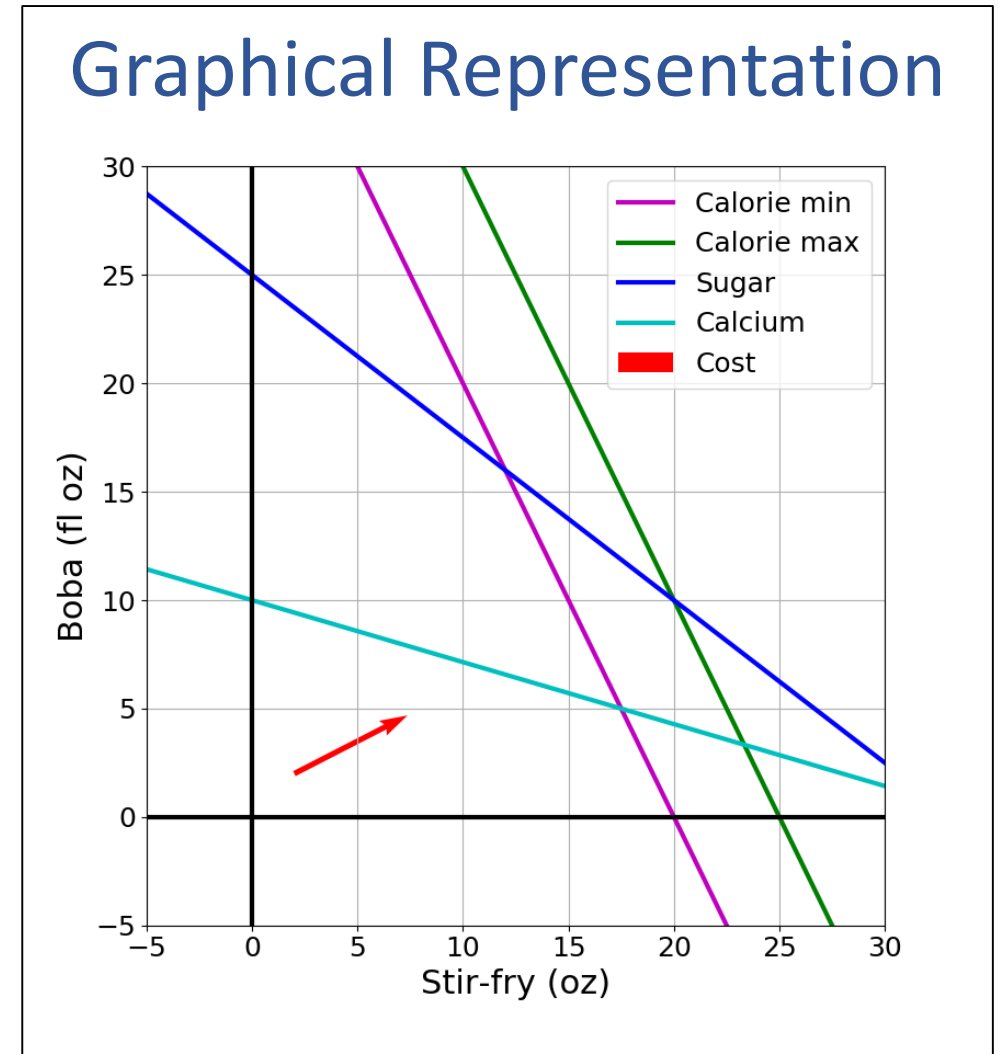
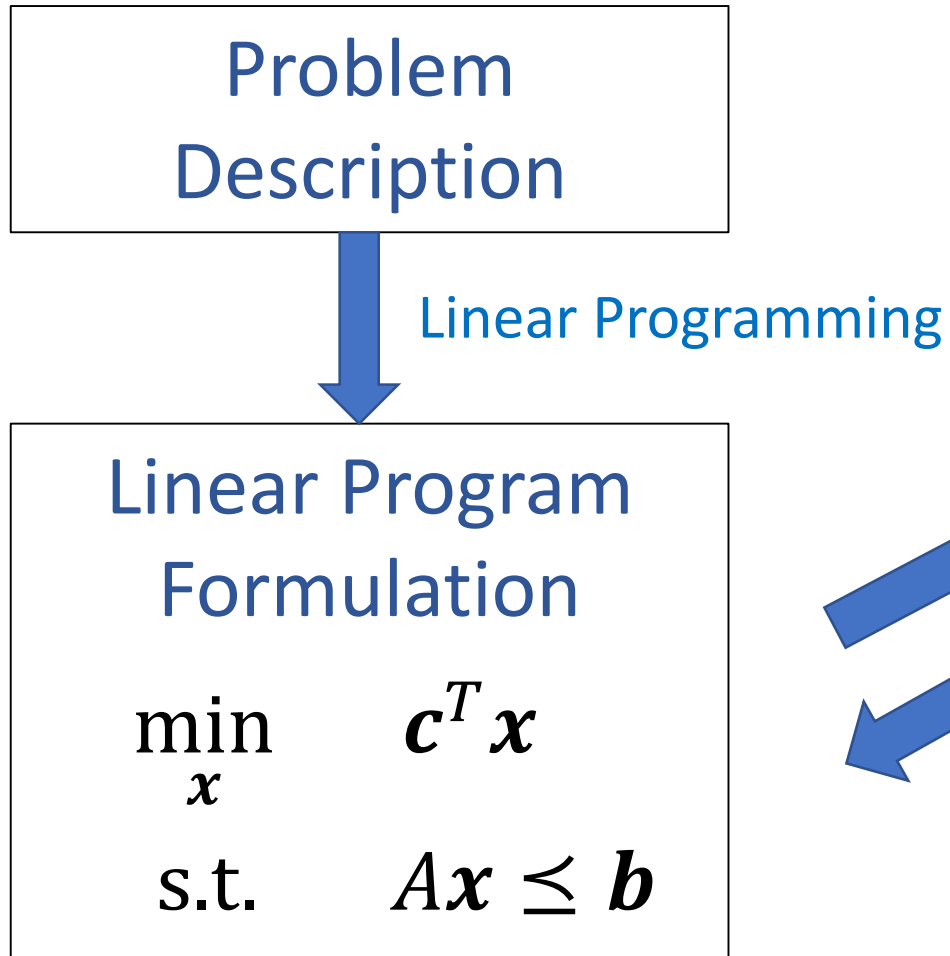
$$\begin{array}{ll} \min. & \mathbf{c}^T \mathbf{x} \quad \text{Objective Function} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b} \quad \text{Feasible Region} \end{array}$$



$$\begin{array}{l} a_{1,1} x_1 + a_{1,2} x_2 \leq b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ a_{4,1} x_1 + a_{4,2} x_2 \leq b_4 \end{array}$$



Focus of Today: (Linear) Optimization Problem



Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of **stir-fry** (ounce) and **boba** (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100 \text{ g}$
- $\text{Calcium} \geq 700 \text{ mg}$

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much **stir-fry** (ounce) and **boba** (fluid ounces) should we buy?

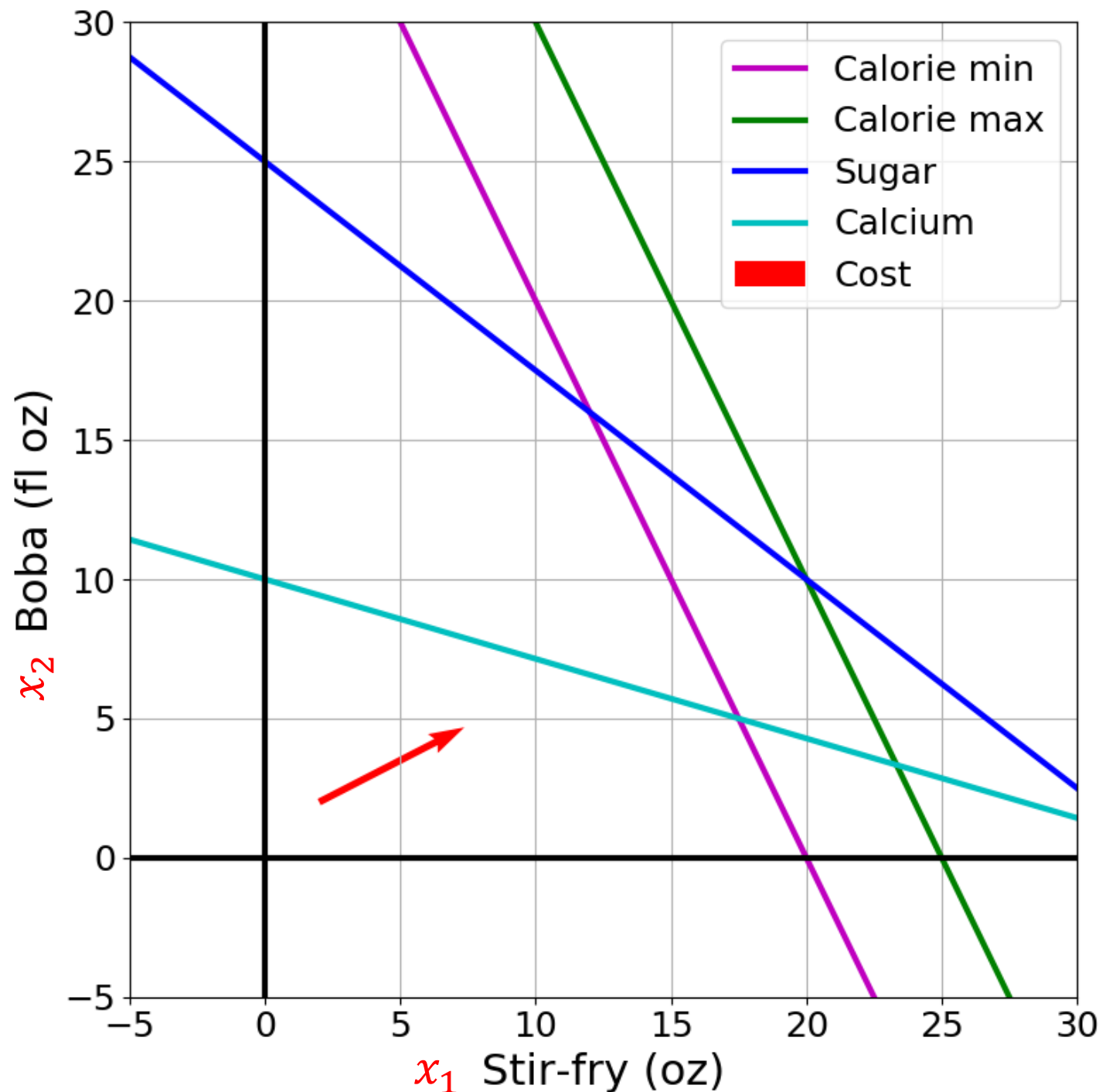
Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100 \text{ g}$
- $\text{Calcium} \geq 700 \text{ mg}$

$$\begin{array}{ll}\min_{x_1, x_2} & 1x_1 + 0.5x_2 \\ \text{s.t.} & 100x_1 + 50x_2 \geq 2000 \\ & 100x_1 + 50x_2 \leq 2500 \\ & 3x_1 + 4x_2 \leq 100 \\ & 20x_1 + 70x_2 \geq 700\end{array}$$

What is the feasible region?

What is the optimal solution?



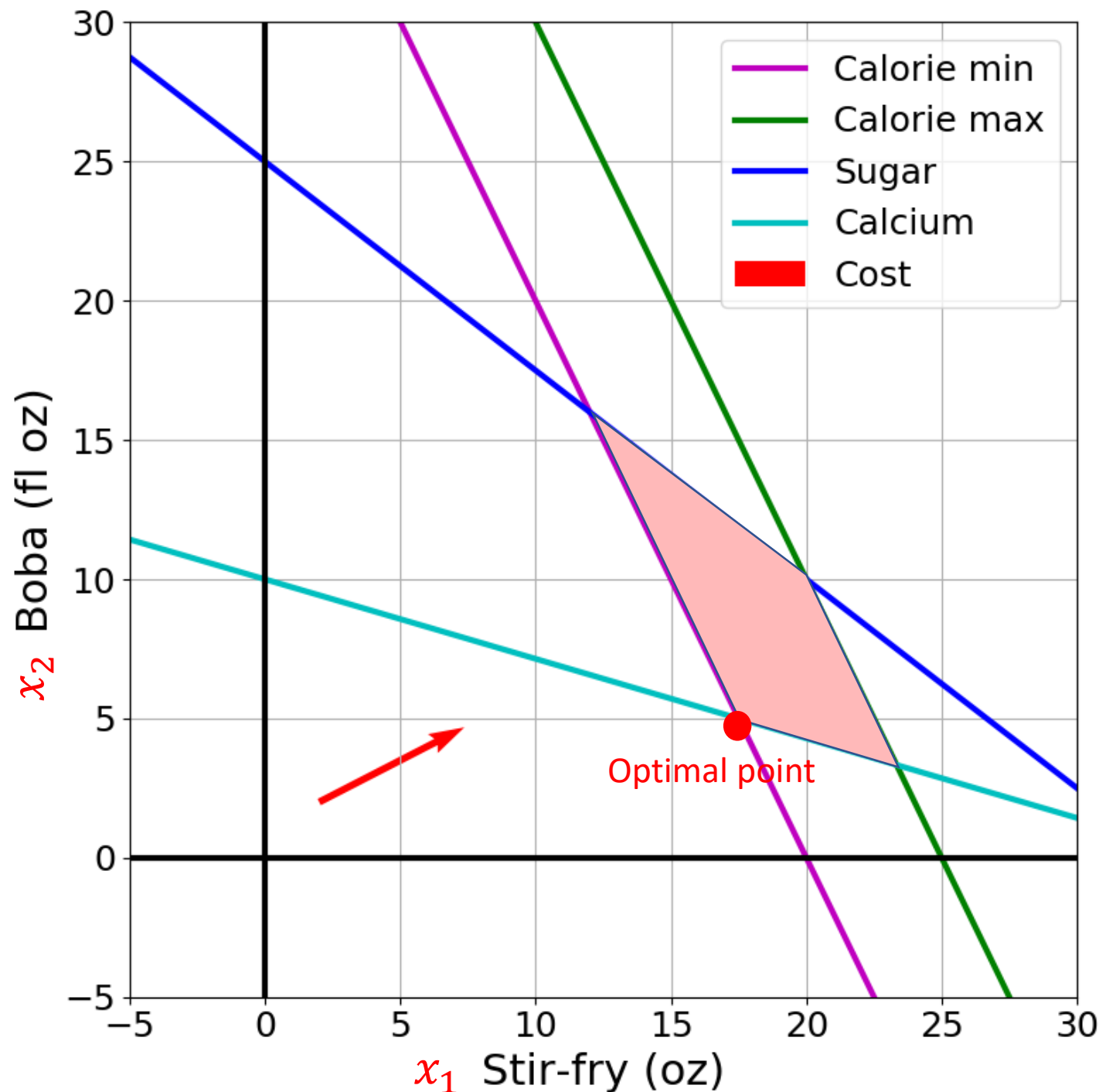
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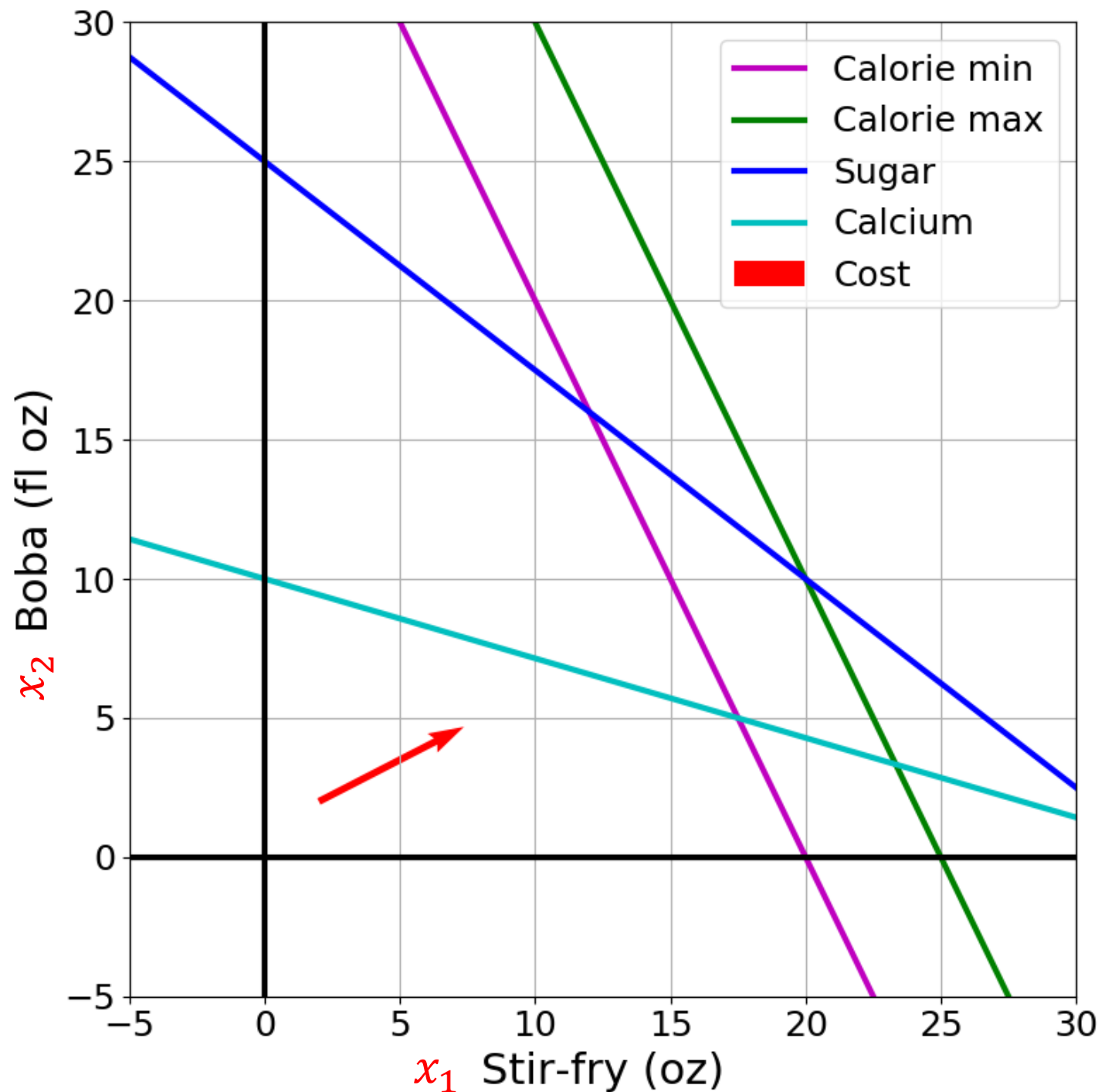
Healthy Squad Goals

- Calories ≥ 2500
- Sugar ≥ 100 g
- Calcium ≥ 700 mg

$$\begin{array}{ll}\min_{x_1, x_2} & 1x_1 + 0.5x_2 \\ \text{s.t.} & 100x_1 + 50x_2 \geq 2500 \\ & 3x_1 + 4x_2 \geq 100\end{array}$$

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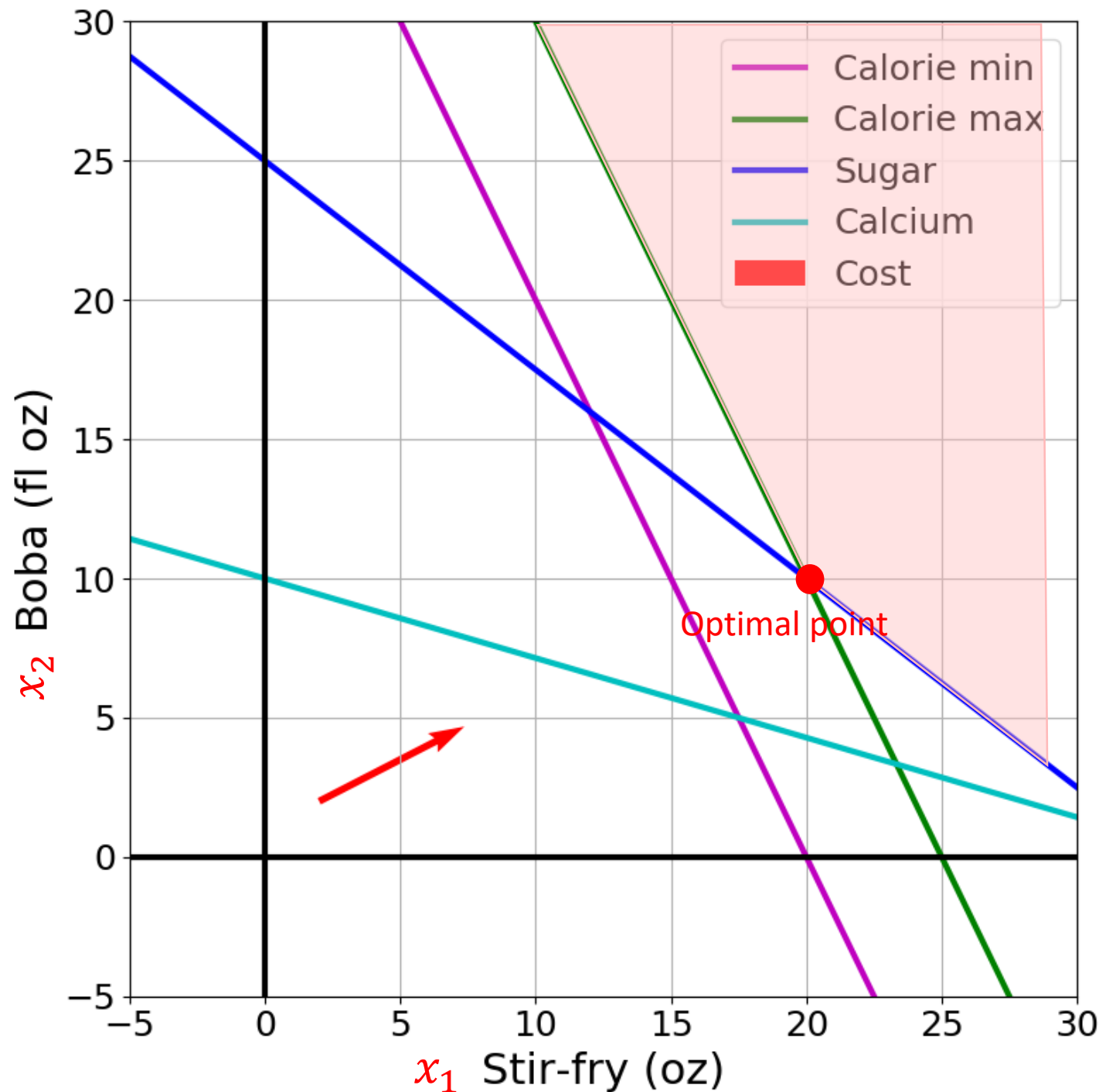
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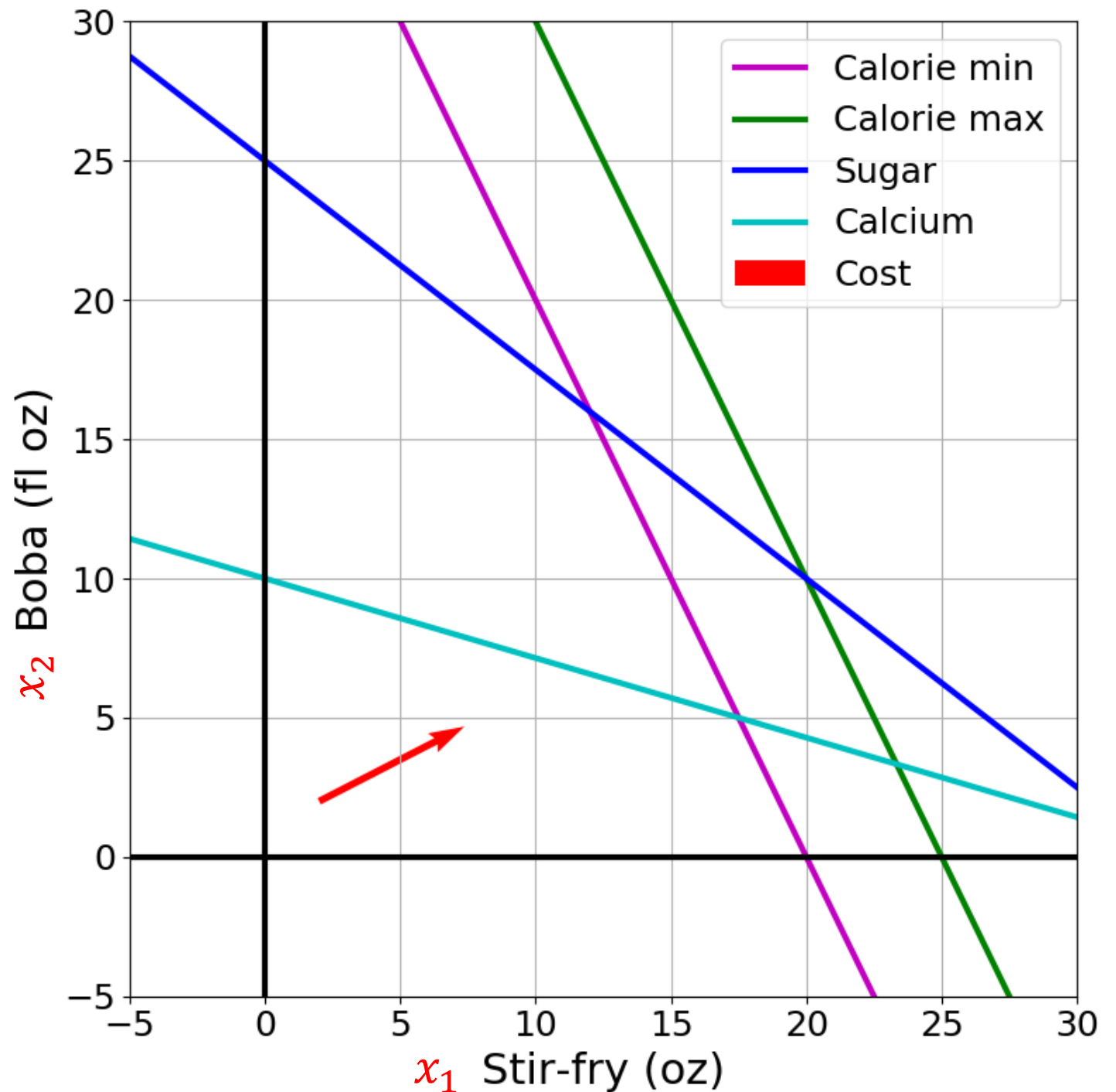
Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\begin{array}{ll}\min_{x_1, x_2} & 1x_1 + 0.5x_2 \\ \text{s.t.} & 100x_1 + 50x_2 \leq 2000 \\ & 3x_1 + 4x_2 \leq 100\end{array}$$

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Healthy Squad Goals

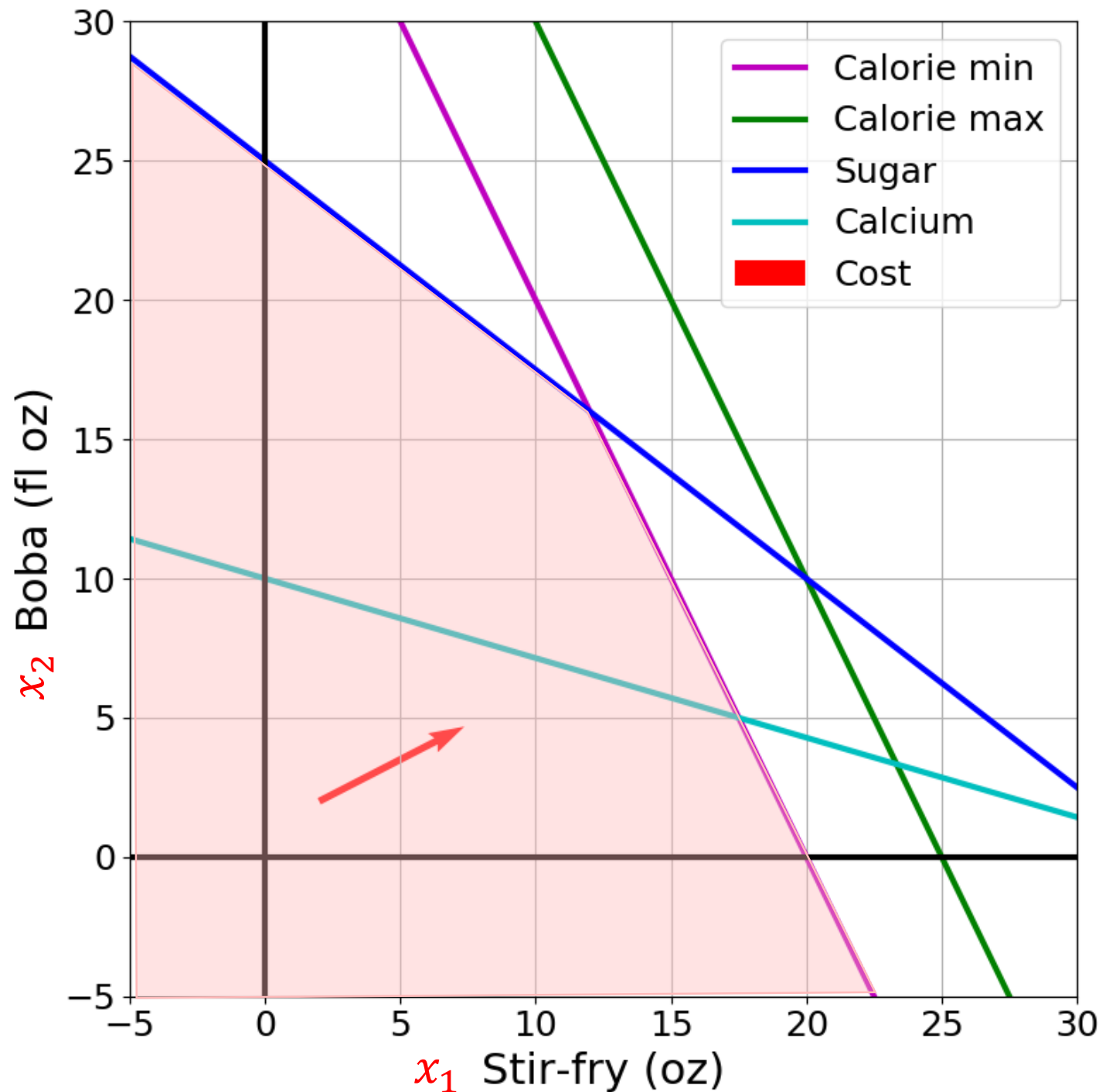
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What is the feasible region?

What is the optimal solution?

Problem unbounded!



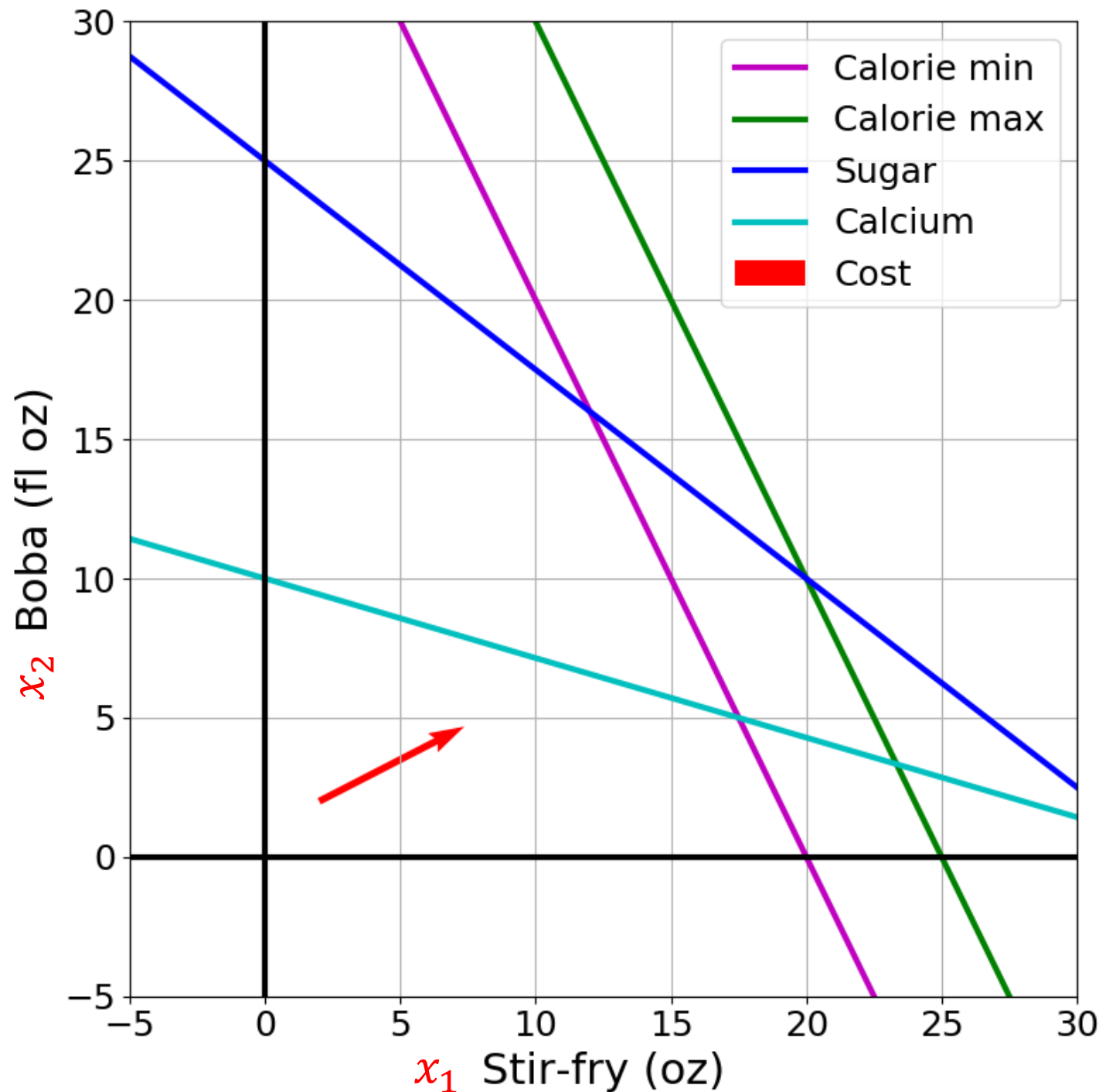
Healthy Squad Goals

- Calories ≤ 2000
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$$\begin{array}{ll}\min & 1x_1 + 0.5x_2 \\ \text{s.t.} & 100x_1 + 50x_2 \leq 2000 \\ & 3x_1 + 4x_2 \leq 100 \\ & x_1 \geq 0 \\ & x_2 \geq 0\end{array}$$

What is the feasible region?

What is the optimal solution?



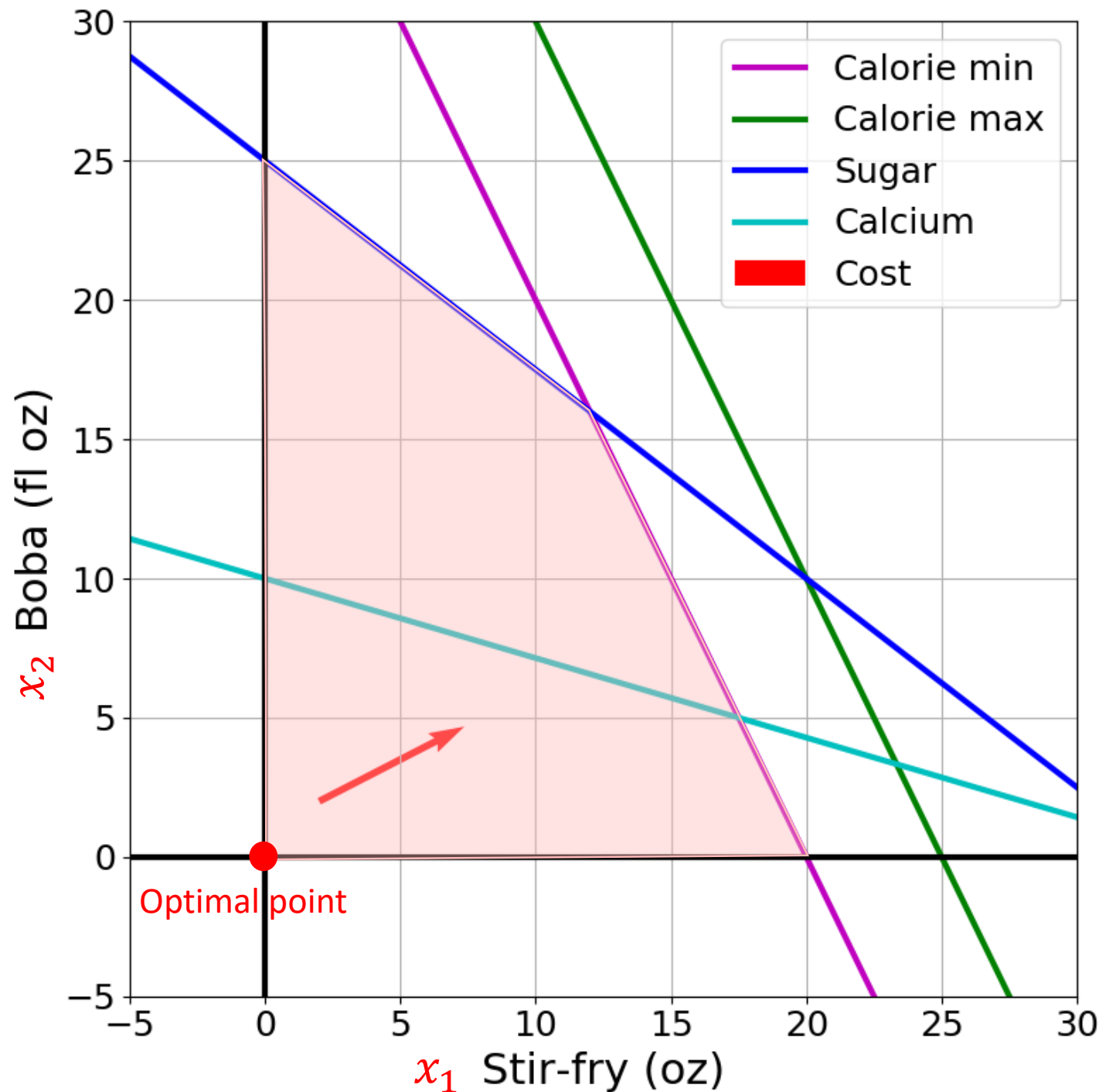
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What is the feasible region?

What is the optimal solution?



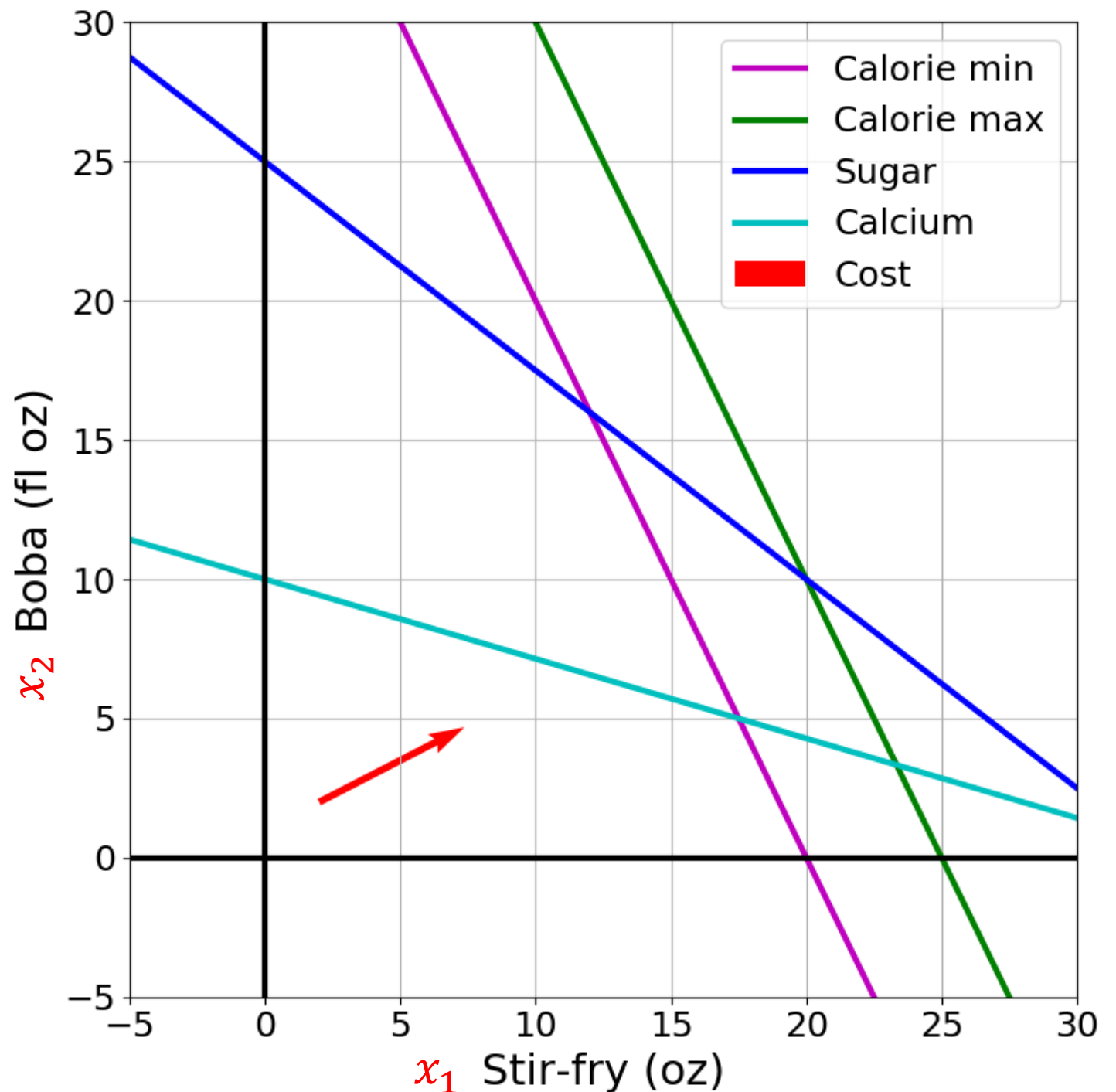
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Healthy Squad Goals

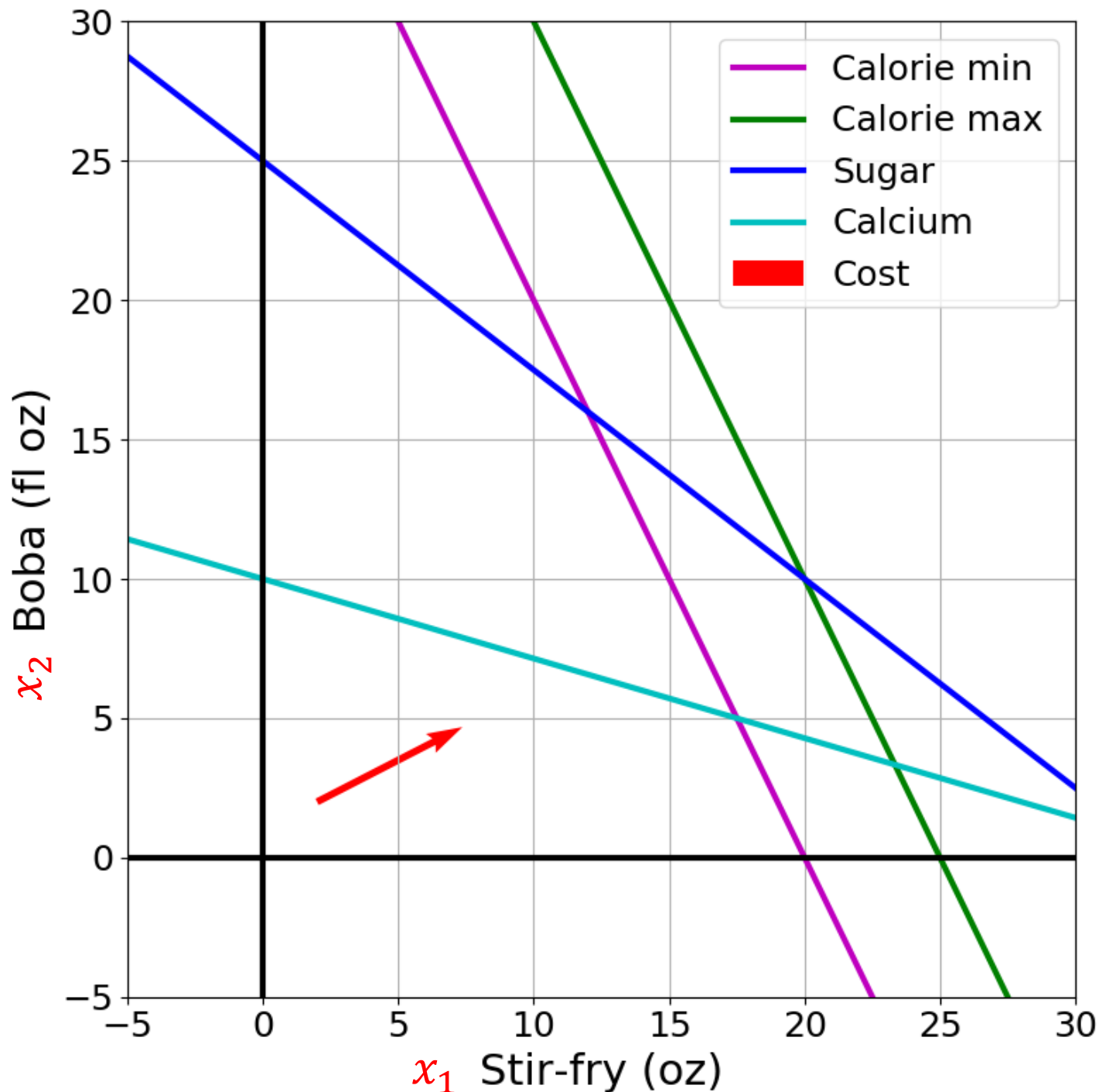
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What is the feasible region?

What is the optimal solution?

Infeasible!

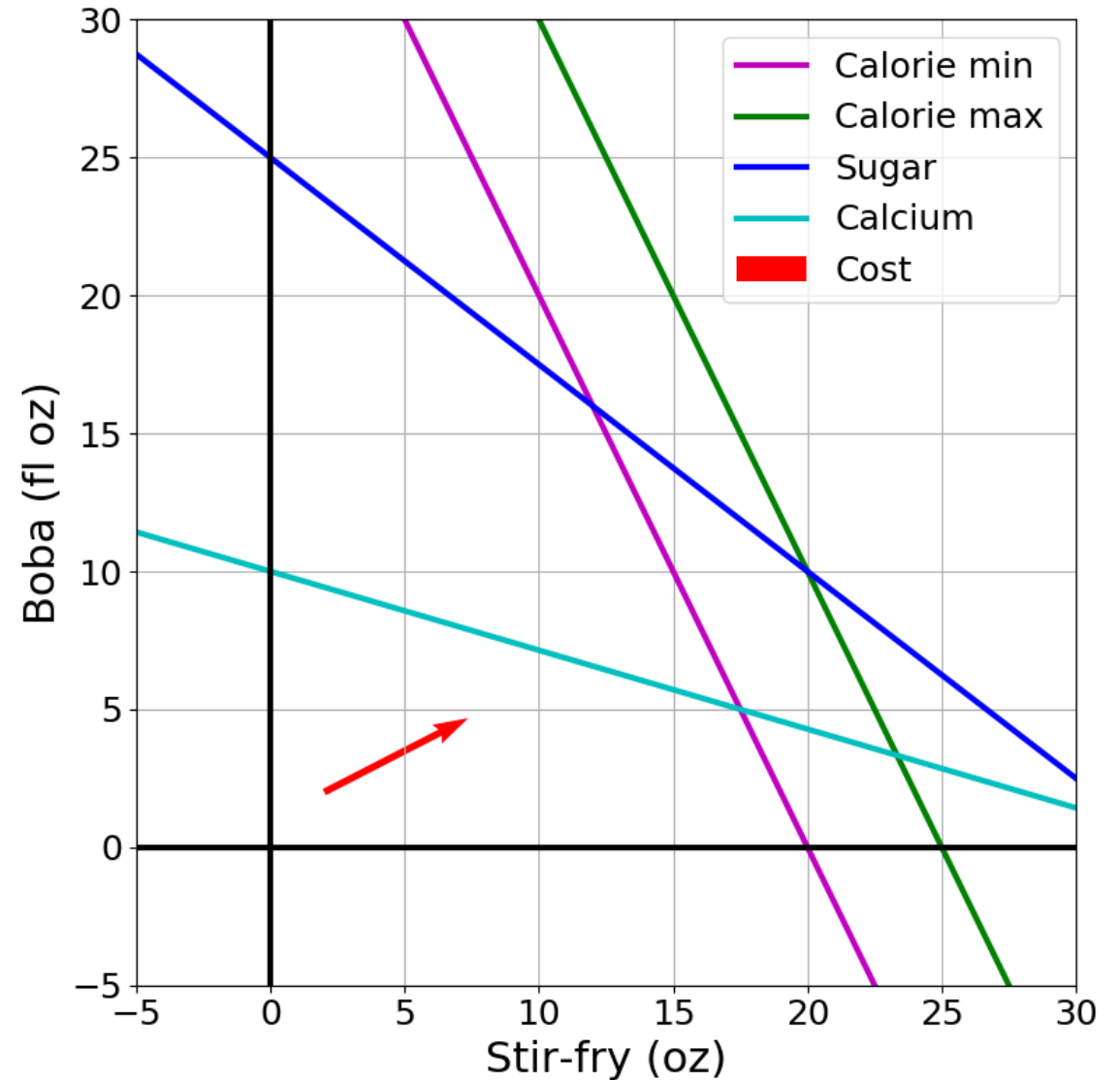


Solving an LP

Solutions are at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value



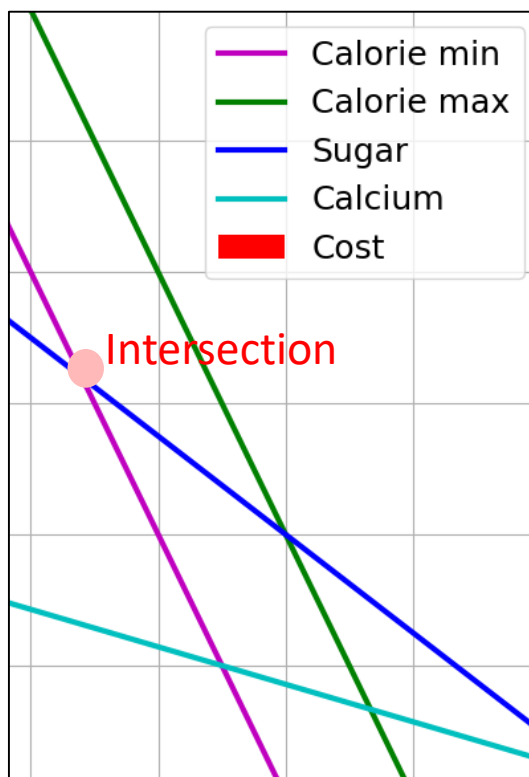
Solving an LP

But, how do we find the intersections?

$$\begin{array}{ll}\min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}\end{array}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



A point is the intersection of two lines

Pick two lines: two rows in $\mathbf{A}$ matrix time $\mathbf{x}$ equals the corresponding rows in $\mathbf{b}$

$$100 x_1 + 50 x_2 = 2000$$

$$3 x_1 + 4 x_2 = 100$$

Not any pair of rows can lead to an intersection

$$100 x_1 + 50 x_2 = 2000$$

$$100 x_1 + 50 x_2 = 2500$$

Solving an LP

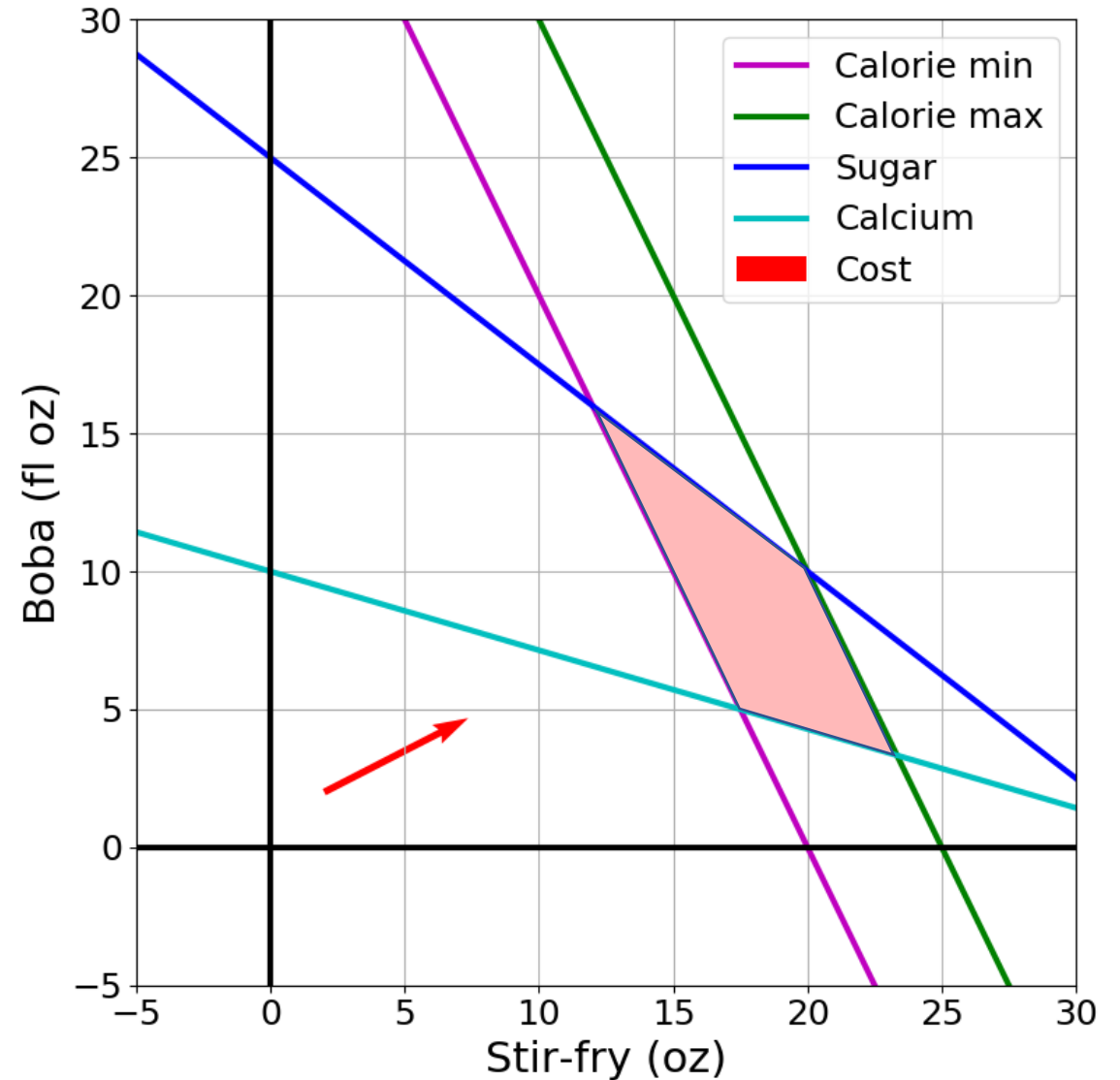
Solutions are at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found

Simplex is most similar to which Local Search algorithm?

Hill Climbing!

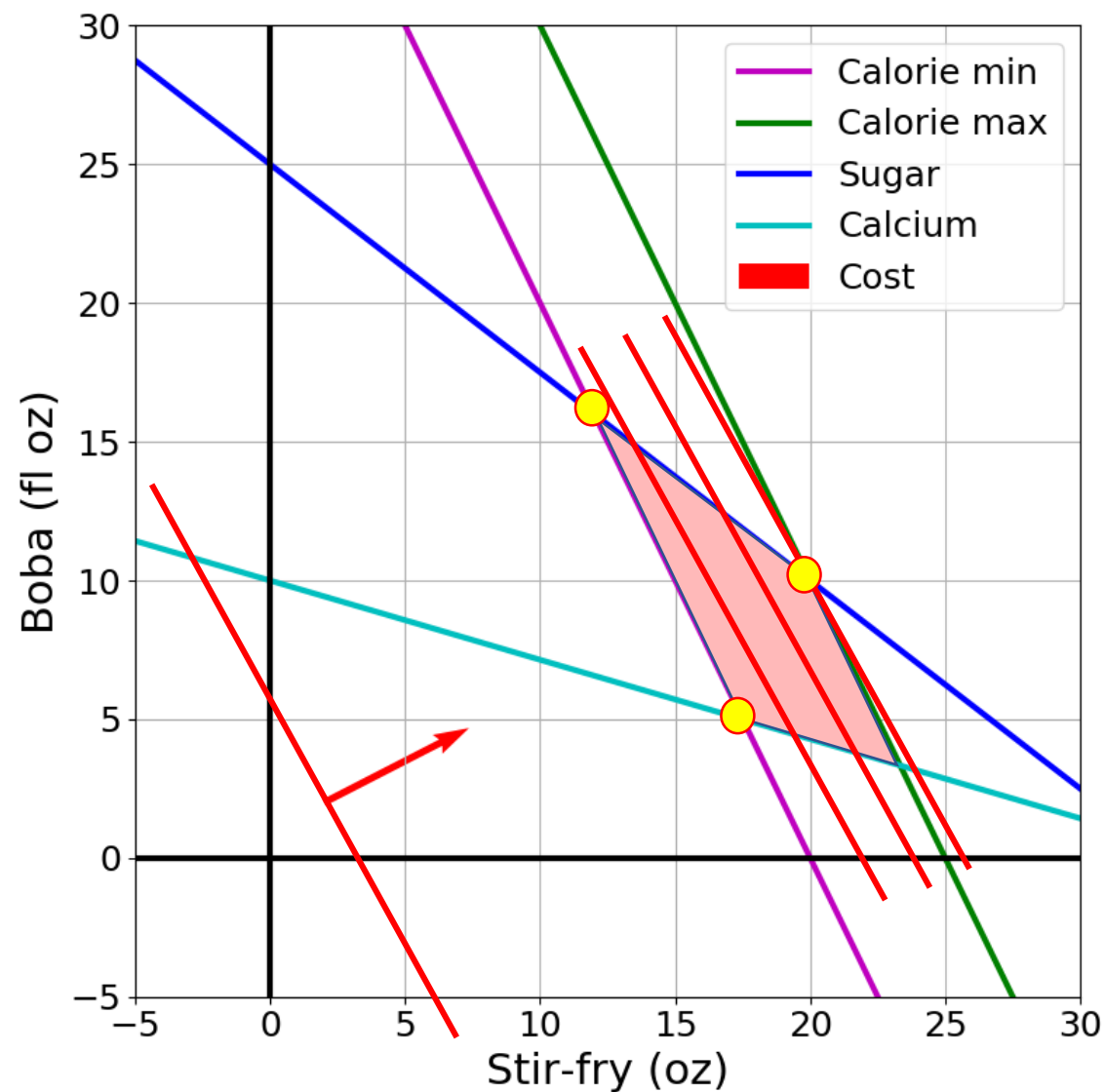


Solving an LP

Solutions are at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found



Solving an LP

But, how do we find a neighboring intersection?

$$\begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Intersection determined by: two rows in $\mathbf{A}$ matrix time $\mathbf{x}$ equals the corresponding rows in $\mathbf{b}$

$$100x_1 + 50x_2 = 2000$$

$$3x_1 + 4x_2 = 100$$

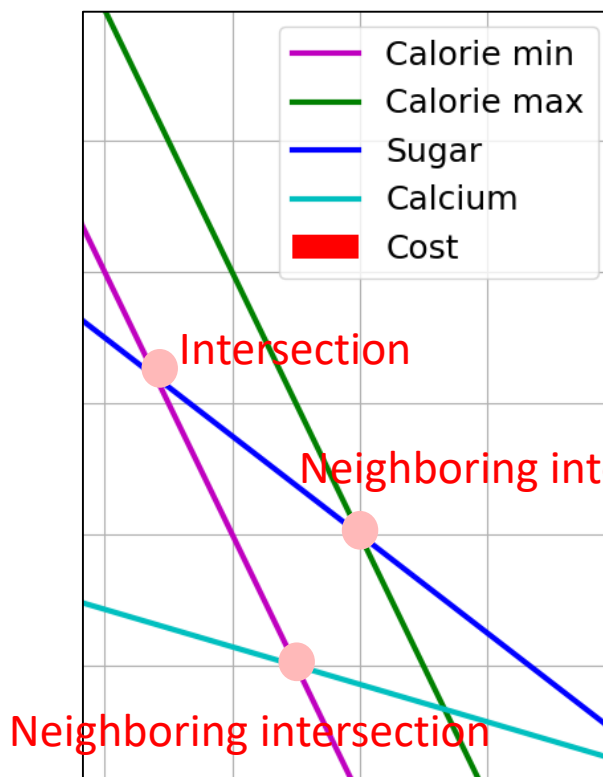
A neighboring intersection only have one different row

$$100x_1 + 50x_2 = 2000$$

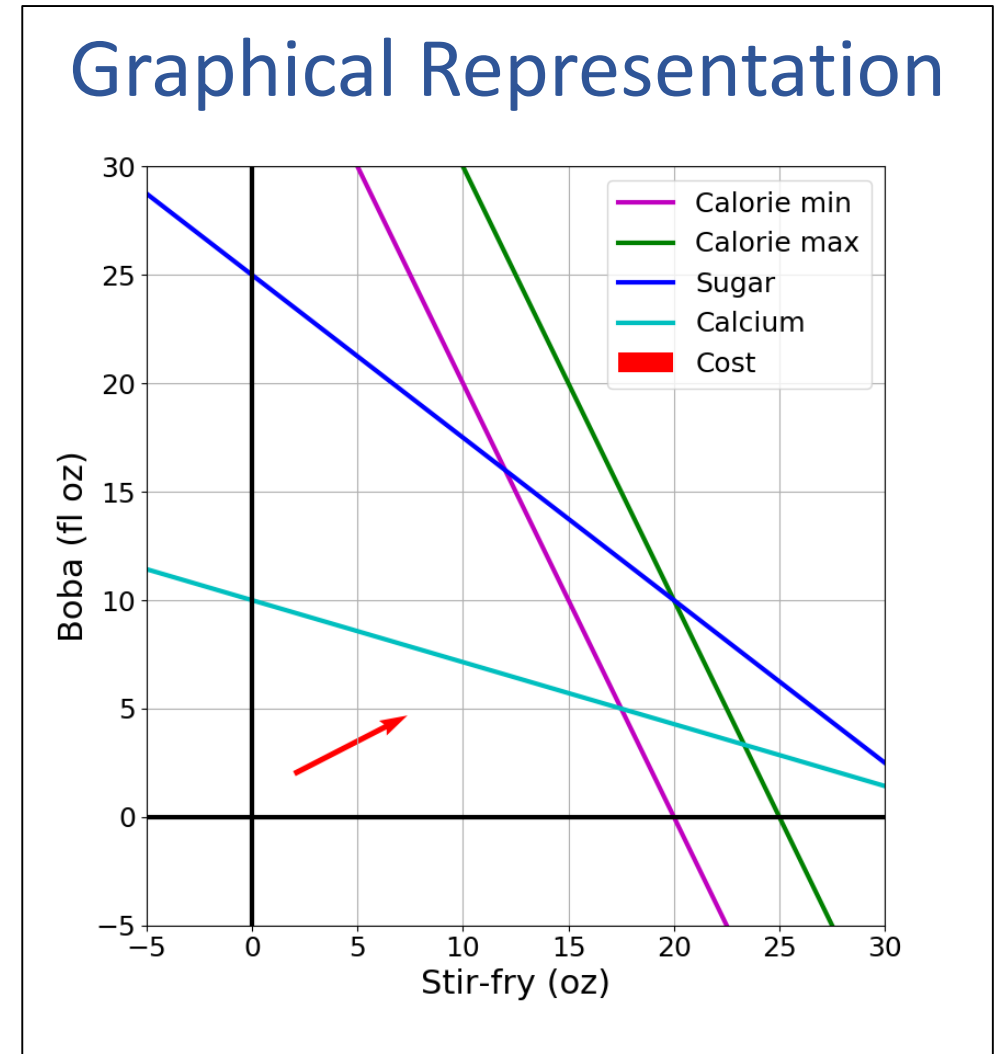
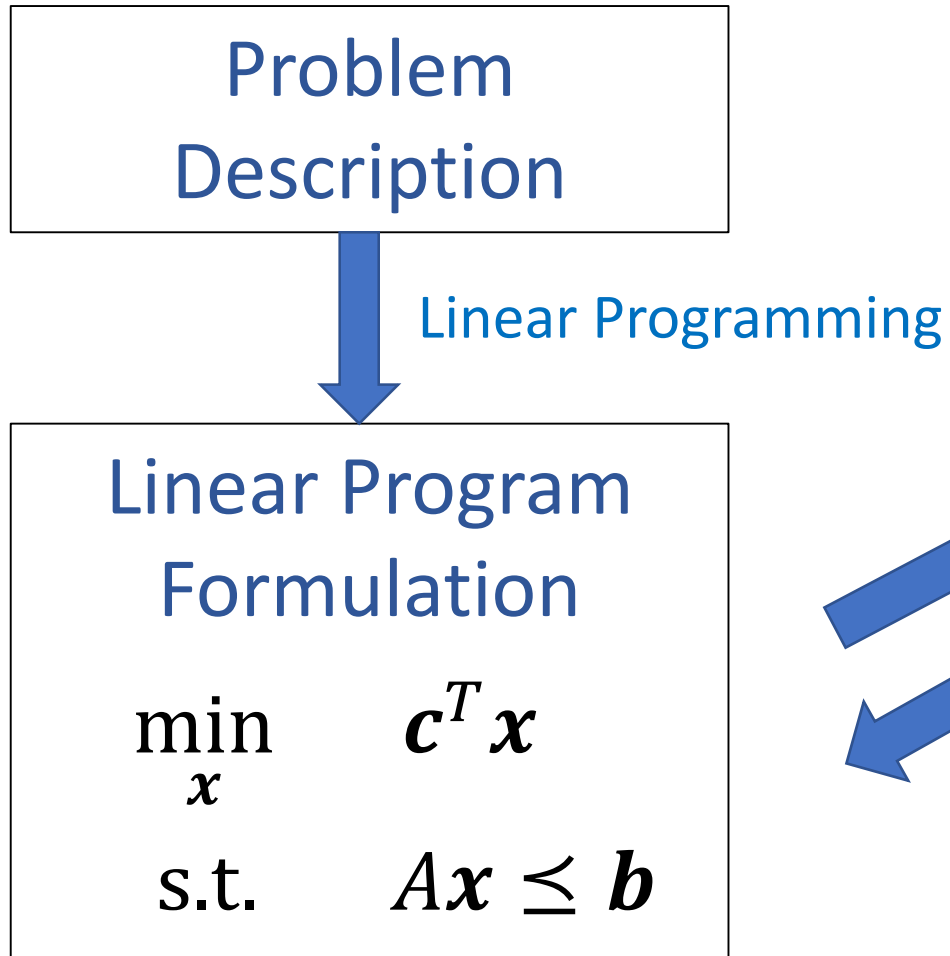
$$20x_1 + 70x_2 = 700$$

$$100x_1 + 50x_2 = 2500$$

$$3x_1 + 4x_2 = 100$$



Focus of Today: (Linear) Optimization Problem



“Marty, your not thinking fourth-dimensionally”



Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$$a_1 x_1 + a_2 x_2 = b_1$$

$$a_1 x_1 + a_2 x_2 \leq b_1$$

$$a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$$

$$a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$$

$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$

$$a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$$

What are intersections in higher dimensions?

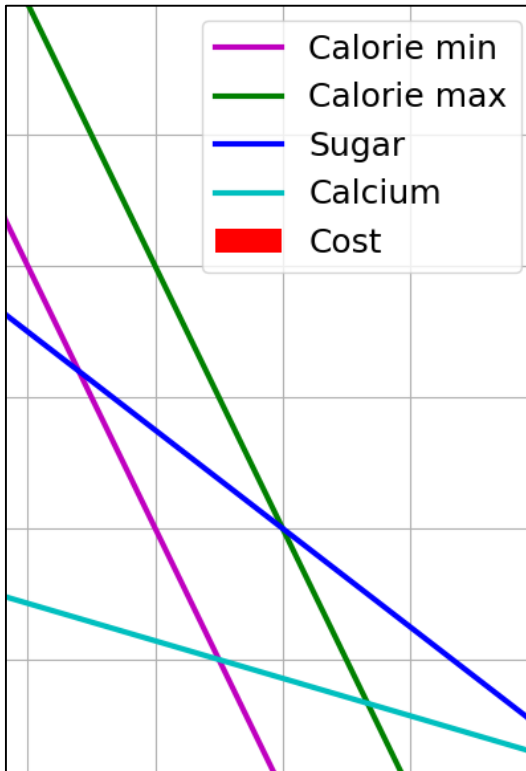
How do these linear shapes extend to 3-D, N-D?

$$\begin{array}{ll}\min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}\end{array}$$

$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



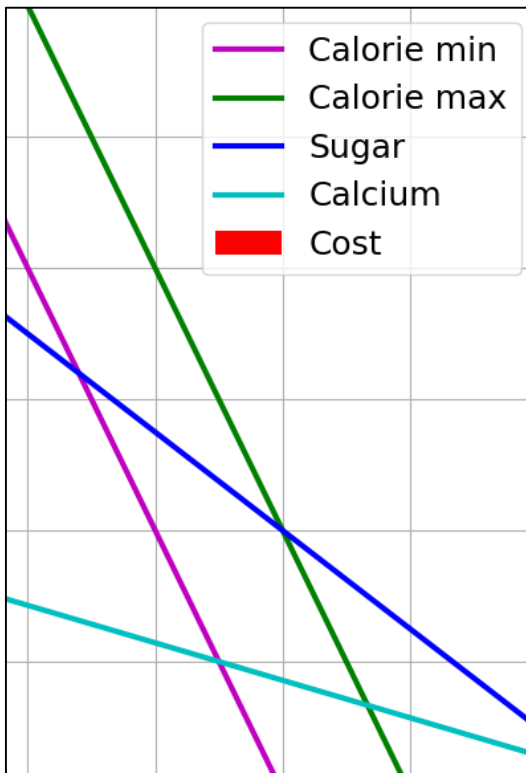
How do we find intersections in higher dimensions?

Still looking at subsets of rows in the A matrix

$$\begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array}$$

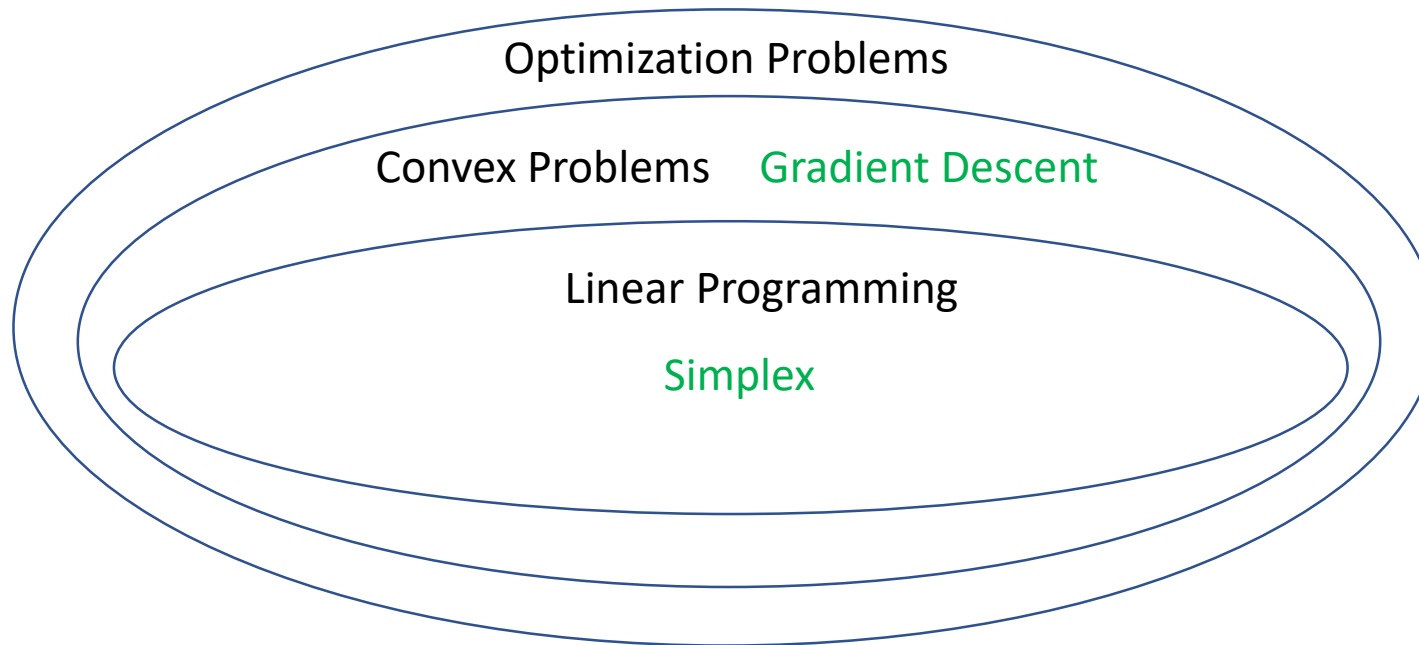
$$\mathbf{A} = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} \quad \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$



Summary

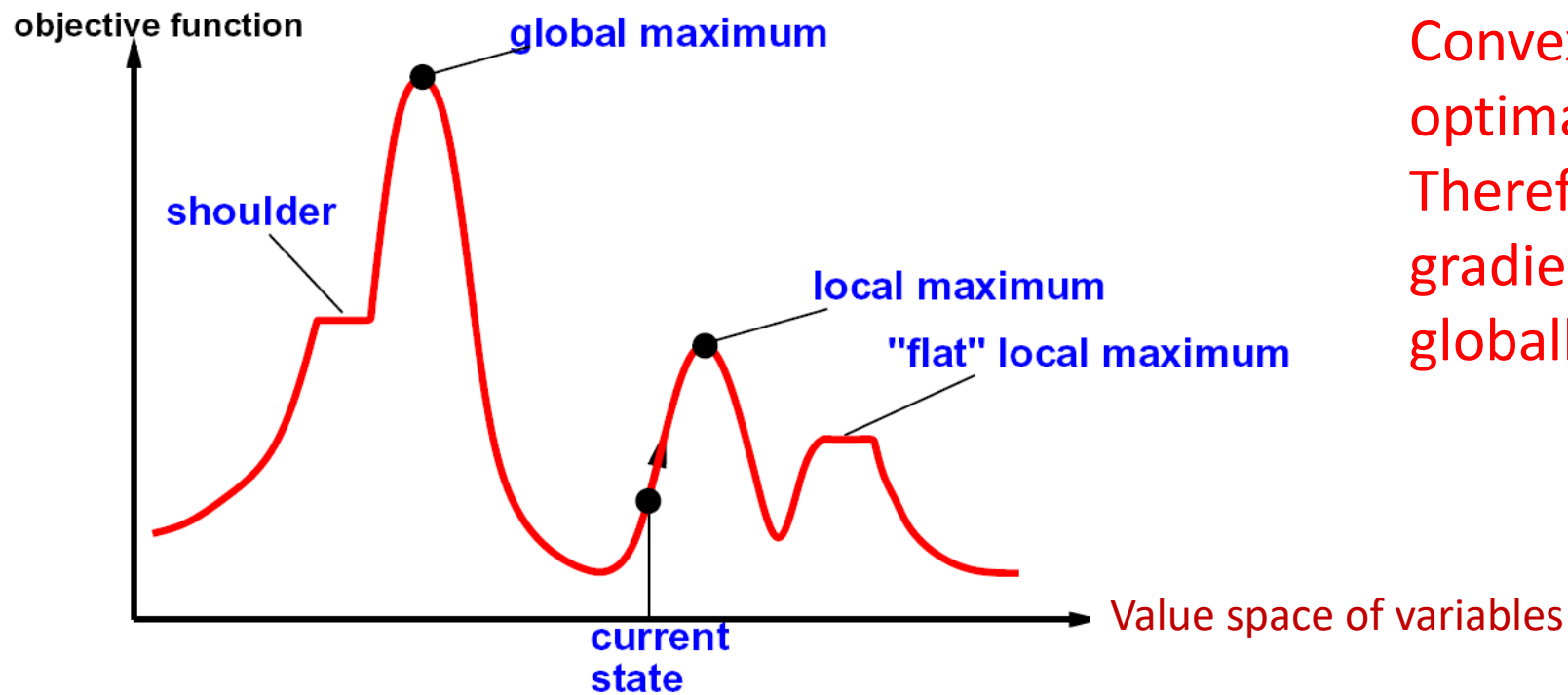
- Linear optimization problem handles continuous space but is closely connected to discrete space (only need to consider vertices)
- For a problem with two variables, graphical representation is helpful
- LP is a special class of optimization problems



Gradient Descent/Ascent

Find $x \in [a, b]$ that minimize / maximizes $f(x)$?

- Gradient descent / ascent
 - Use gradient to find best direction
 - Use the magnitude of the gradient to determine how big a step you move



Convex Problems: Any local optima is global optima. Therefore, for convex problems, gradient descent/ascent leads to globally optimal solution.