Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

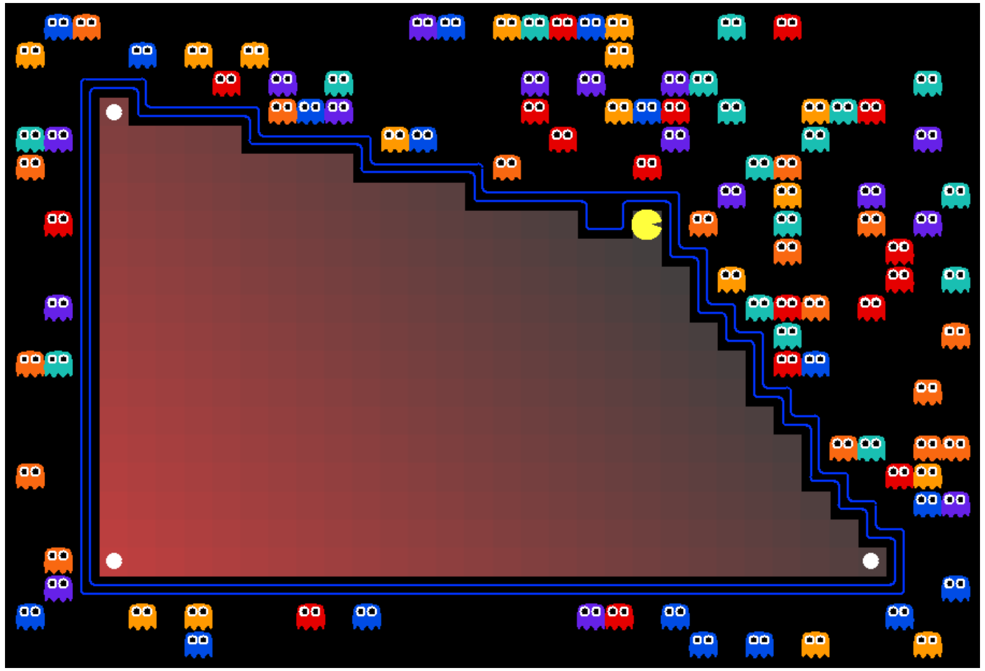
Announcements

Assignments:

- P2
  - Due ~~Thu 10/3, 10 pm~~ Sat 10/5, 10 pm
- P3
  - Will be released later today
  - Due Thu 10/17, 10 pm
- HW5 (written)
  - Released Tue 10/1
  - Due Tue 10/8, 10 pm

AI: Representation and Problem Solving

Optimization & Linear Programming



Instructors: Fei Fang & Pat Virtue

Slide credits: CMU AI, http://ai.berkeley.edu

Learning Objectives

- Formulate a problem as a Linear Program (LP)
- Convert a LP into a required form, e.g., inequality form
- Plot the graphical representation of a linear optimization problem with two variables and find the optimal solution
- Understand the relationship between optimal solution of an LP and the intersections of constraints
- Describe and implement a LP solver based on vertex enumeration
- Describe the high-level idea of Simplex algorithm

Next Lecture

Recap

What have we learned so far?

Search ; Adversarial Search , Local Search

Constraint S.P.

Logic / Classical Planning

What do they have in common?

Discrete , Search-related

This lecture:

(1) Continuous space

(2) General formulation

Recap

What have we learned so far?

- Search: Depth/Breadth-first search, A* search, local search
- Constraint Satisfaction Problem: 8-queen, graph coloring
- Logic and Planning: Propositional Logic, SAT, First-order Logic

What do they have in common?

  - Variables/Symbols, Finite options

This lecture:

(1) Move to continuous space (with connections to the discrete space)

(2) Provide a general formulation that can be used to represent many of the previously seen problems

Focus of Today: (Linear) Optimization Problem

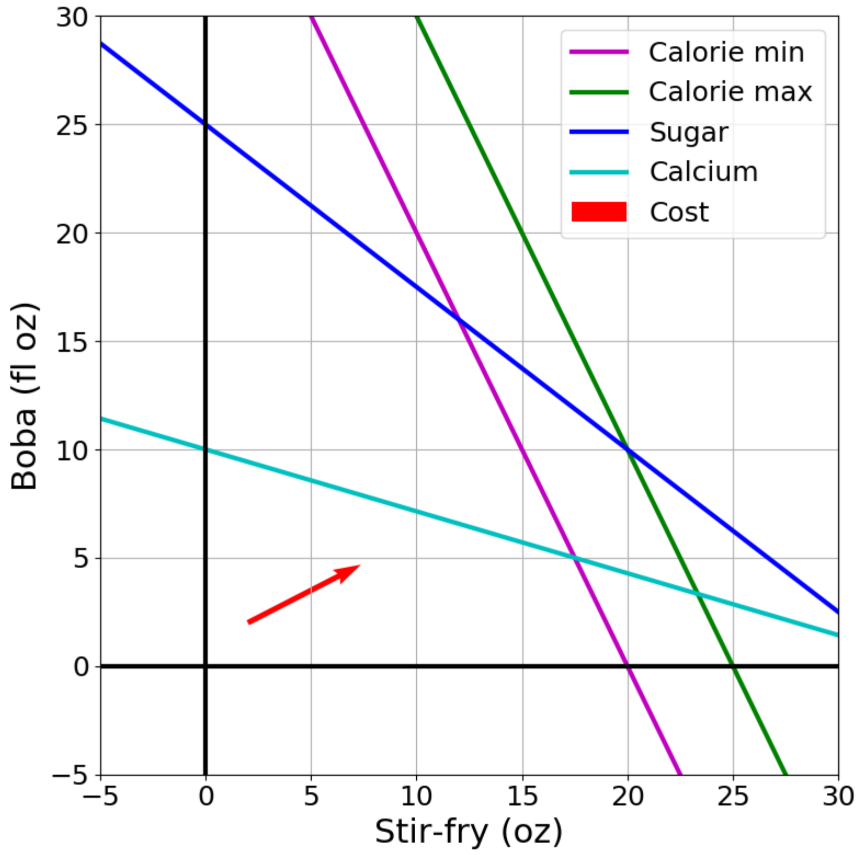
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Graphical Representation



Notation Alert!

Diet Problem: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

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Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation

Can we formulate it as a Constraint Satisfaction Problem? → Discrete

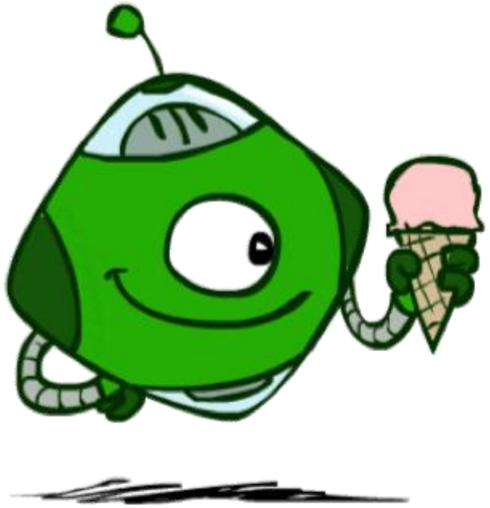
Variable: x_1, x_2

Domain: $[0, +\infty)$

Constraint: $2000 \leq 100x_1 + 50x_2 \leq 2500$

$3x_1 + 4x_2 \leq 100$

$20x_1 + 70x_2 \geq 700$



What are the issues with this CSP formulation?

Healthy Squad Goals

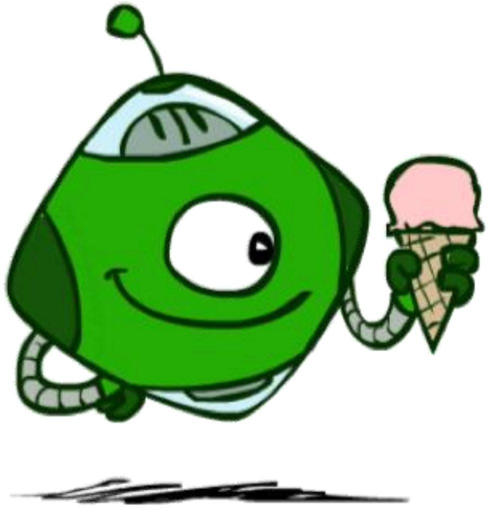
- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

	Food	Cost	Calories	Sugar	Calcium
x_1	(amount of) Stir-fry (per oz)	1	100	3	20
x_2	Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation



Can we formulate it as a Constraint Satisfaction Problem?

Variable: x_1 (ounces for stir-fry), x_2 (ounces for boba)

Domain: $[0, +\infty)$

Constraint: Implicit: $100\ x_1 + 50\ x_2 \in [2000,2500]$

$$3\ x_1 + 4\ x_2 \leq 100,$$

$$20\ x_1 + 70\ x_2 \geq 700$$

What are the issues with this CSP formulation?

Healthy Squad Goals

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- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
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Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation

Optimization problem: Finding the **best** solution from all **feasible** solutions

From CSP to Optimization Problem

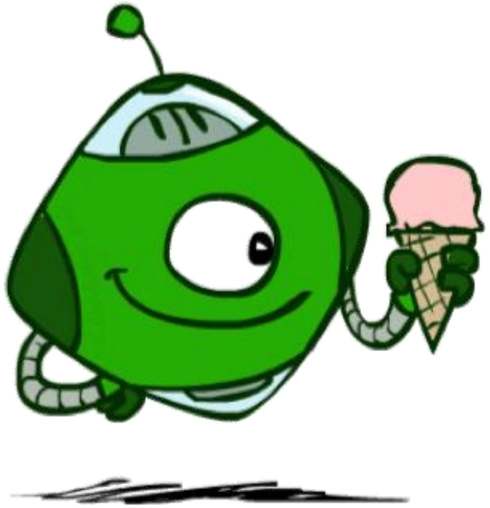
- Variable ✓
- ~~Domain~~ $[0,+\infty) \Rightarrow x_1 \geq 0,\ x_2 \geq 0$
- Constraint
- objective

$$\min_{x} \quad f(x) \qquad \text{Objective}$$

$$\text{s.t.} \quad x \ \ satisfies\ \ constraints$$

subject to

Notation Alert!



Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Optimization problem: Finding the best solution from all feasible solutions

From CSP to Optimization Problem

(Optimization) Variable

~~Domain~~ Can be represented as constraints

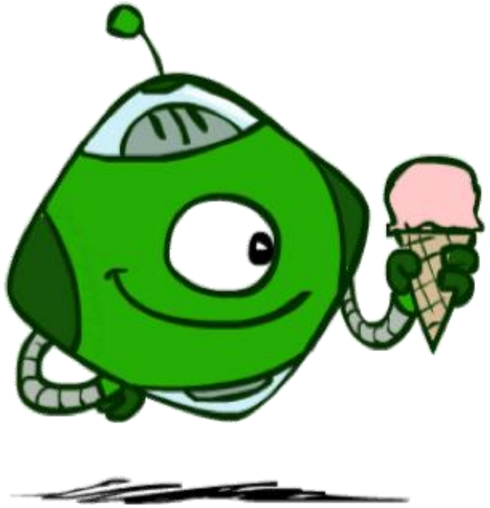
Constraint

Optimization Objective

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \qquad \text{Objective}$$

$$\text{s.t.} \quad \boldsymbol{x} \;\; satisfies \;\; constraints$$

Notation Alert!



Healthy Squad Goals

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- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x})$$

$$\text{s.t.} \quad \boldsymbol{x} \ \ satisfies \ \ constraints$$

Variable: x_1, x_2

Objective: $\min_{x_1, x_2} \ 1 \cdot x_1 + 0.5 \cdot x_2$

Constraints:

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

	Food	Cost	Calories	Sugar	Calcium
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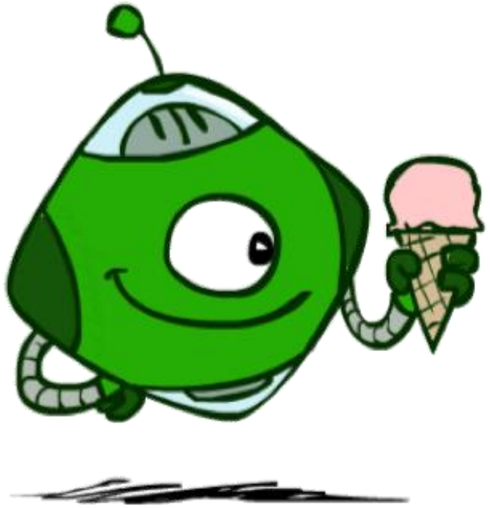

Variable: $\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

Objective: $\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$

Constraints:

$$\text{calories}(\boldsymbol{x}) \; in \; required \; range$$

$$\text{sugar}(\boldsymbol{x}) \leq limit$$

$$\text{calcium}(\boldsymbol{x}) \geq limit$$

Can be ignored in this problem. Why?

~~$x_1 \geq 0, x_2 \geq 0$~~

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$$
$$\text{s.t.} \quad \text{calories}(\boldsymbol{x}) \textit{ in required range}$$
$$\text{sugar}(\boldsymbol{x}) \leq \textit{limit}$$
$$\text{calcium}(\boldsymbol{x}) \geq \textit{limit}$$

$\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

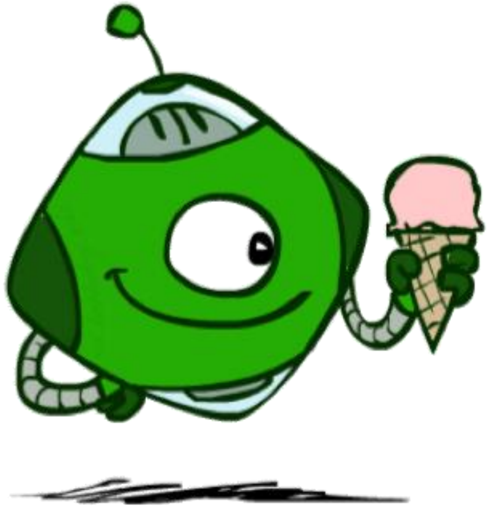
What is the expression of

$\text{cost}(\boldsymbol{x})$

$\text{calories}(\boldsymbol{x})$

$\text{sugar}(\boldsymbol{x})$

$\text{calcium}(\boldsymbol{x})$



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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$$
$$\text{s.t.} \quad \text{calories}(\boldsymbol{x}) \textit{ in required range}$$
$$\text{sugar}(\boldsymbol{x}) \leq \textit{limit}$$
$$\text{calcium}(\boldsymbol{x}) \geq \textit{limit}$$

$\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

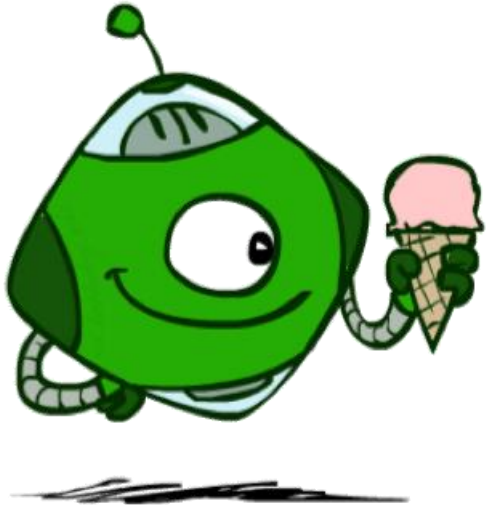
What is the expression of

$$\text{cost}(\boldsymbol{x}) = x_1 + 0.5x_2$$

$$\text{calories}(\boldsymbol{x}) = 100x_1 + 50x_2$$

$$\text{sugar}(\boldsymbol{x}) = 3x_1 + 4x_2$$

$$\text{calcium}(\boldsymbol{x}) = 20x_1 + 70x_2$$



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Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Formulate Diet Problem as an optimization problem

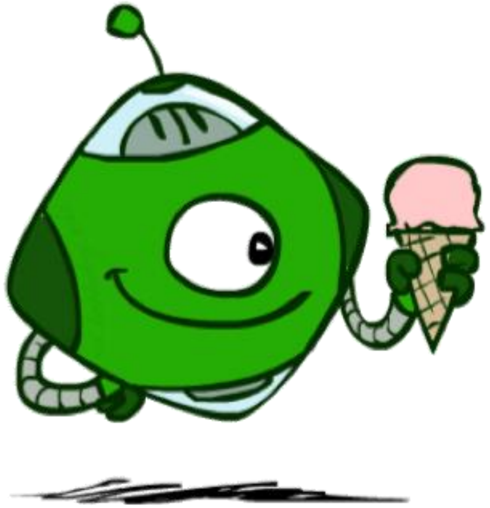
$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2 \quad \text{(cost)}$$

$$\text{s.t.} \quad 100\, x_1 + 50\, x_2 \geq 2000 \quad \text{(calories LB)}$$

$$100\, x_1 + 50\, x_2 \leq 2500 \quad \text{(UB)}$$

$$3\, x_1 + 4\, x_2 \leq 100 \quad \text{(sugar)}$$

$$20\, x_1 + 70\, x_2 \geq 700 \quad \text{(calcium)}$$



Healthy Squad Goals

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Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Piazza Poll 1

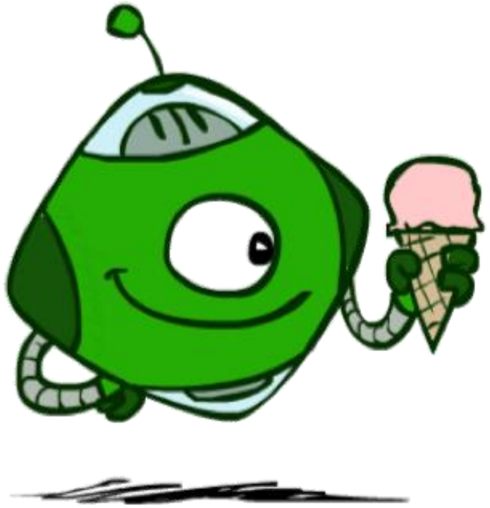
Is $x_1 = 10, x_2 = 30$ a feasible solution of the following optimization problem?

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

100×10 + 50×30 = 2500

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000 \quad ✓$$
$$100\,x_1 + 50\,x_2 \leq 2500 \quad ✓$$
$$3\,x_1 + 4\,x_2 \leq 100 \quad ✗$$
$$20\,x_1 + 70\,x_2 \geq 700$$

3×10 + 4×30



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
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Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Linear Programming

Linear Programming:

- Technique for the optimization of a linear objective function subject to linear equality and linear inequality constraints

Example Linear program

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$$\text{s.t.} \quad 100\, x_1 + 50\, x_2 \geq 2000$$
$$100\, x_1 + 50\, x_2 \leq 2500$$
$$3\, x_1 + \; 4\, x_2 \leq 100$$
$$20\, x_1 + 70\, x_2 \geq 700$$

Linear program in Inequality Form

$$\min_{\boldsymbol{x}} . \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a} + \boldsymbol{b}$? $\boldsymbol{a} - \boldsymbol{b}$?

$$a \pm b = \begin{bmatrix} a_1 \pm b_1 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

What does $\boldsymbol{a} \leq 0$ mean?

$a_1 \leq 0$

$a_2 \leq 0$

$\vdots$

$a_n \leq 0$

$\forall i, \quad a_i \leq 0$

(Note: It is fine if you directly write $\boldsymbol{a} \leq 0$)

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a} + \boldsymbol{b}$? $\boldsymbol{a} - \boldsymbol{b}$?

$$\boldsymbol{a} \pm \boldsymbol{b} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

What does $\boldsymbol{a} \leq 0$ mean?

$$\forall i, a_i \leq 0$$

(Note: It is fine if you directly write $\boldsymbol{a} \leq 0$)

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a}^T \boldsymbol{b}$?

$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$$Ab = \begin{bmatrix} a_{11} b_1 + \cdots + a_{1m} b_m \\ \\ a_{n1} b_1 + \cdots a_{nm} b_m \end{bmatrix}$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $A\boldsymbol{b}$?

$$\begin{bmatrix} a_{11} b_1 + a_{21} b_2 + \cdots a_{n1} b_n \\ a_{12} b_1 + a_{22} b_2 + \cdots a_{n2} b_n \\ \vdots \\ a_{1m} b_1 + a_{2m} b_2 + \cdots a_{nm} b_n \end{bmatrix}$$

Ab

$n \times 1$

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a}^T\boldsymbol{b}$?

$$\boldsymbol{a}^T\boldsymbol{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$
$$= \sum_{i=1}^{n} a_ib_i$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $A\boldsymbol{b}$?

$$A\boldsymbol{b} = \begin{bmatrix} a_{1,1}b_1 + a_{1,2}b_2 + \cdots a_{1,m}b_m \\ \vdots \\ a_{n,1}b_1 + a_{n,2}b_2 + \cdots a_{n,m}b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1,i}b_i \\ \vdots \\ \sum_{i=1}^{m} a_{n,i}b_i \end{bmatrix}$$

Problem Formulation

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\begin{aligned} \text{s.t.} \quad 100\,x_1 + 50\,x_2 &\geq 2000 \\ 100\,x_1 + 50\,x_2 &\leq 2500 \\ 3\,x_1 + \quad 4\,x_2 &\leq 100 \\ 20\,x_1 + 70\,x_2 &\geq 700 \end{aligned}$$

$(-1)*(100x_1 + 50x_2) \geq (2000)(-1)$? $\Rightarrow -100x_1 - 50x_2 \leq -2000$

$\Rightarrow -20x_1 - 70x_2 \leq -700$

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$c^T \mathbf{x}$ $\quad c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Problem Formulation

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$\boldsymbol{c}^T \boldsymbol{x}$ with $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\text{s.t.} \quad 100\, x_1 + 50\, x_2 \geq 2000$$

$$-100\, x_1 - 50\, x_2 \leq -2000$$

$$100\, x_1 + 50\, x_2 \leq 2500$$

$$3\, x_1 + 4\, x_2 \leq 100$$

$$20\, x_1 + 70\, x_2 \geq 700$$

$$-20\, x_1 - 70\, x_2 \leq -700$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Problem Formulation

$$\min_{x_1, x_2} \quad \boldsymbol{c}^T \boldsymbol{x}$$

s.t.

$$-100x_1 - 50x_2 \leq -2000$$

$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + \;\; 4\,x_2 \leq 100$$

$$-20\,x_1 - 70\,x_2 \leq -700$$

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Problem Formulation

$$\min_{x_1, x_2} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad -100\, x_1 - 50\, x_2 \le -2000$$

$$100\, x_1 + 50\, x_2 \le 2500$$

$$3\, x_1 + \quad 4\, x_2 \le 100$$

$$-20\, x_1 - 70\, x_2 \le -700$$

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Piazza Poll 2

What has to increase to add more nutrition constraints?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) height A

D) width A

E) length $\boldsymbol{b}$

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- …(More Constraints)

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Piazza Poll 2

What has to increase to add more nutrition constraints?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

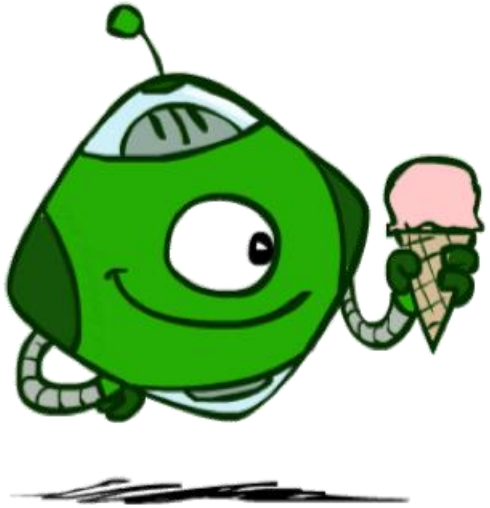
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

$-3 \quad -4 \qquad -20$

$3x_1 + 4x_2 \geq 20$

$-3x_1 - 4x_2 \leq -20$



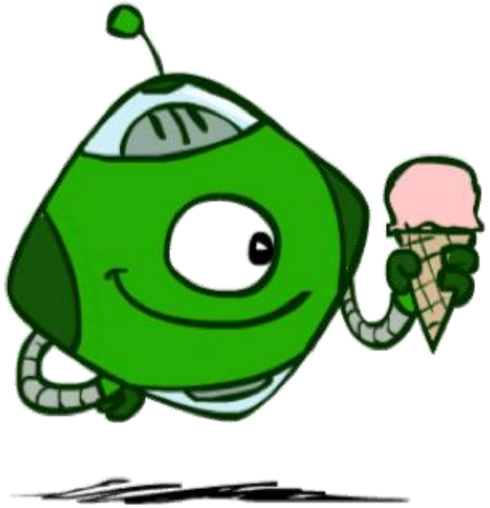
Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- Sugar ≥ 20 g

	Food	Cost	Calories	Sugar	Calcium
x_1	Stir-fry (per oz)	1	100	3	20
x_2	Boba (per fl oz)	0.5	50	4	70

Piazza Poll 2

What has to increase to add more nutrition constraints?



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

$$3x_1 + 4x_2 \geq 20$$

$$-3x_1 - 4x_2 \leq -20$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \\ -3 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \\ -20 \end{bmatrix}$$

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- Sugar ≥ 20 g

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Question

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

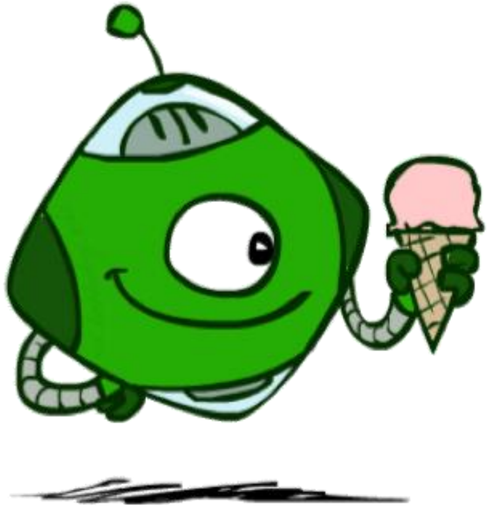
✓ A) length $\boldsymbol{x}$

✓ B) length $\boldsymbol{c}$

C) height A

✓ D) width A

E) length $\boldsymbol{b}$



Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
New Item	…	…	…	…

Question

What has to increase to add more menu items?

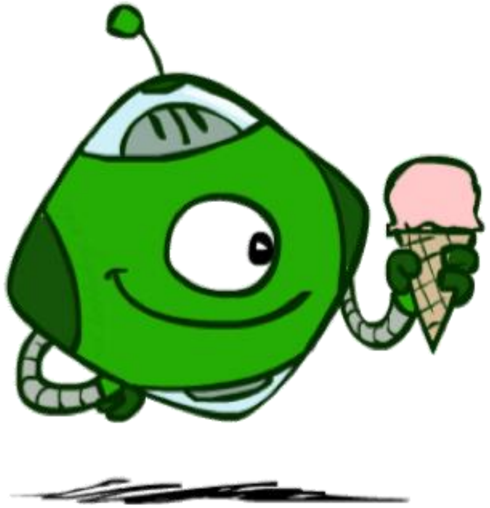
$x_1 + 0.5x_2 + 2x_3$

$100x_1 + 50x_2 + 80x_3 \leq 2500$

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 & \\ 100 & 50 & 80 \\ 3 & 4 & \\ -20 & -70 & \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

	Food	Cost	Calories	Sugar	Calcium
x_1	Stir-fry (per oz)	1	100	3	20
x_2	Boba (per fl oz)	0.5	50	4	70
x_3	Beef (per oz)	2	80	1	30

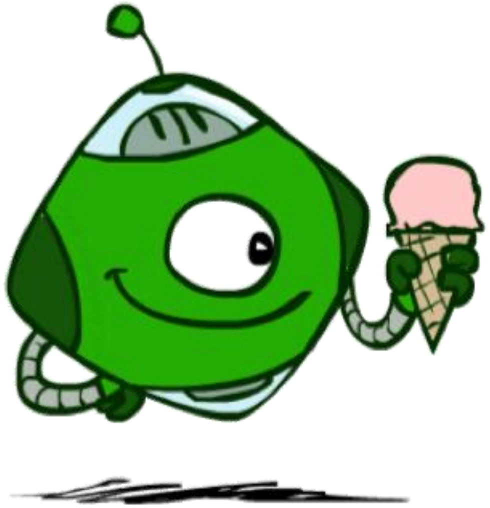
Question

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 & -80 \\ 100 & 50 & 80 \\ 3 & 4 & 1 \\ -20 & -70 & -30 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
Beef (per oz)	2	80	1	30

Question

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

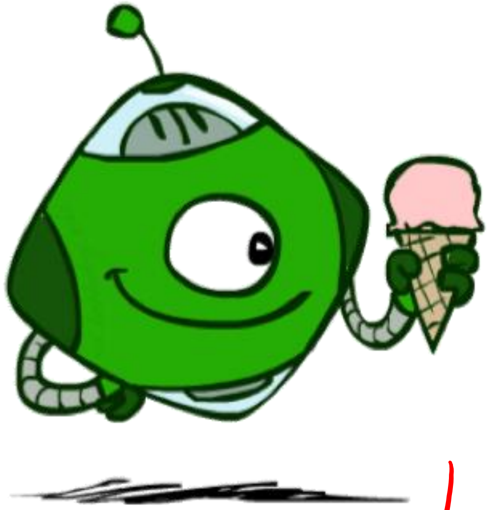
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$



Linear Programming

Different forms

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Important to pay attention to form!

Can switch between formulations!

Note: Different books have different definitions. We will refer to the forms in this slide (also consistent with B&V book).

Focus of Today: (Linear) Optimization Problem

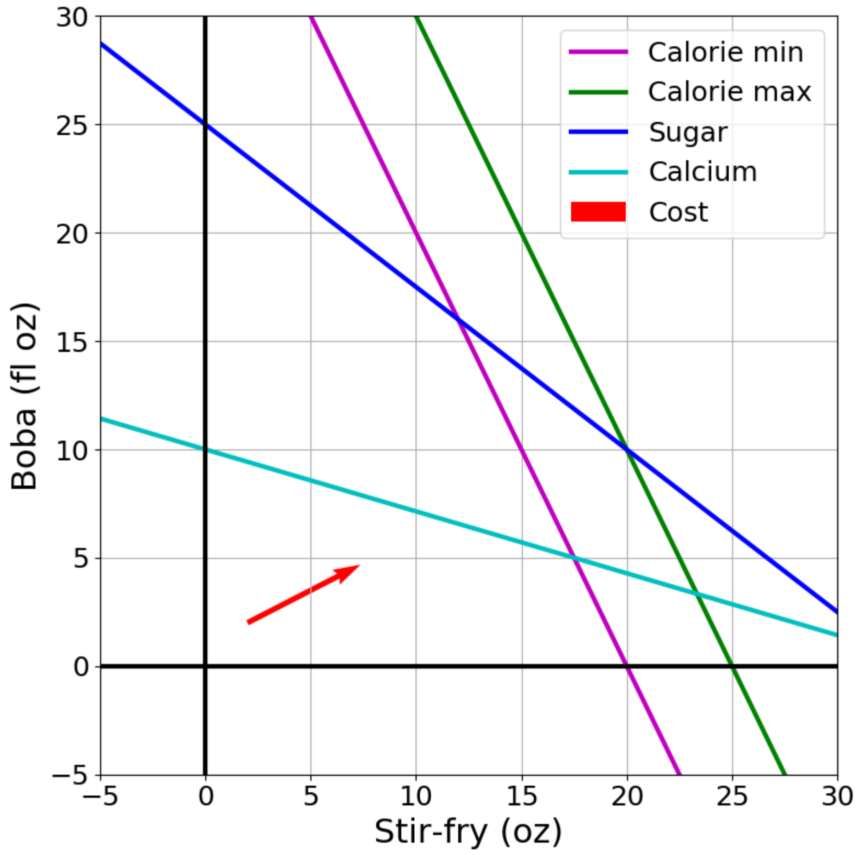
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Graphical Representation



Shape of Linear Equality

Geometry / Algebra Question

What is the equality present this line?

$$x_2 = -\frac{2}{3}x_1 + 2$$

$$x_1 = 3 \Rightarrow x_2 = 0$$

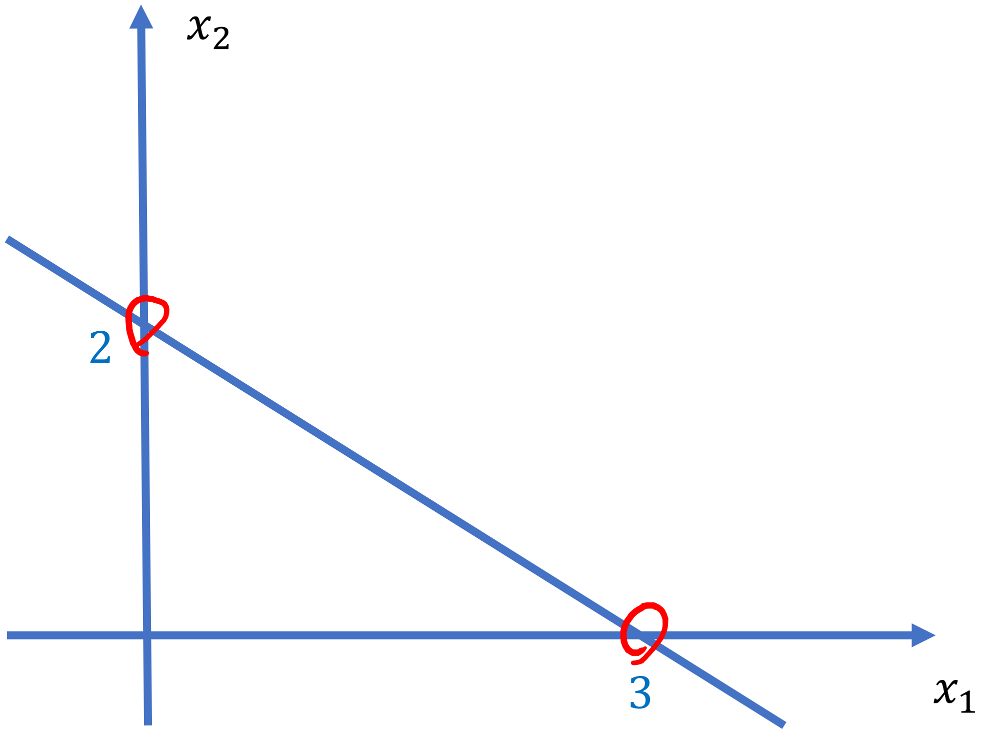
$$x_1 = 0 \Rightarrow x_2 = 2$$

$$3x_2 = -2x_1 + 6$$

$$2x_1 + 3x_2 = 6$$

$$\leq$$

$$\geq$$



Shape of Linear Equality

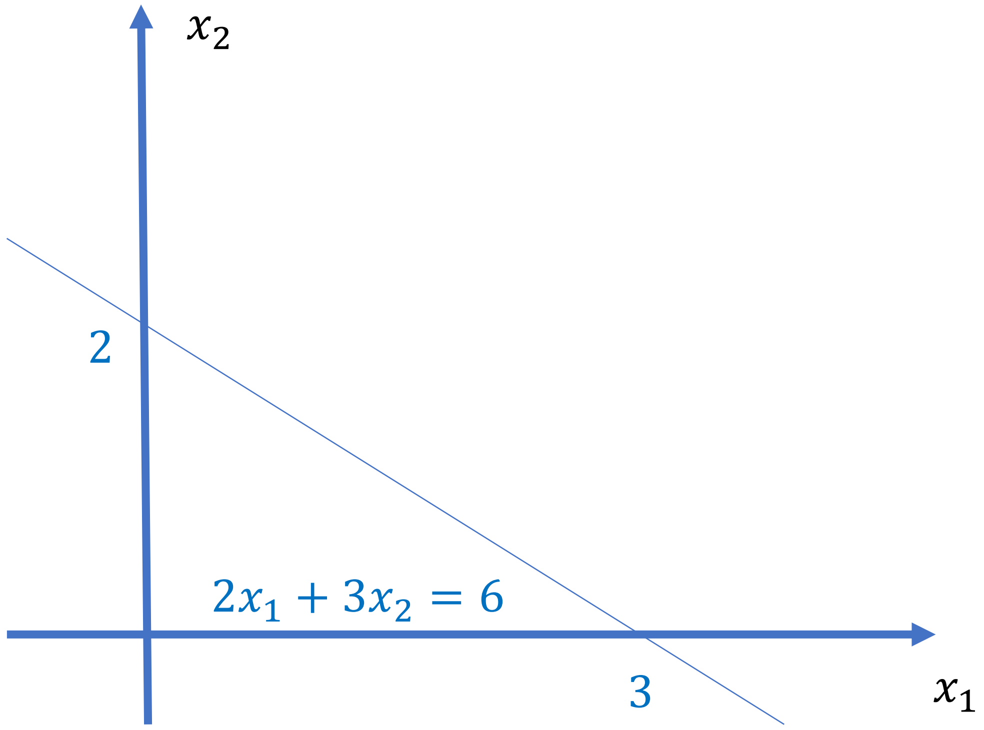
Geometry / Algebra Question

What is the equality present this line?

$$x_2 = -\frac{2}{3}x_1 + 2$$

$$3x_2 + 2x_1 = 6$$

To make sure the line intersects with the axes at (3,0) and (0,2)



Shape of Linear Inequality

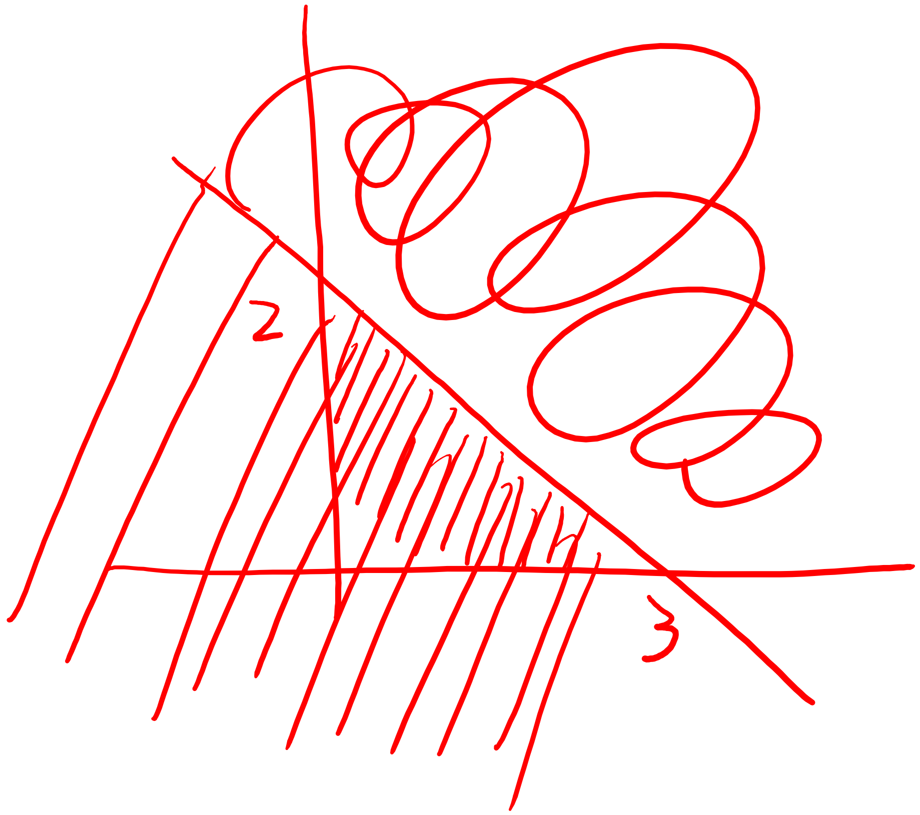
Geometry / Algebra Question

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

$2x_1 + 3x_2 \leq 6$

$-2x_1 - 3x_2 \leq -6$



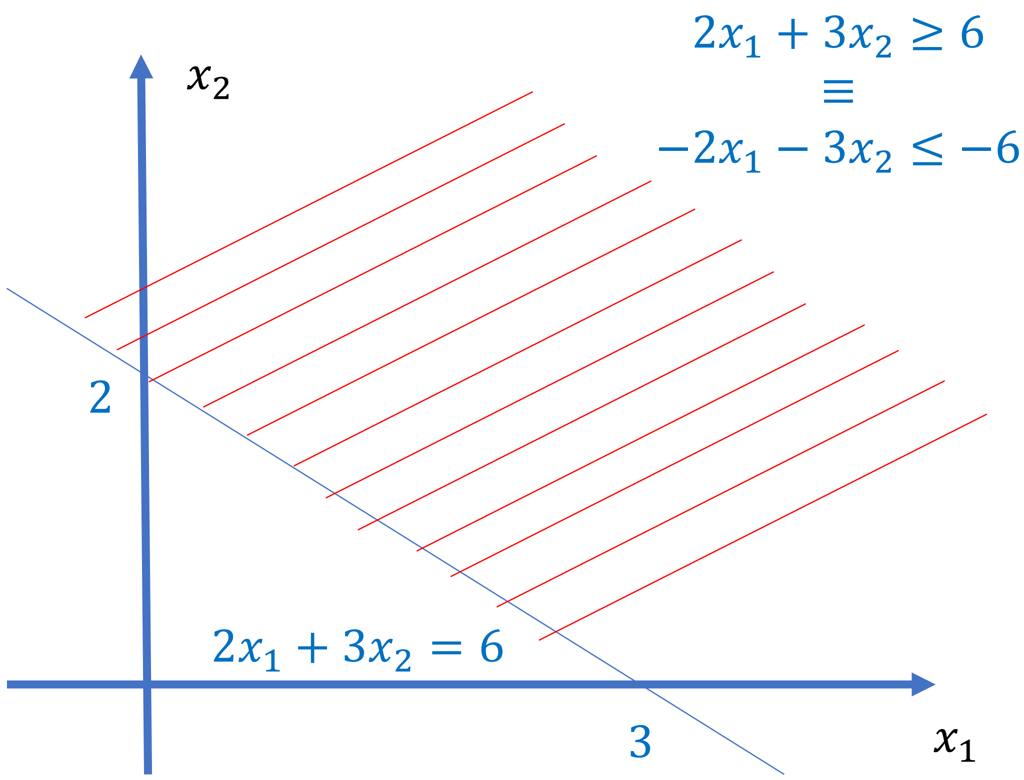
Shape of Linear Inequality

Geometry / Algebra Question

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \le b_1$$ Half Plane

$$a_1\, x_1 + a_2\, x_2 = b_1$$ Line



Shape of Linear Inequality

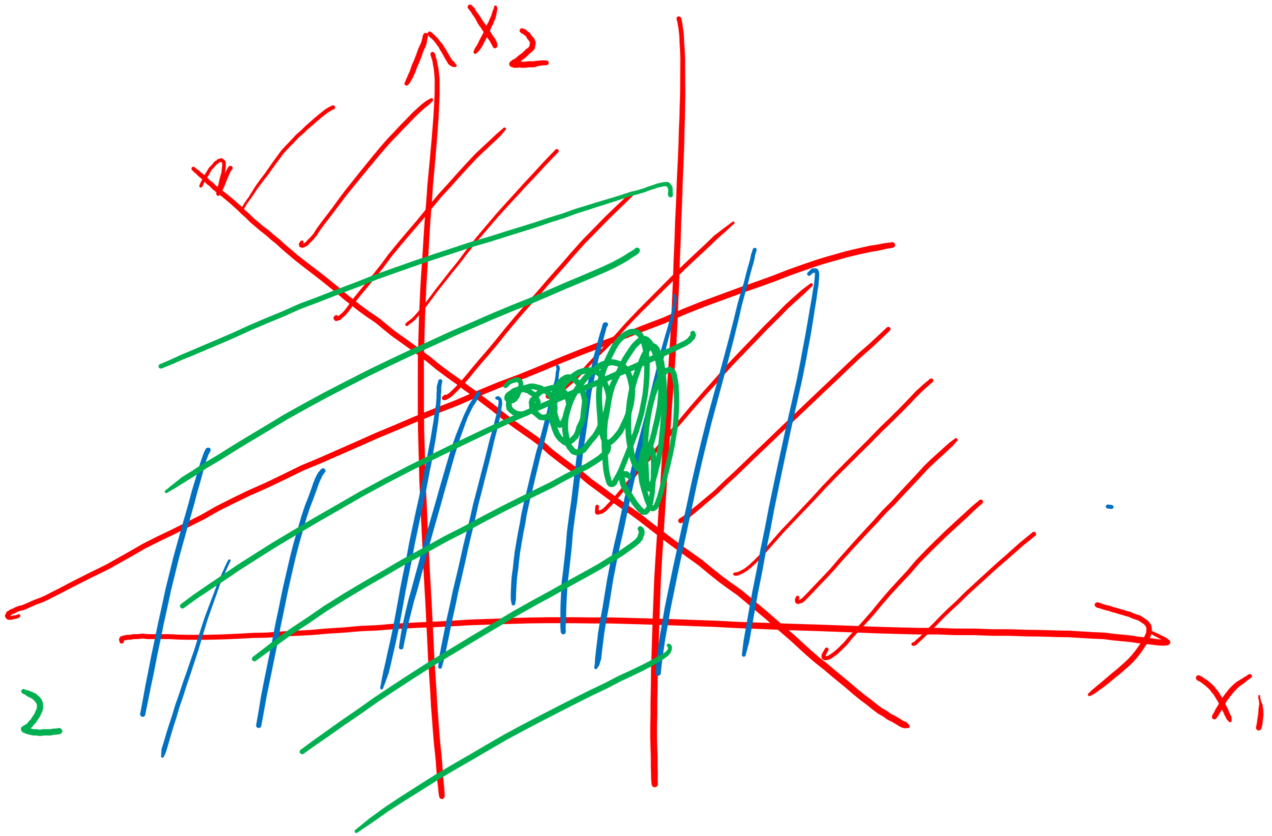
Geometry / Algebra Question

What shape does these inequalities jointly represent?

$$a_{1,1}\,x_1 + a_{1,2}\,x_2 \le b_1$$
$$a_{2,1}\,x_1 + a_{2,2}\,x_2 \le b_2$$
$$a_{3,1}\,x_1 + a_{3,2}\,x_2 \le b_3$$
$$a_{4,1}\,x_1 + a_{4,2}\,x_2 \le b_4$$

$x_1 + x_2 \le 1$

$-x_1 - x_2 \le -2 \iff x_1 + x_2 \ge 2$

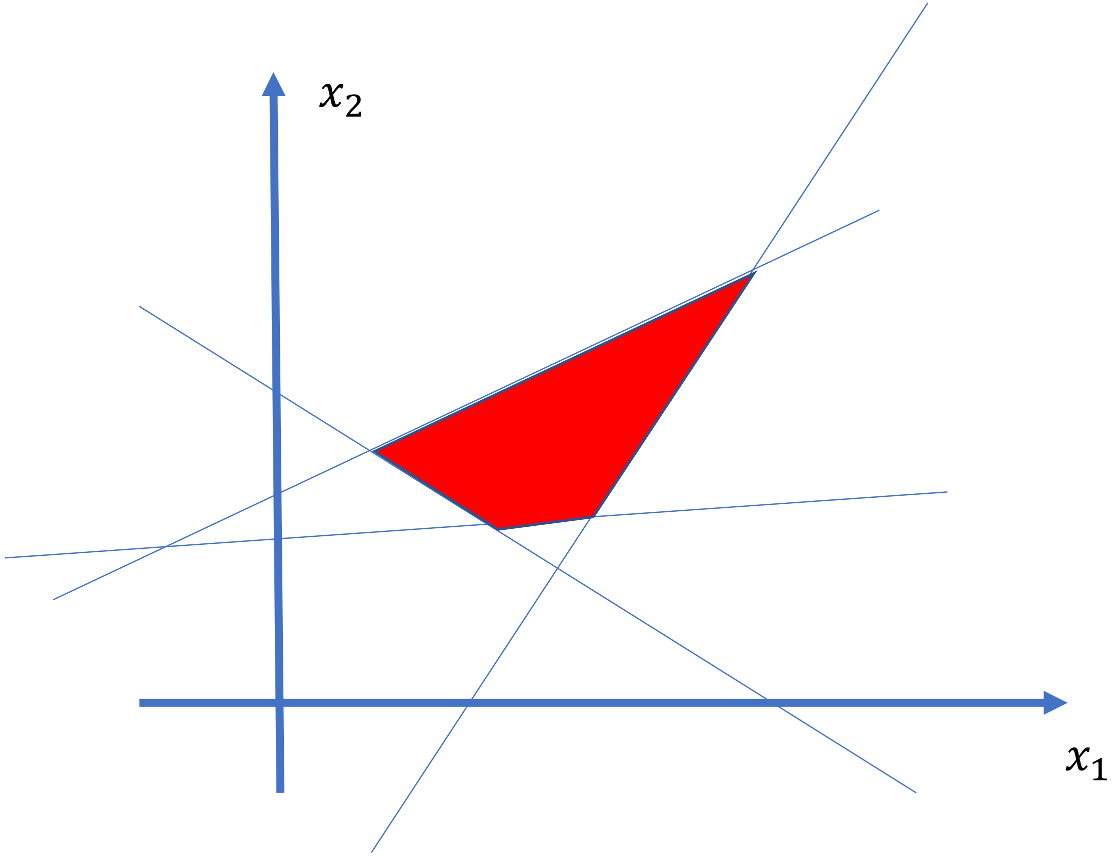


Shape of Linear Inequality

Geometry / Algebra Question

What shape does these inequalities jointly represent?

$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



Intersection of half planes

Could be polyhedron

Could be empty

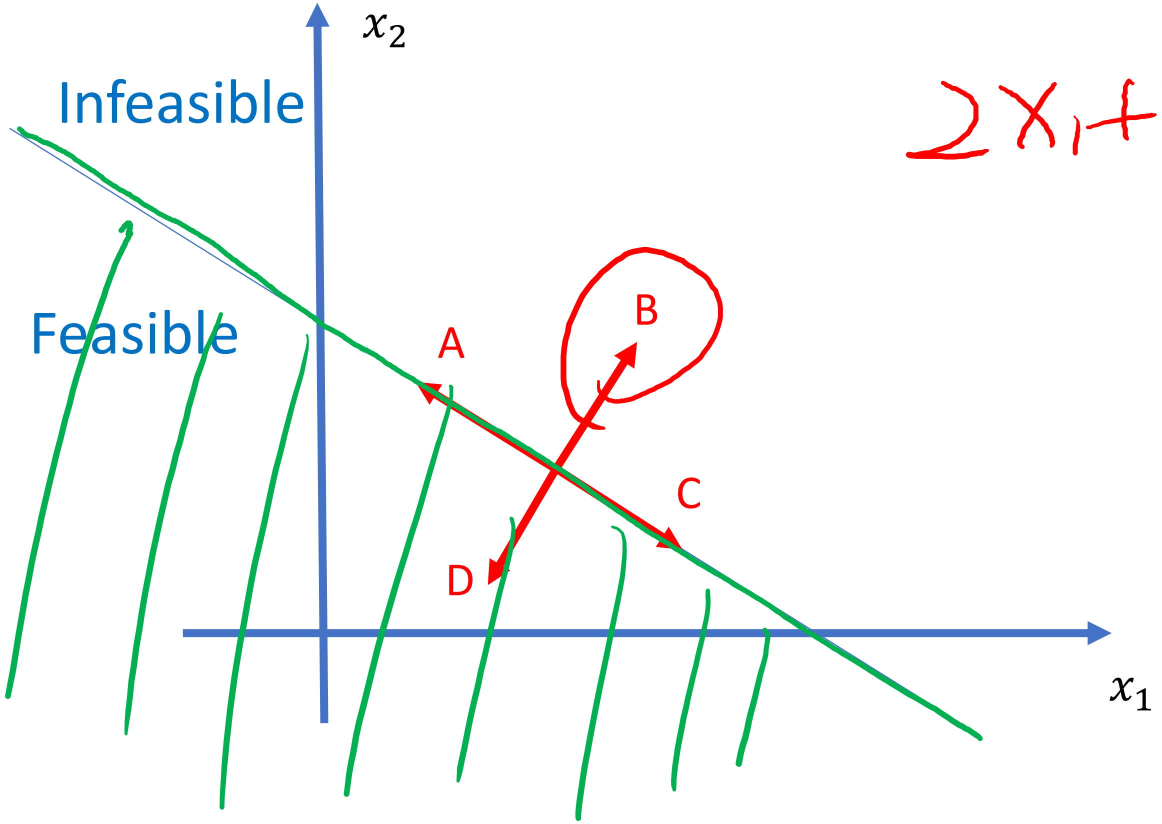
Piazza Poll 3

What is the relationship between the half plane:

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

and the vector:

$$[a_1, a_2]^T$$



$$2x_1 + 3x_2 \leq 6$$

$$[2, 3]^T$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

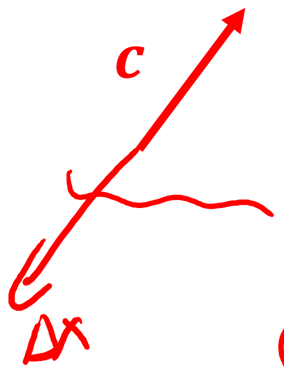
Lowering Cost

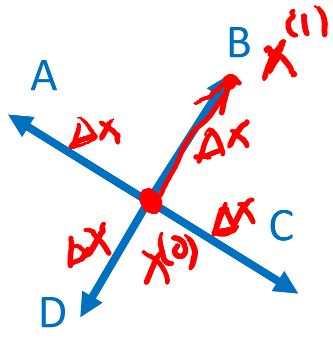
Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle\,\boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle\,\boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?

$c^T x^{(1)}$





$$c^T x^{(1)} = c^T (x^{(0)} + \Delta x)$$

$$= c^T x^{(0)} + c^T \Delta x$$

$$c^T \Delta x = \|c\| \|\Delta x\| \cos(c, \Delta x)$$

$$\cos(c, \Delta x) = -1$$

Lowering Cost

Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle \boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?

$$\boldsymbol{c}^T \boldsymbol{x}^{(1)} = \boldsymbol{c}^T \left(\boldsymbol{x}^{(0)} + \triangle \boldsymbol{x} \right) = \boldsymbol{c}^T \boldsymbol{x}^{(0)} + \boldsymbol{c}^T \triangle \boldsymbol{x}$$

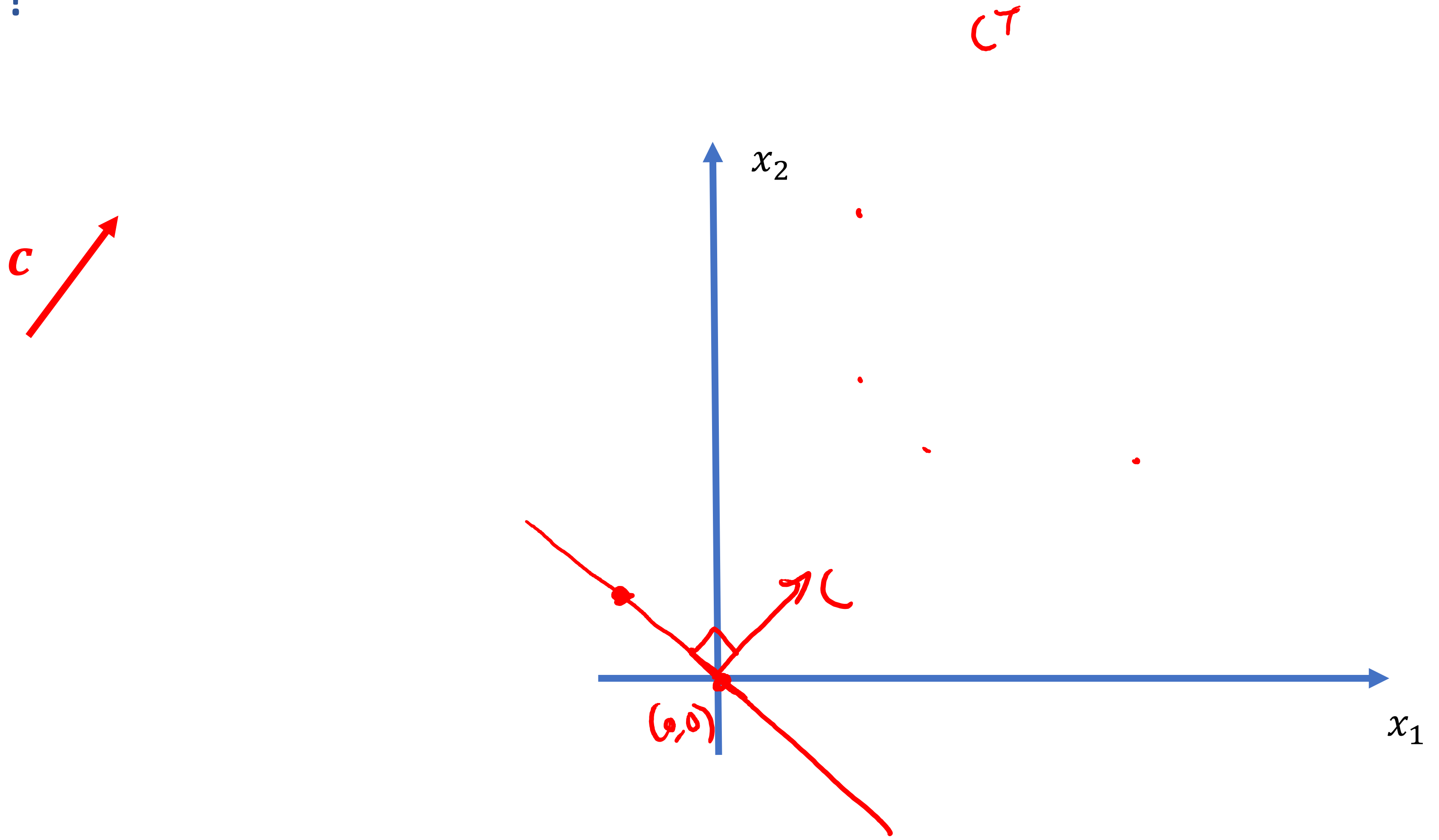
$$\boldsymbol{c}^T \triangle \boldsymbol{x} = |\boldsymbol{c}||\triangle \boldsymbol{x}| \cos(\boldsymbol{c}, \triangle \boldsymbol{x})$$

$\cos(\boldsymbol{c}, \triangle \boldsymbol{x}) = -\mathbf{1}$ when $\boldsymbol{c}$ and $\triangle \boldsymbol{x}$ have opposite directions

Cost Contours

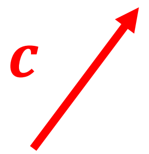
Given the cost vector $[c_1, c_2]^T$ where will

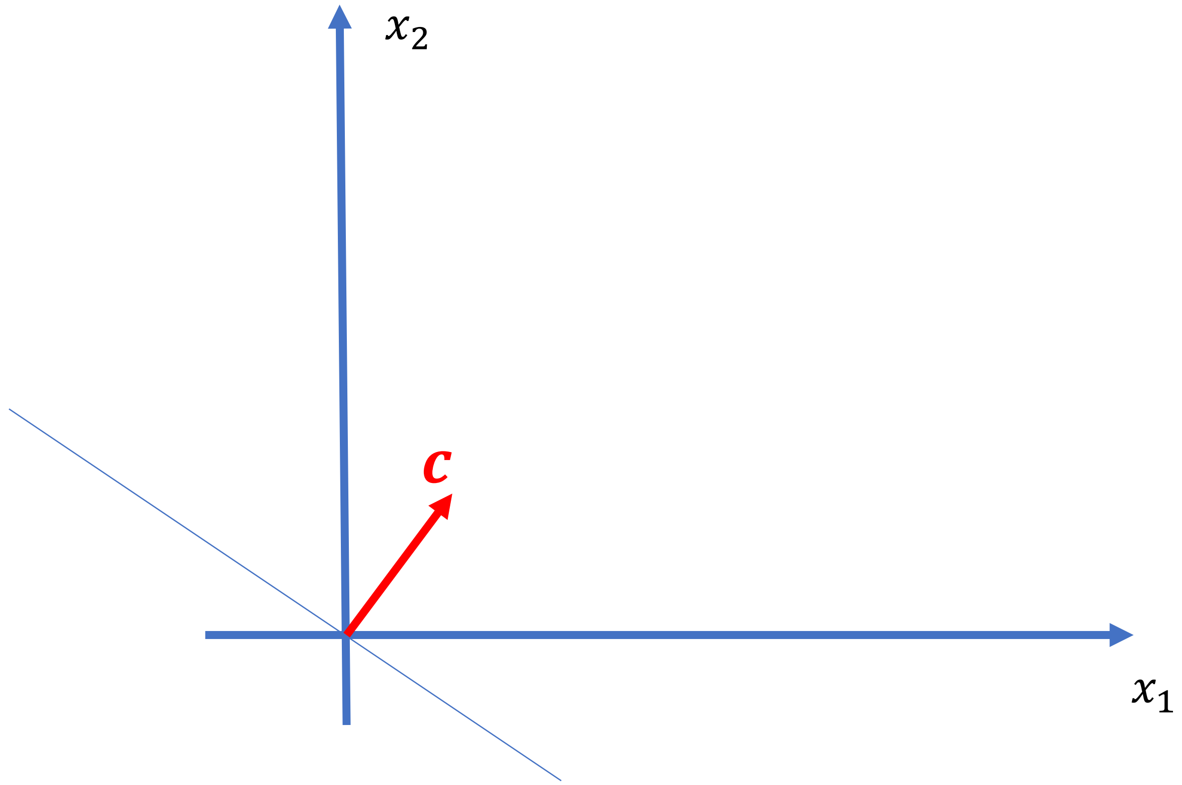
$\boldsymbol{c}^T \boldsymbol{x} = 0$?



Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will $\boldsymbol{c}^T\boldsymbol{x} = 0$?





Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

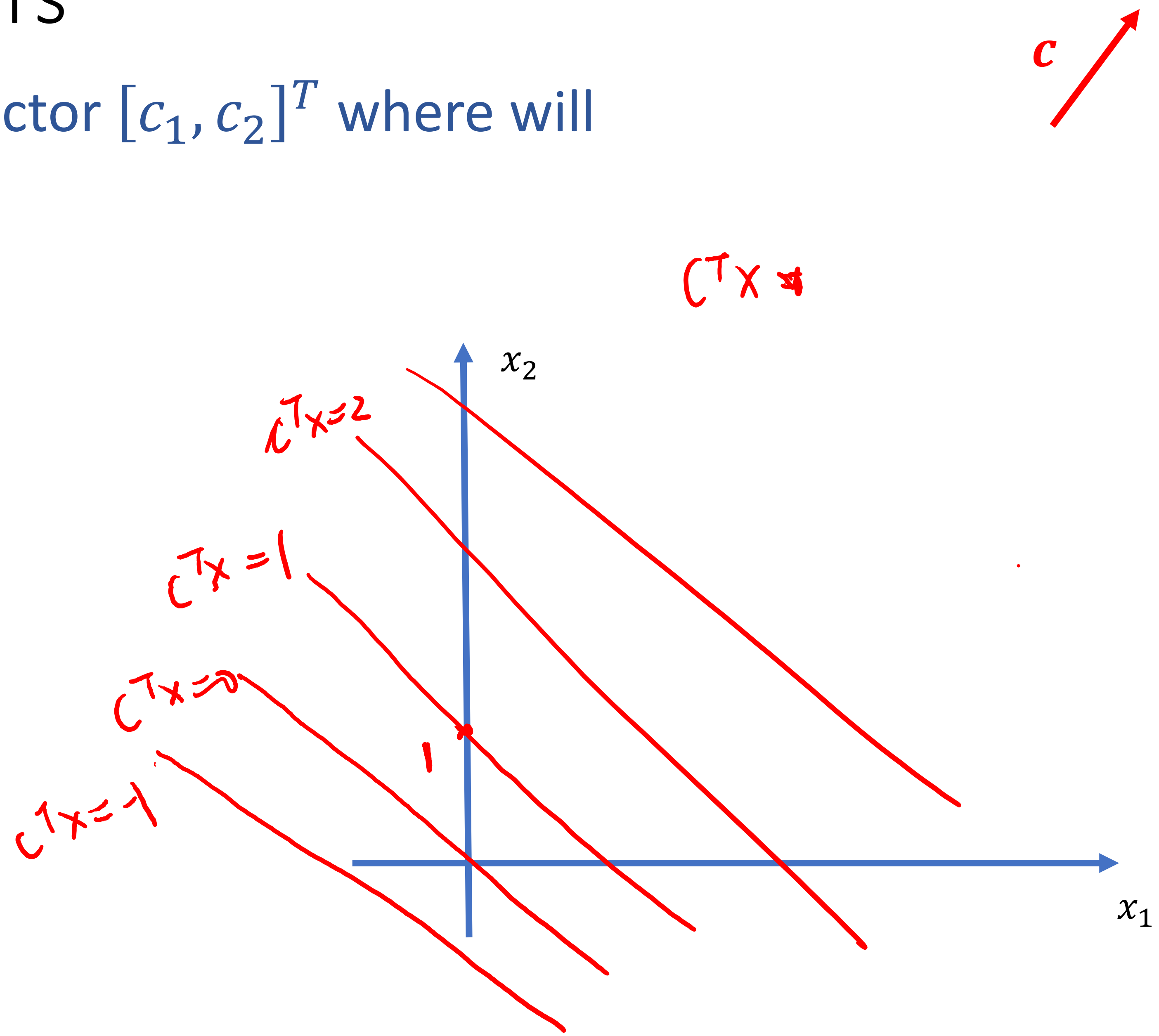
$\boldsymbol{c}^T\boldsymbol{x} = 0$?

$\boldsymbol{c}^T\boldsymbol{x} = 1$?

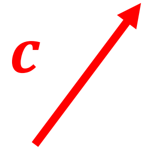
$\boldsymbol{c}^T\boldsymbol{x} = 2$?

$\boldsymbol{c}^T\boldsymbol{x} = -1$?

$\boldsymbol{c}^T\boldsymbol{x} = -2$?



Cost Contours



Given the cost vector $[c_1, c_2]^T$ where will

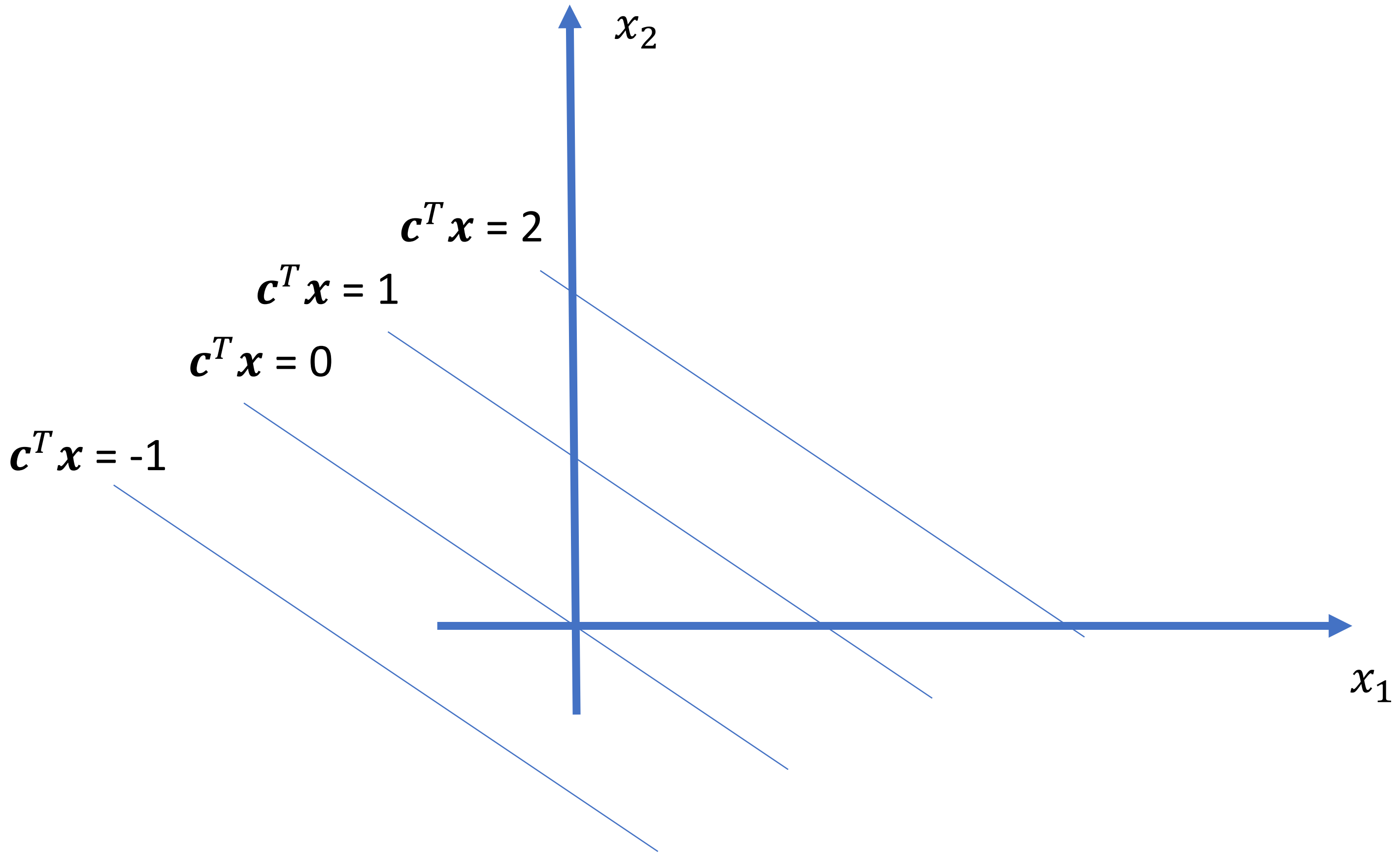
$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

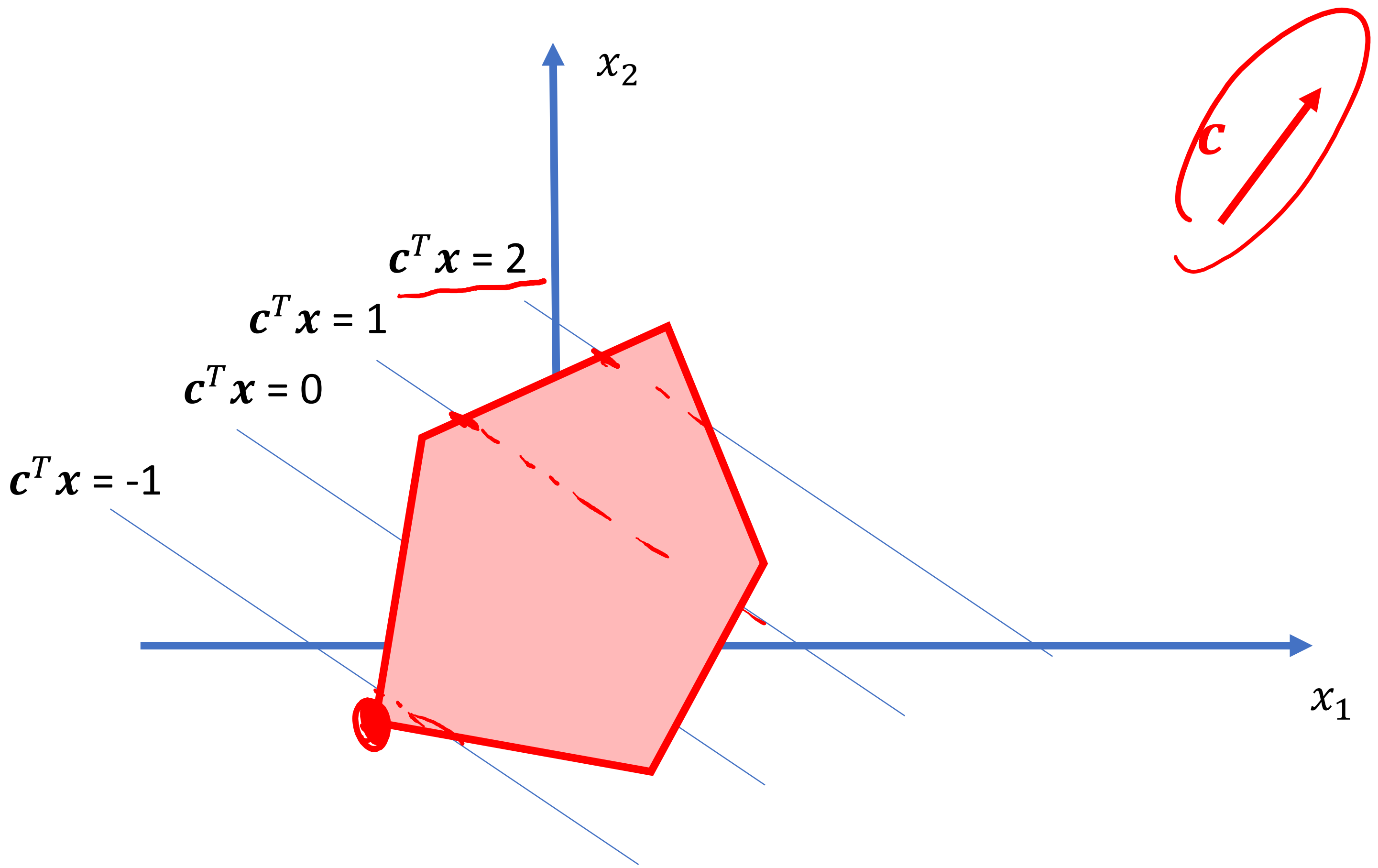
$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -2 ?



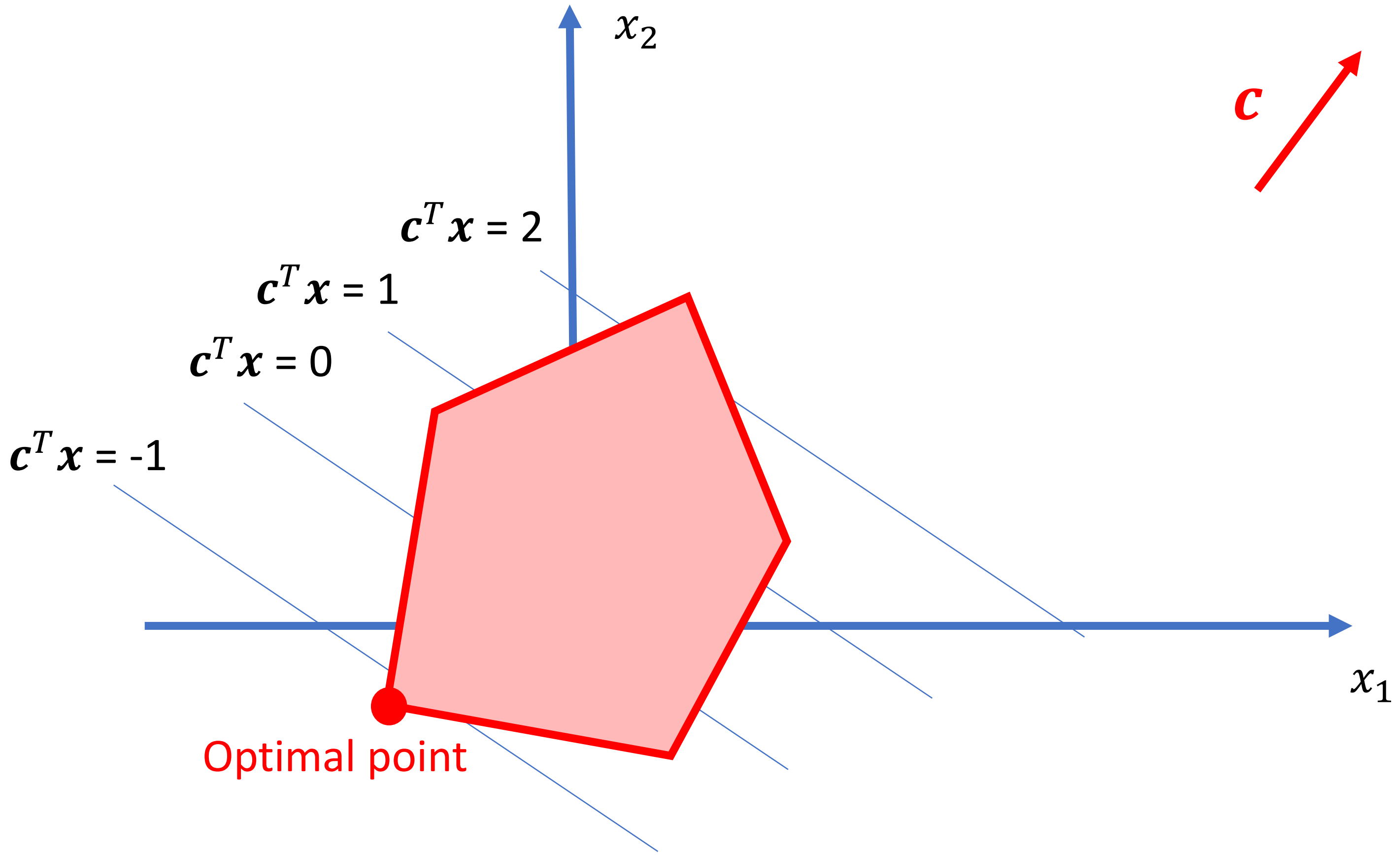
Optimizing Cost

Given the cost vector $[c_1, c_2]^T$, which point in the red polyhedron below can minimize $\boldsymbol{c}^T\boldsymbol{x}$?



Optimizing Cost

Given the cost vector $[c_1, c_2]^T$, which point in the red polyhedron below can minimize $\boldsymbol{c}^T\boldsymbol{x}$?



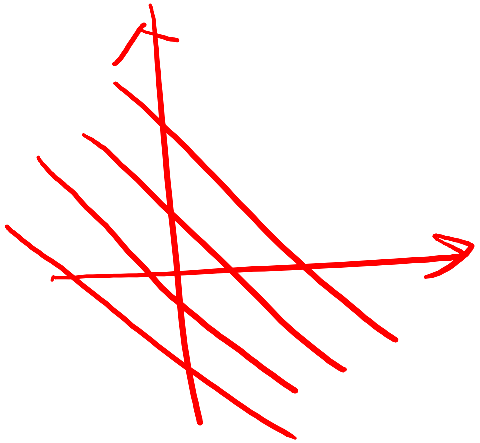
LP Graphical Representation

Inequality form

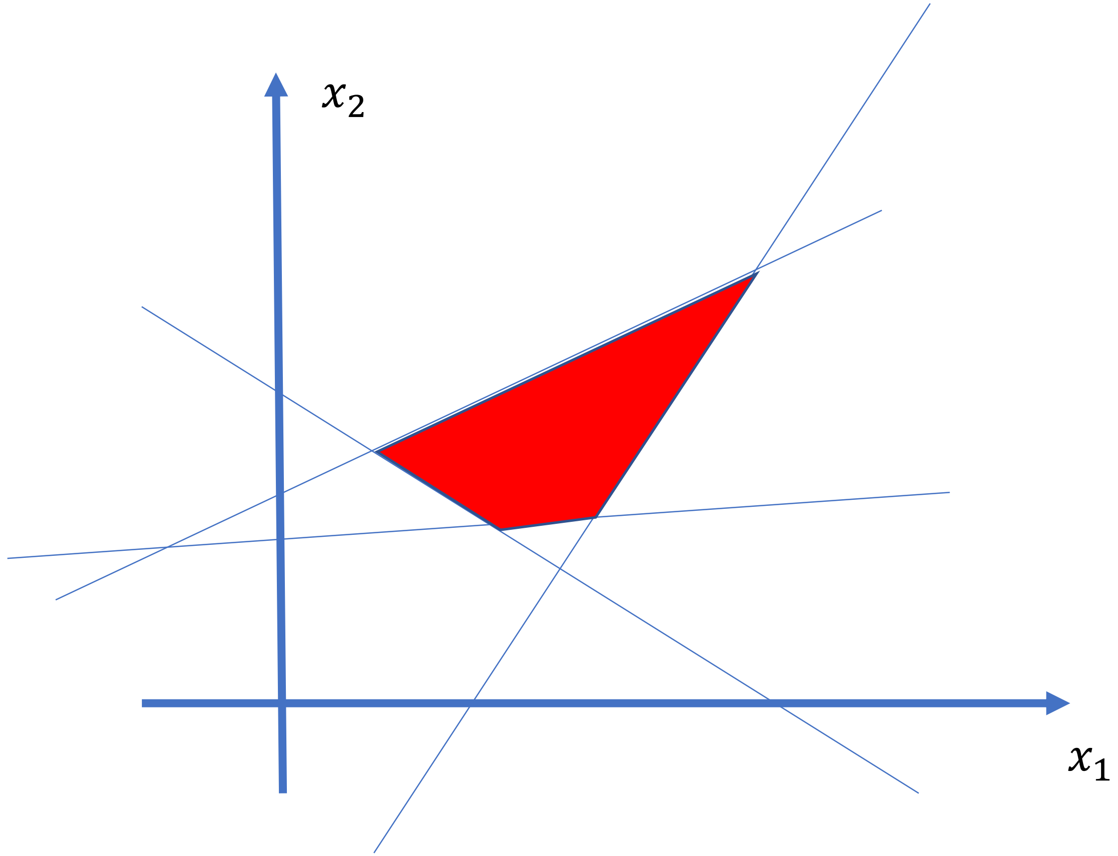
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x} \quad \text{Objective Function}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b} \quad \text{Feasible Region}$$

$c_1 x_1 + c_2 x_2$



$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



Piazza Poll 4

$c = [0, 0]^T$

True or False: An minimizing LP with exactly one constraint, will always have a minimum objective of $-\infty$.

value

$$\min_{x} \quad c^T x$$
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \le b$$

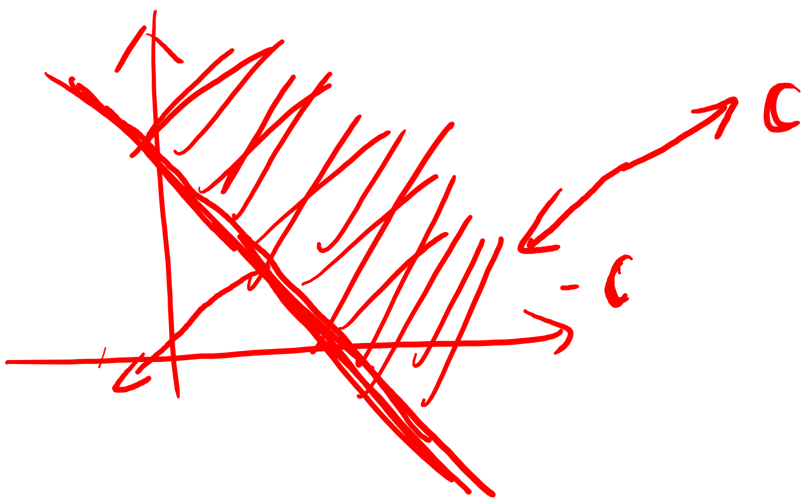
$a_1 < 0, \; a_2 < 0$

$-2x_1 - 3x_2 \le -6$

$c = [2, 3]^T$

$2x_1 + 3x_2 \ge 6$

$\min \; 2x_1 + 3x_2$

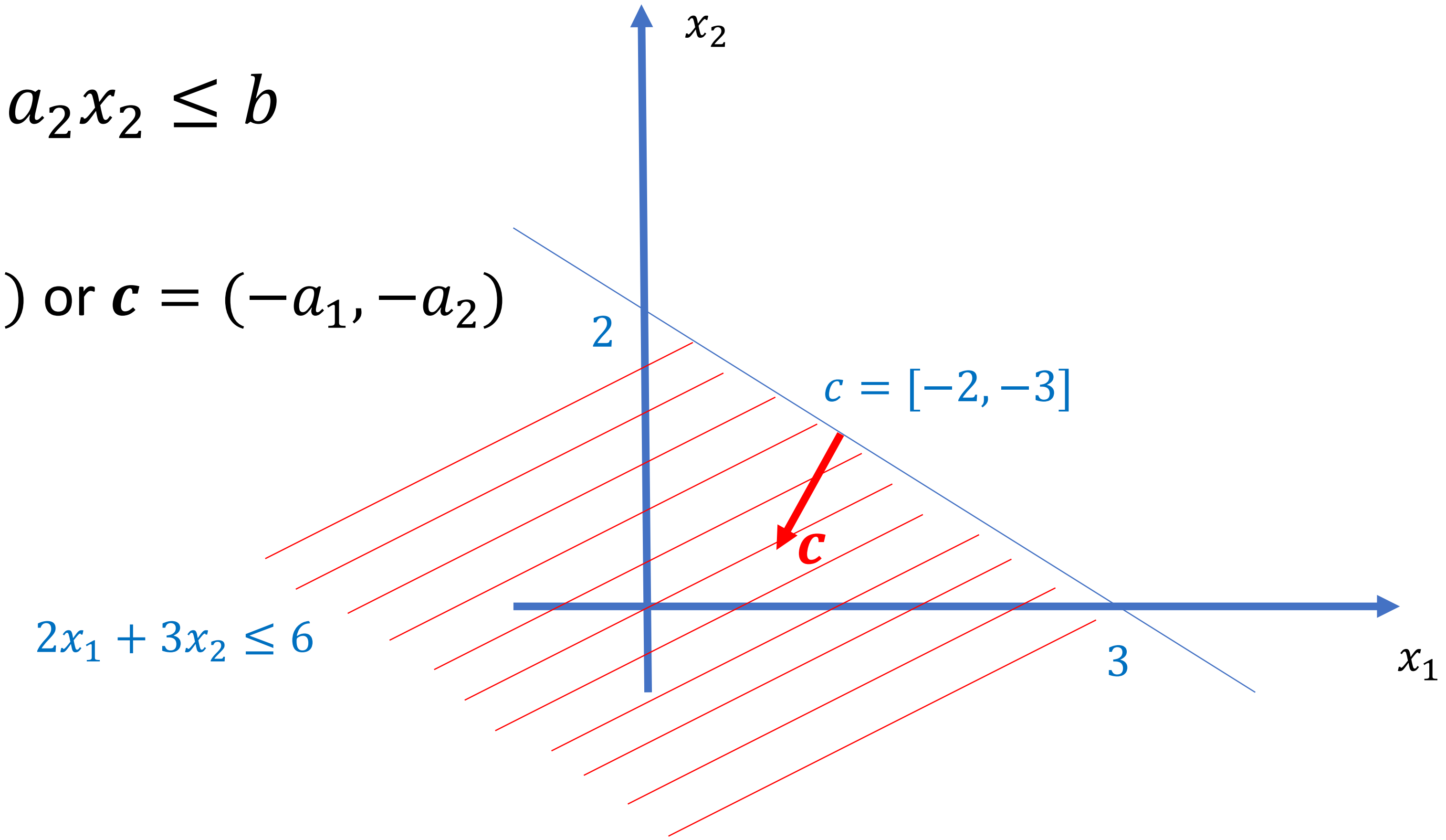


Piazza Poll 4

True or False: An minimizing LP with exactly one constraint, will always have a minimum objective value of $-\infty$.

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad a_1x_1 + a_2x_2 \leq b$$

False: $\boldsymbol{c} = (0{,}0)$ or $\boldsymbol{c} = (-a_1, -a_2)$



Focus of Today: (Linear) Optimization Problem

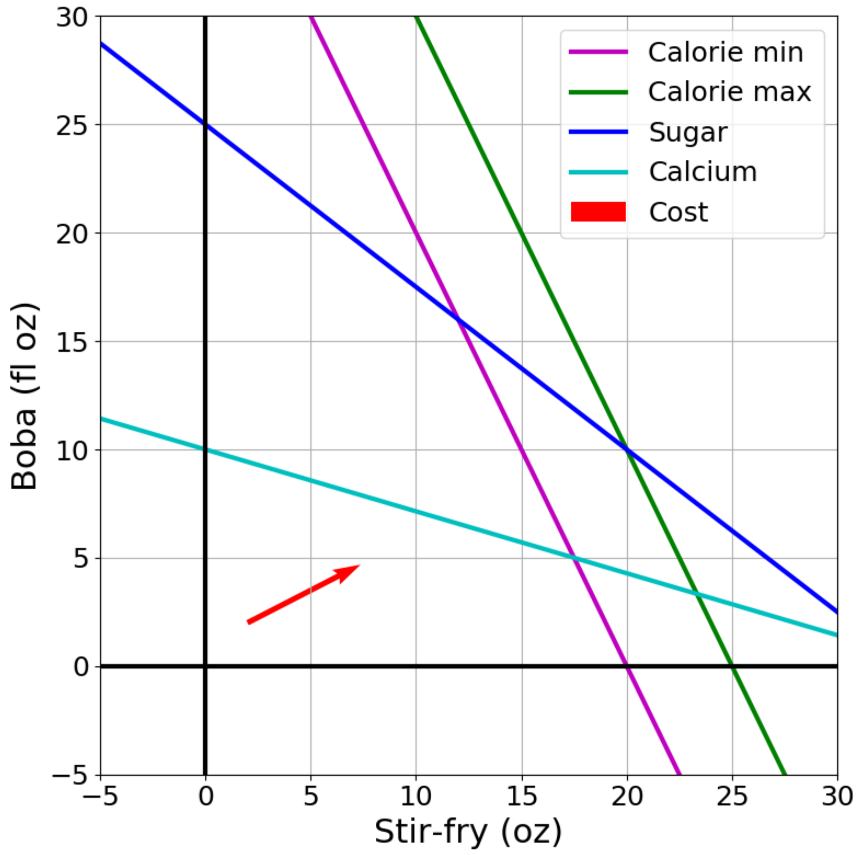
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar $\leq$ 100 g
- Calcium $\geq$ 700 mg

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$

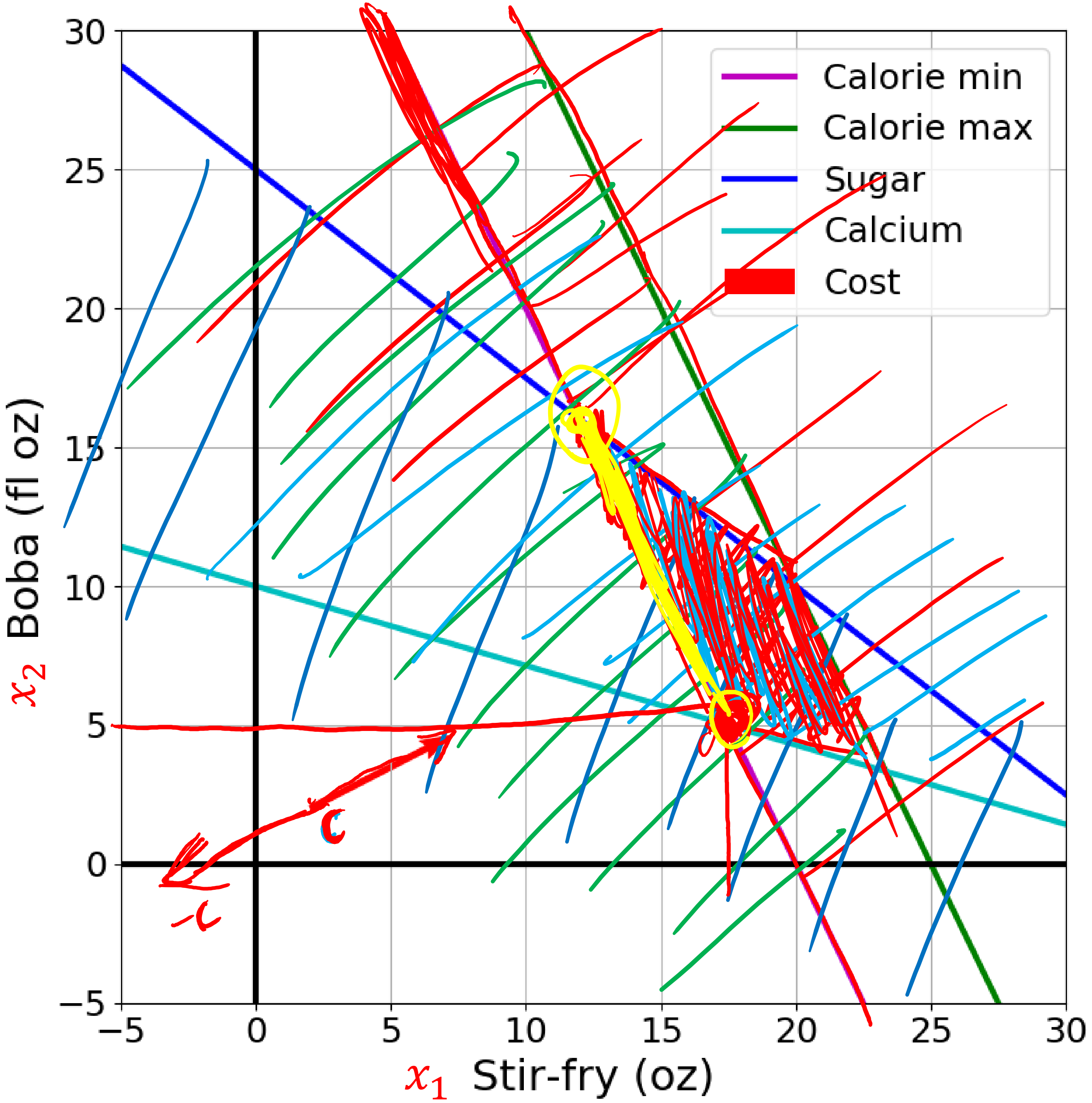
$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$$\text{s.t.} \quad 100\, x_1 + 50\, x_2 \geq 2000$$

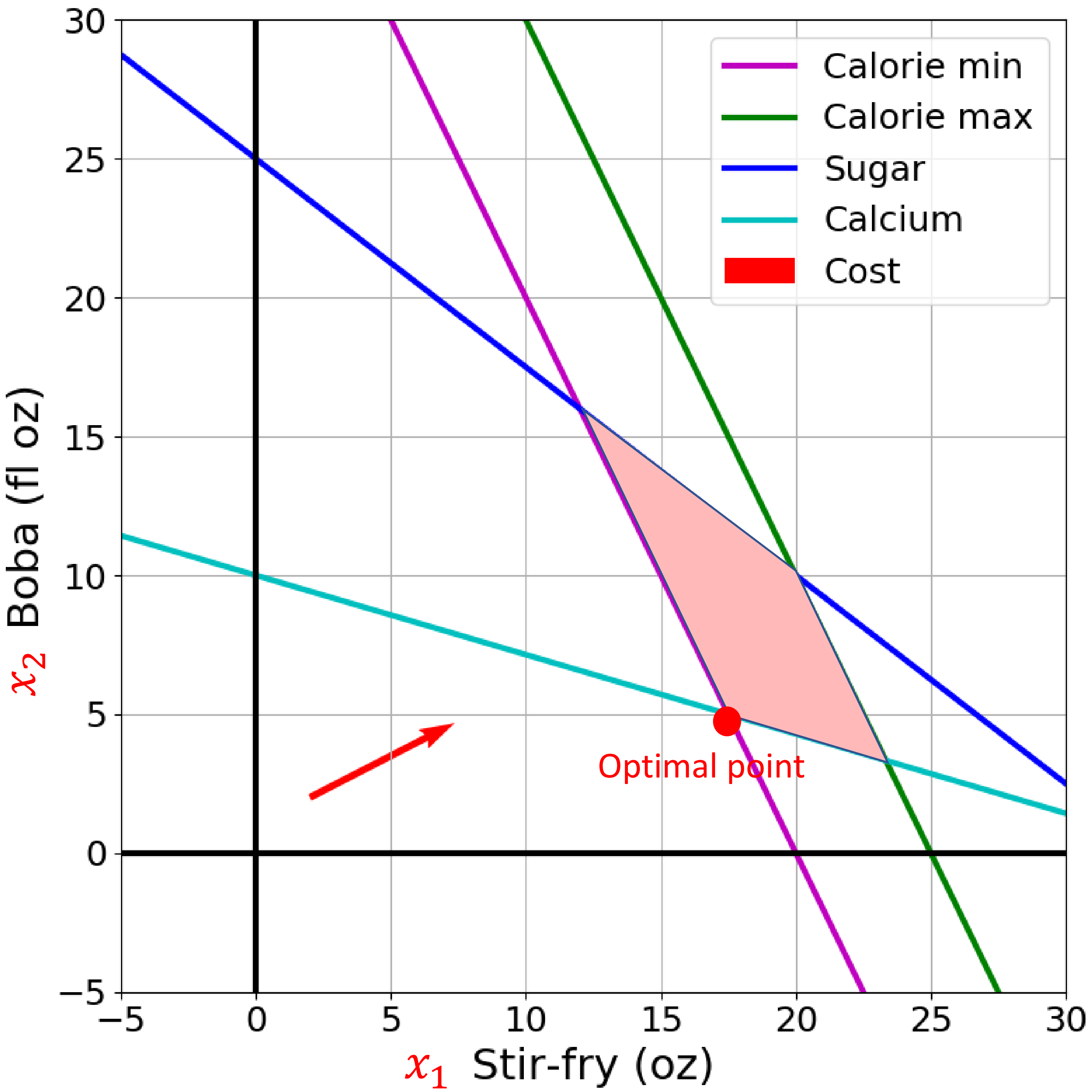
$$100\, x_1 + 50\, x_2 \leq 2500$$

$$3\, x_1 + 4\, x_2 \leq 100$$

$$20\, x_1 + 70\, x_2 \geq 700$$

What is the feasible region?

What is the optimal solution?

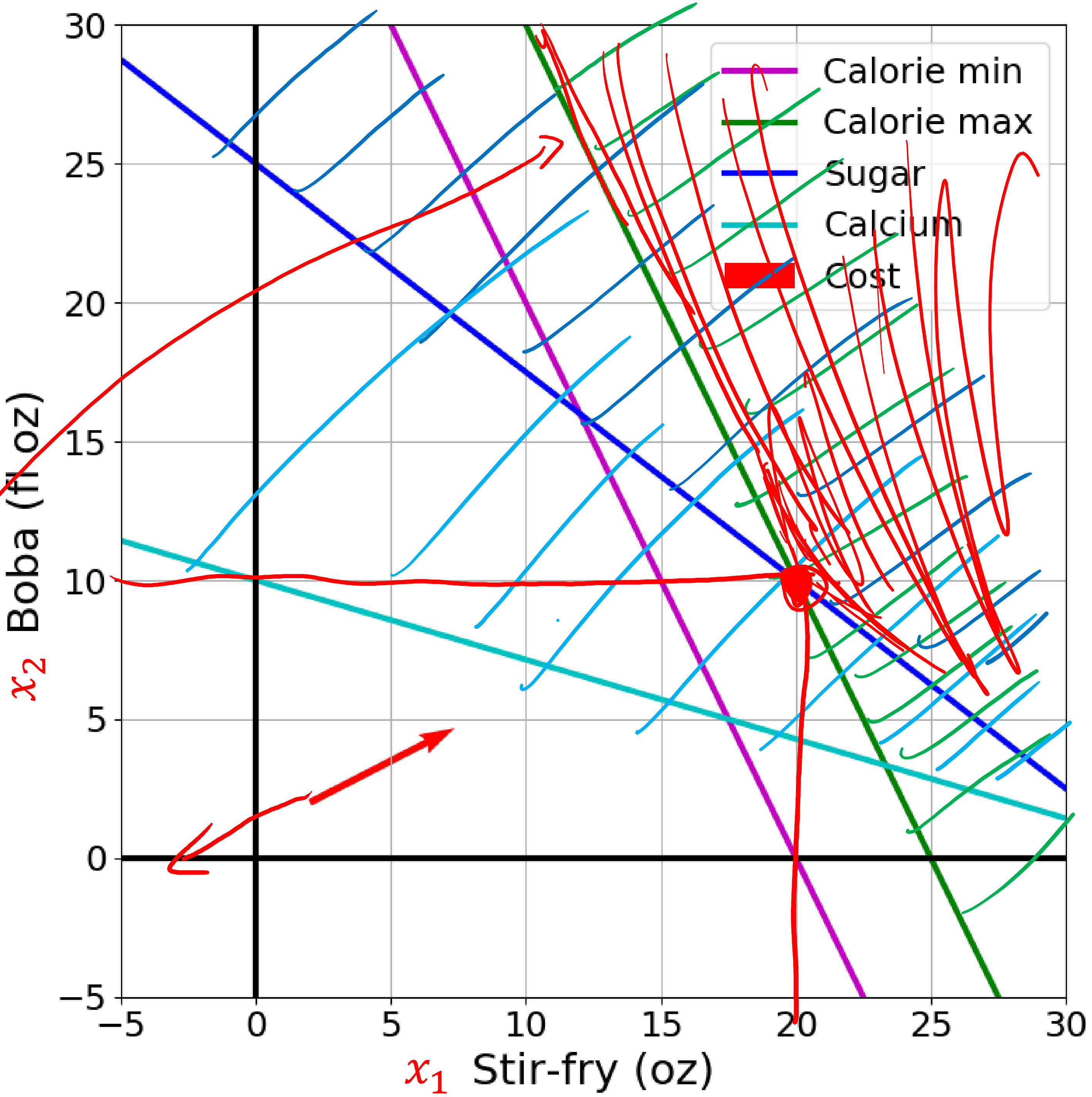


Healthy Squad Goals

- Calories ≥ 2500
- Sugar ≥ 100 g
- Calcium ≥ 700 mg

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$
$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2500$$
$$3\,x_1 + 4\,x_2 \geq 100$$

$x_1 + 50x_2 = 25$



What is the feasible region?

What is the optimal solution?

Healthy Squad Goals

- Calories ≥ 2500
- Sugar ≥ 100 g
- Calcium ≥ 700 mg

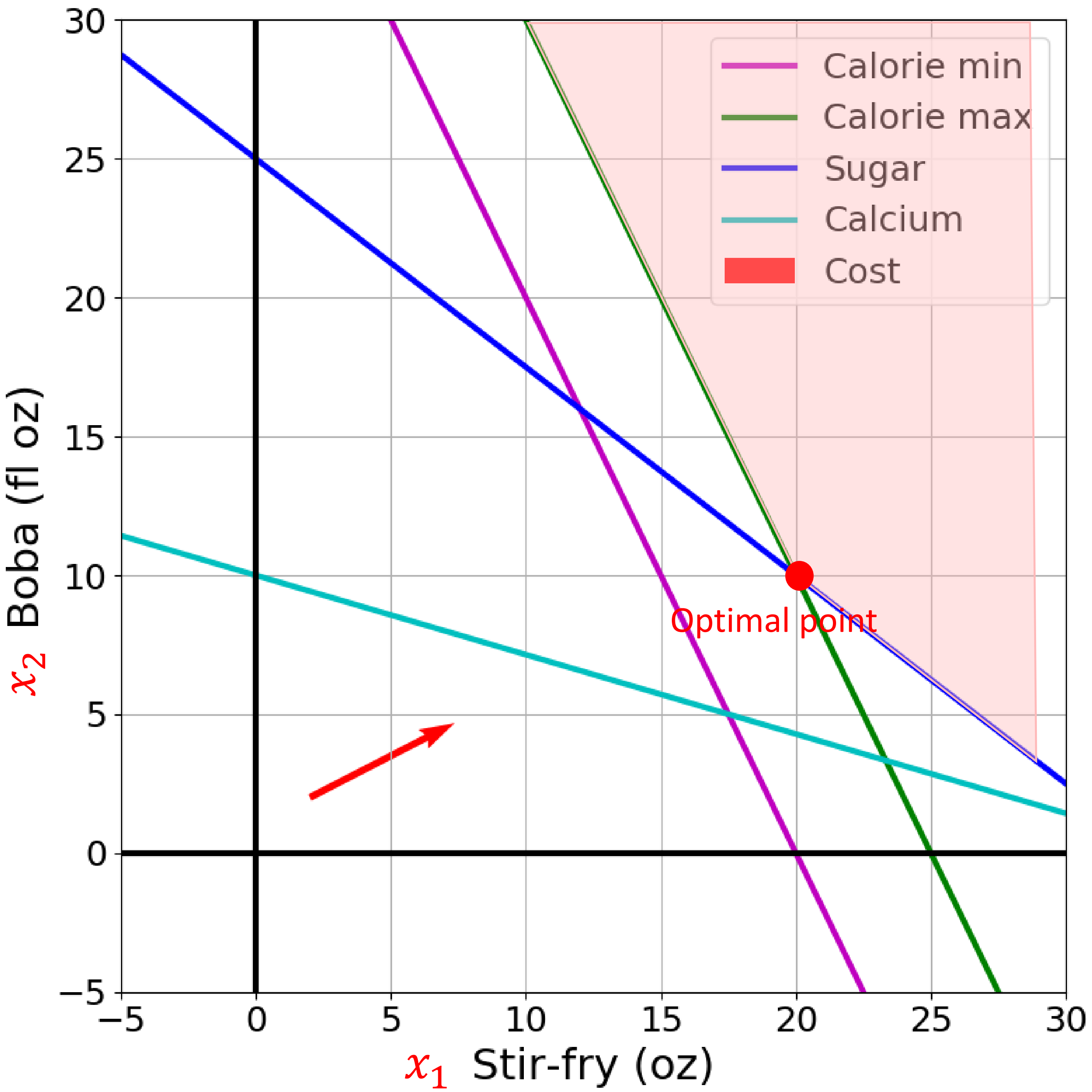
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2500$$

$$3\,x_1 + \ \ 4\,x_2 \geq 100$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \le 2000$$

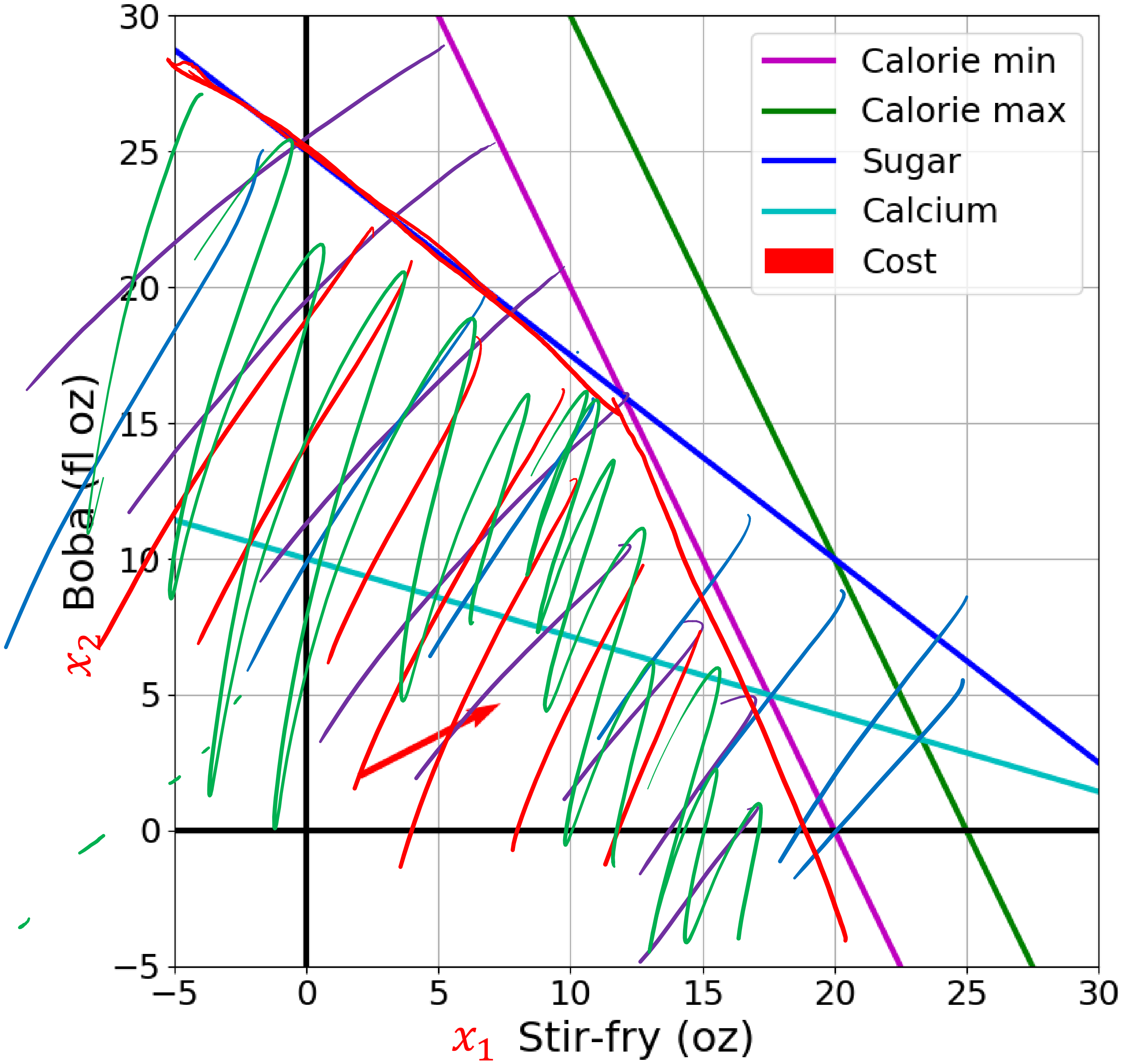
$$3\,x_1 + 4\,x_2 \le 100$$

~~$x_1 \ge 0,\ x_2 \ge 0$~~

What is the feasible region?

What is the optimal solution?

Unbounded



Healthy Squad Goals

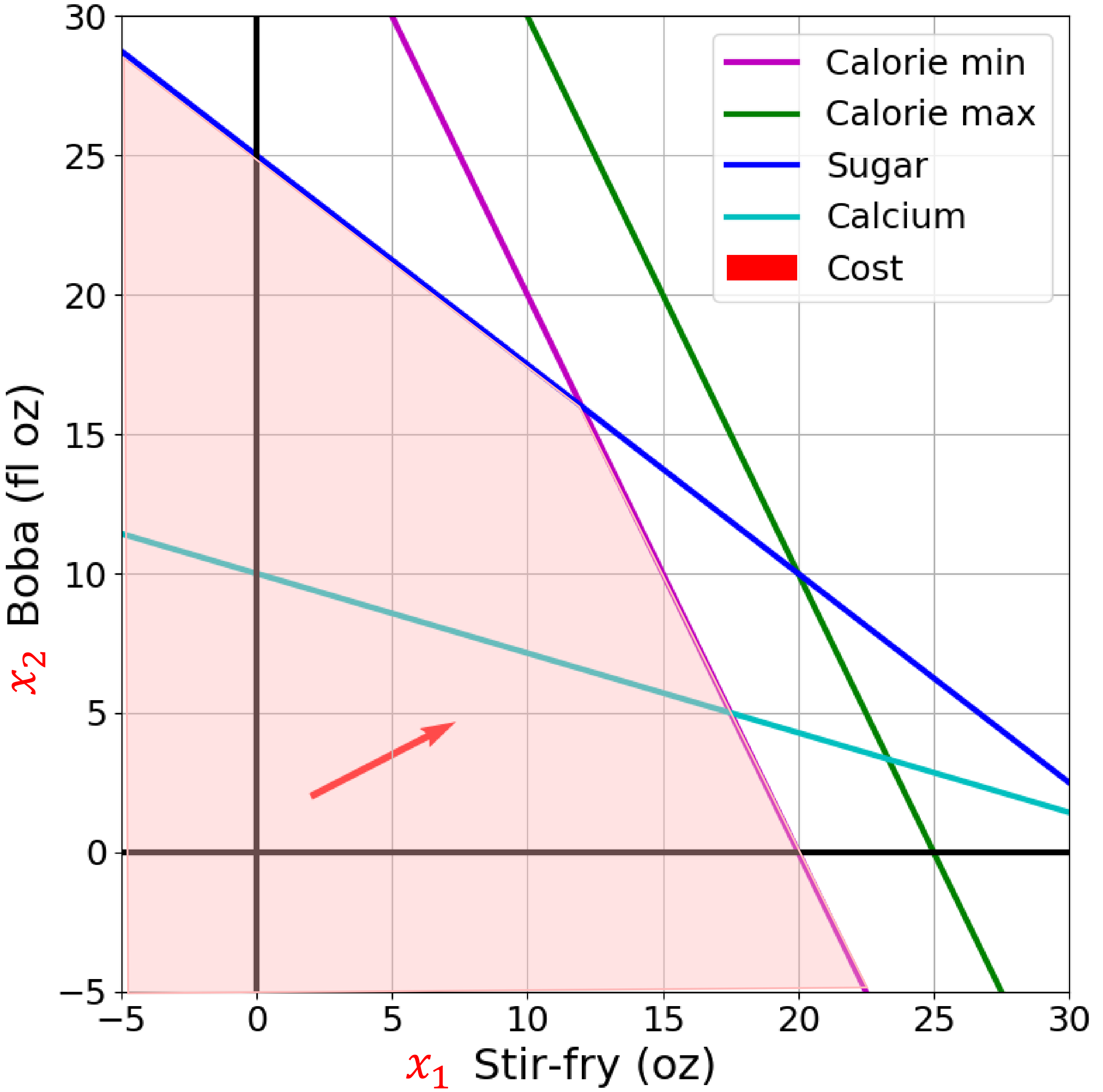
- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$
$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \leq 2000$$
$$3\,x_1 + 4\,x_2 \leq 100$$

What is the feasible region?

What is the optimal solution?

Problem unbounded!



Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \leq 2000$$

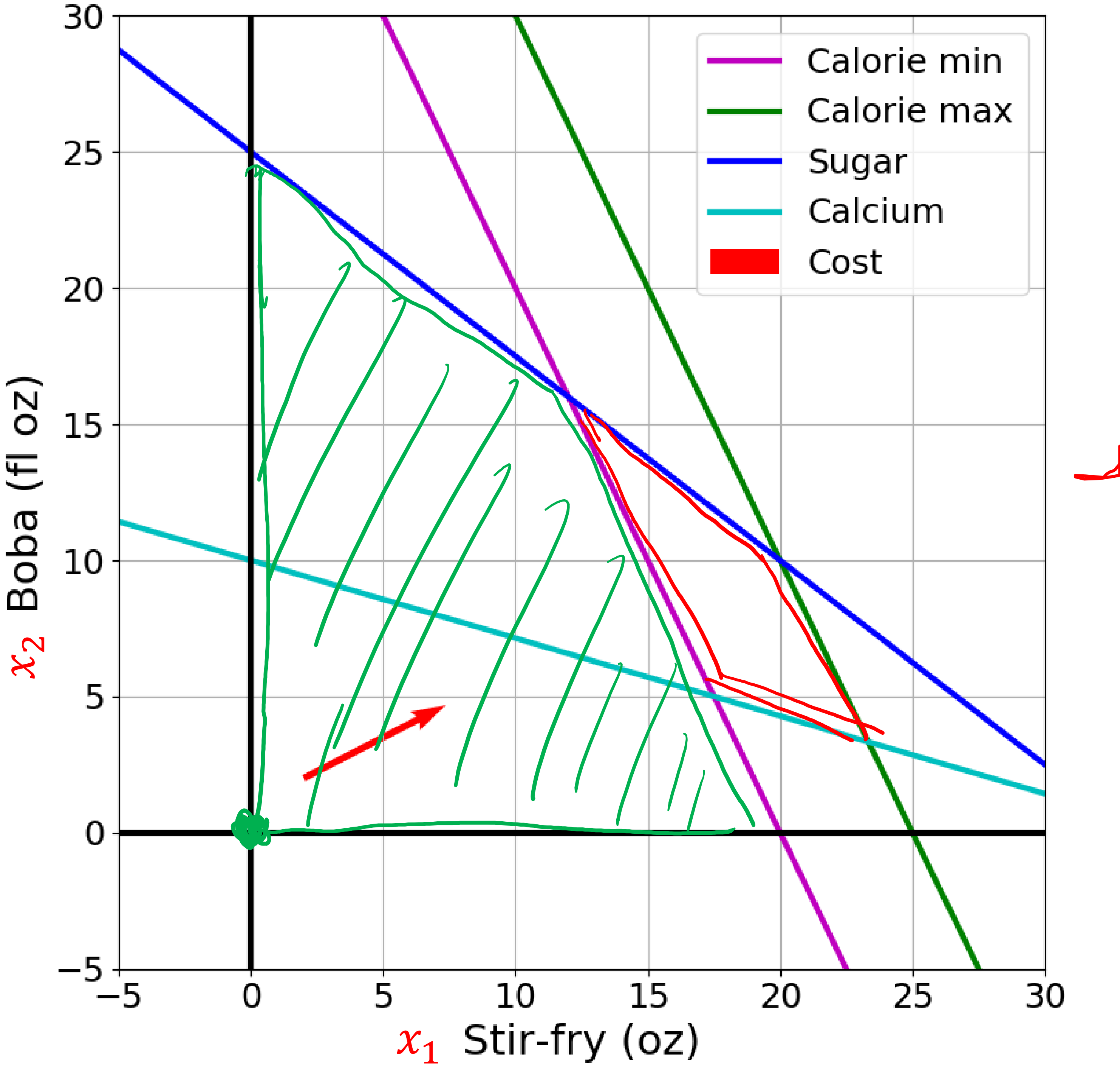
$$3\,x_1 + 4\,x_2 \leq 100$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \leq 2000$$

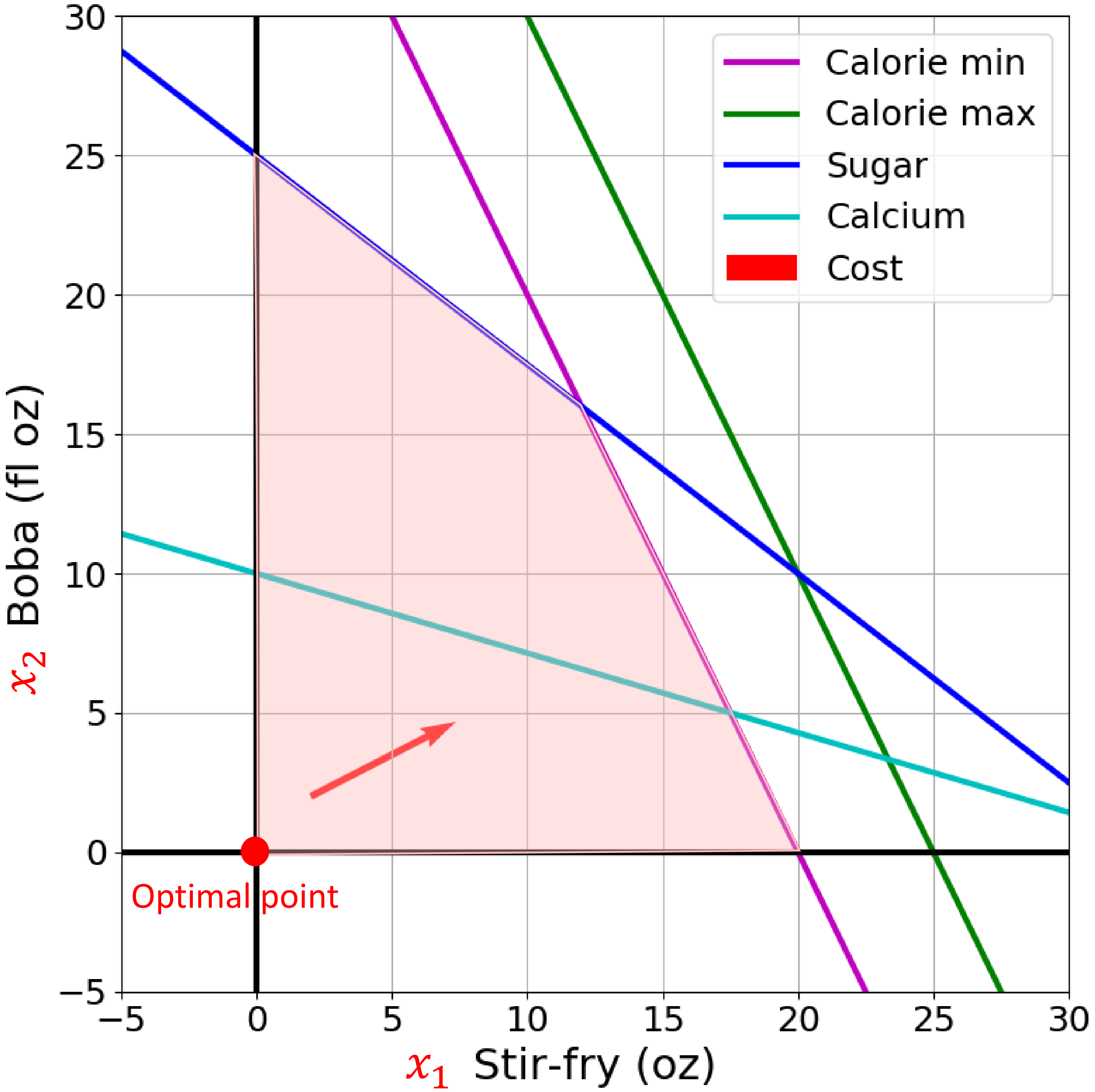
$$3\,x_1 + 4\,x_2 \leq 100$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≥ 100 g
- Calcium ≤ 700 mg

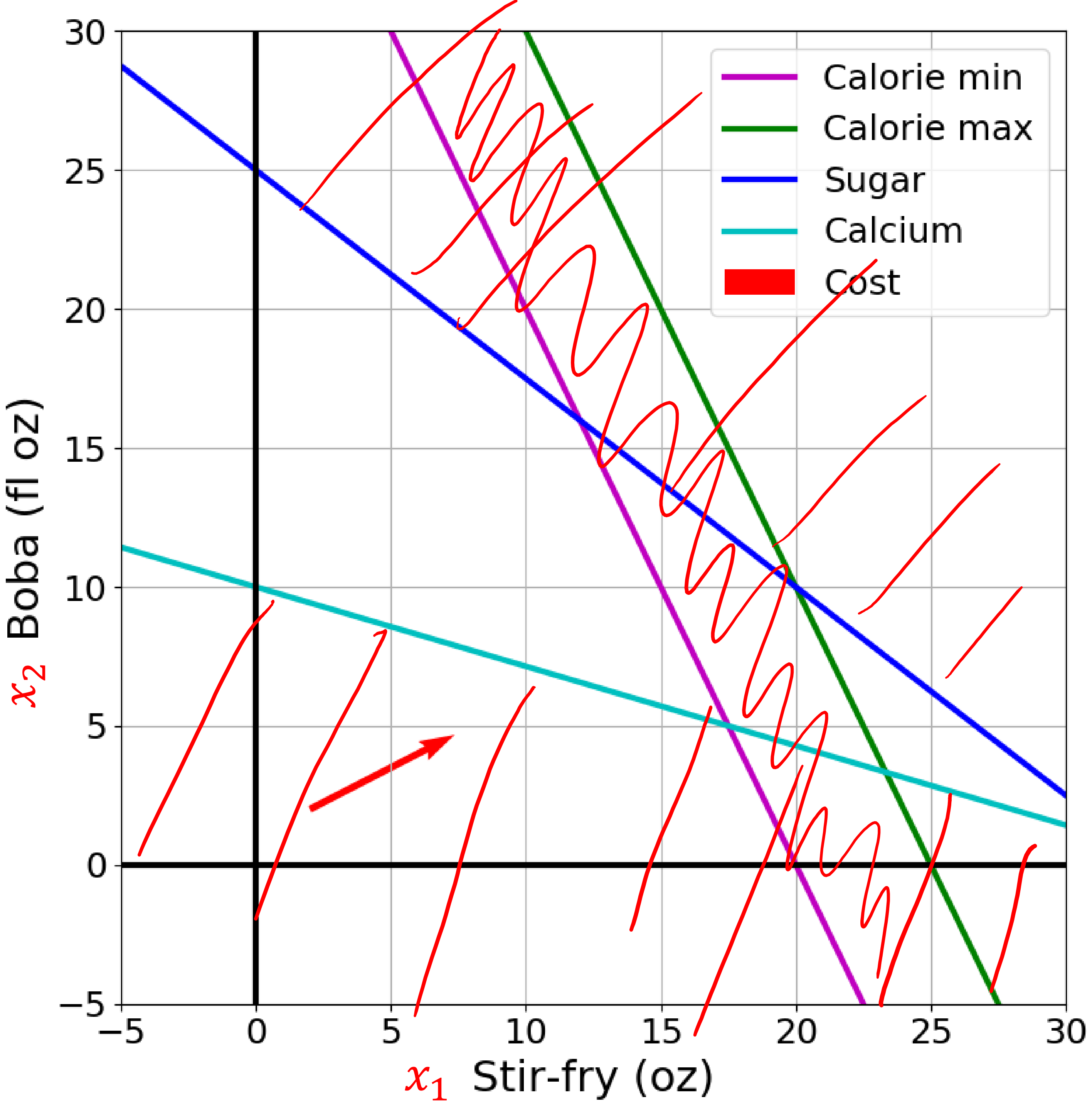
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
$$3\,x_1 + 4\,x_2 \geq 100$$
$$20\,x_1 + 70\,x_2 \leq 700$$

What is the feasible region?

What is the optimal solution?

Infeasible!



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
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$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$$\begin{aligned} \text{s.t.} \quad & 100\, x_1 + 50\, x_2 \geq 2000 \\ & 100\, x_1 + 50\, x_2 \leq 2500 \\ & 3\, x_1 + 4\, x_2 \geq 100 \\ & 20\, x_1 + 70\, x_2 \leq 700 \end{aligned}$$

What is the feasible region?

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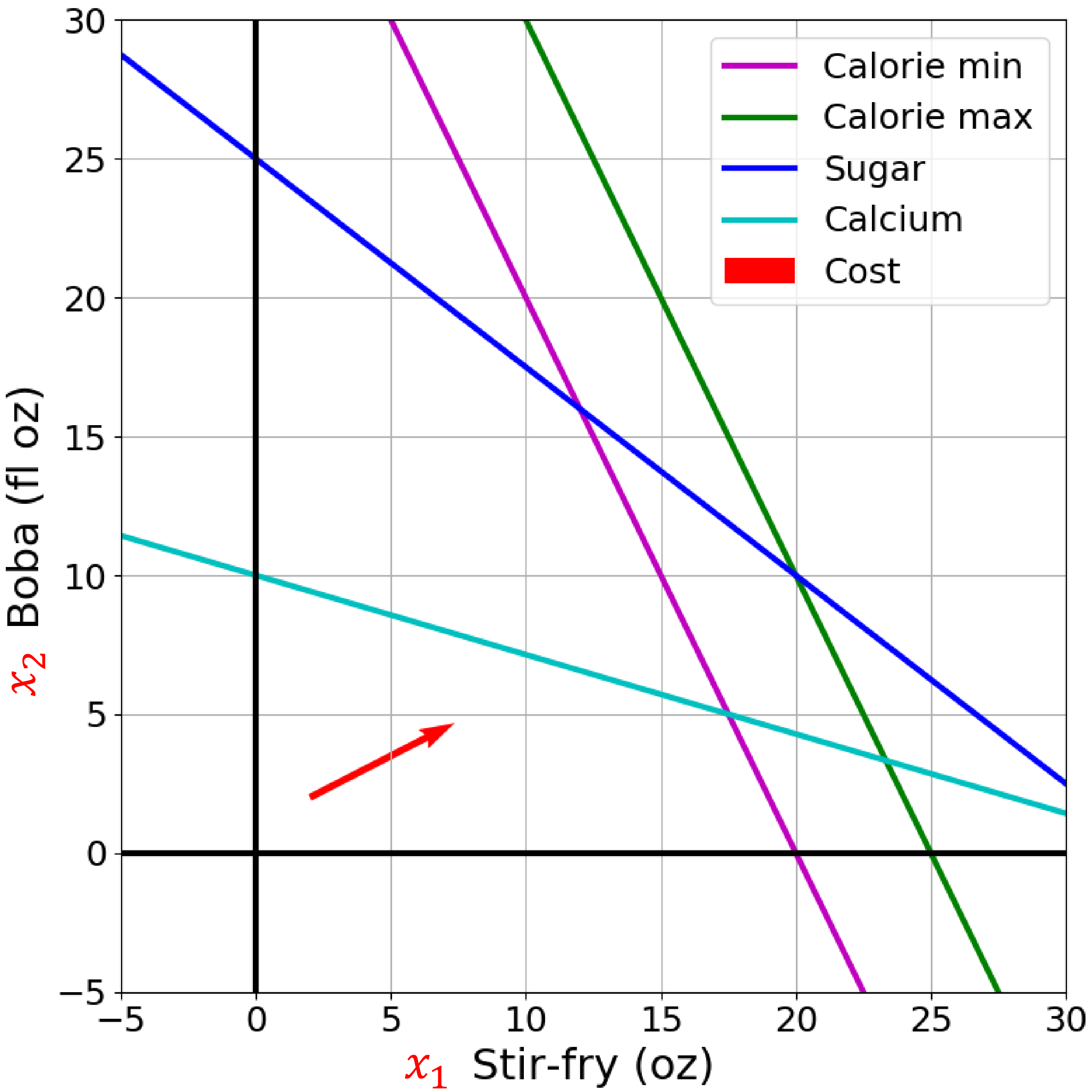
Infeasible!

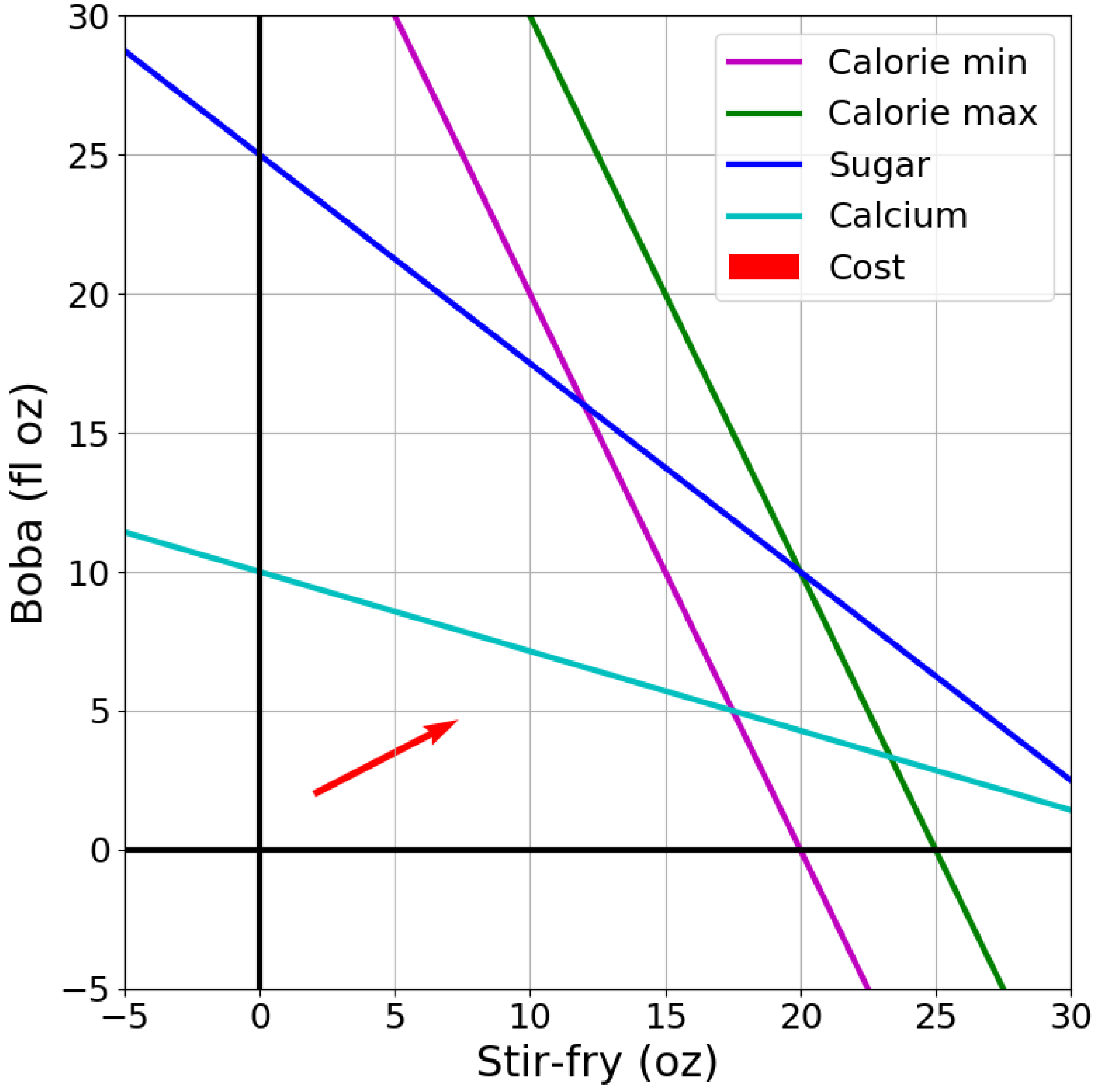


Solving an LP

If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value



Solving an LP

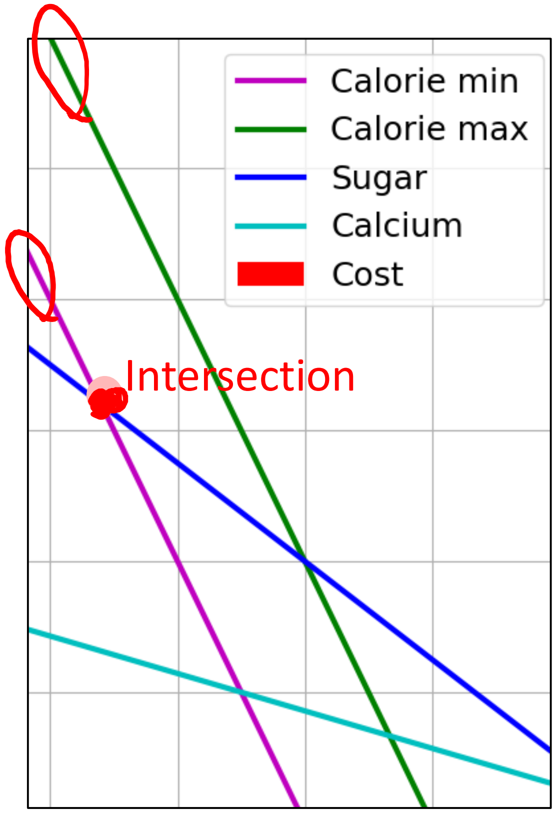
$M \times N$ $\quad \binom{M}{2}$ intersections

But, how do we find the intersections?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



A point is the intersection of two lines

Pick two lines: two rows in A matrix time $\boldsymbol{x}$ equals the corresponding rows in $\boldsymbol{b}$

$$100\,x_1 + 50\,x_2 = 2000$$
$$3\,x_1 + 4\,x_2 = 100$$

$A[(1,3),:]\,x = b[(1,3)]$

Not any pair of rows can lead to an intersection

$$100\,x_1 + 50\,x_2 = 2000$$
$$100\,x_1 + 50\,x_2 = 2500$$

Solving an LP

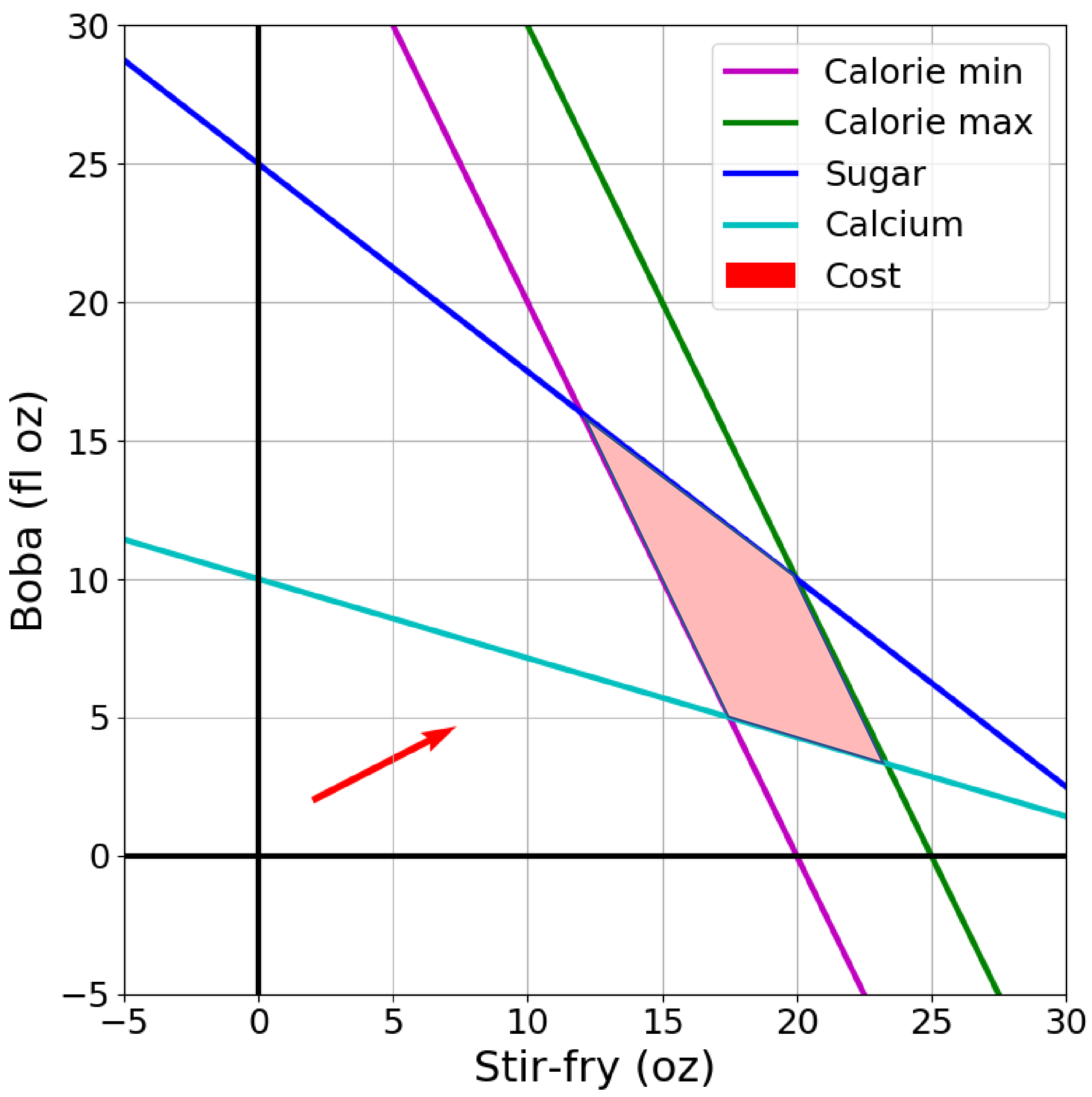
If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found

Simplex is most similar to which Local Search algorithm?

Hill Climbing!



Solving an LP

If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
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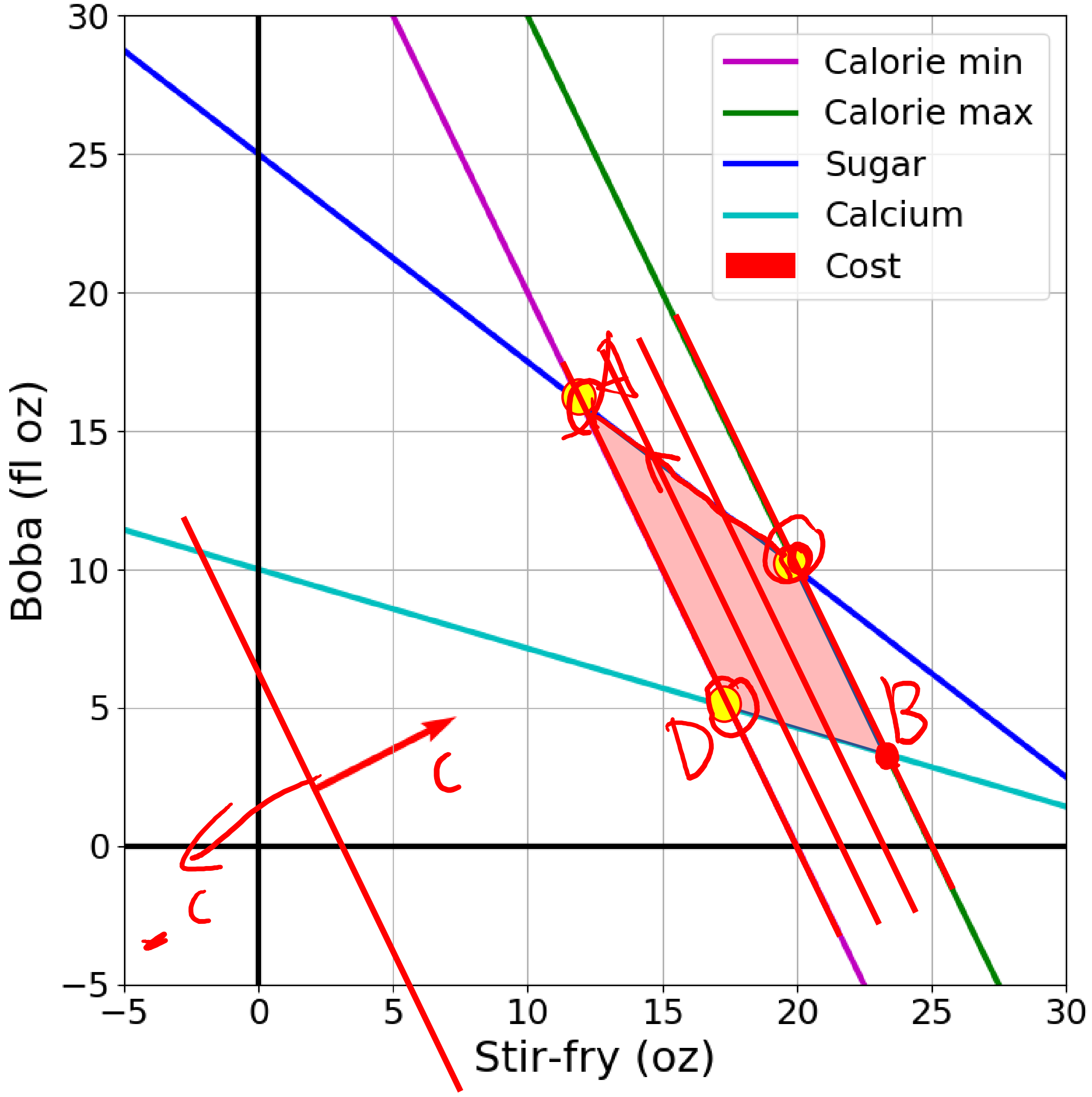
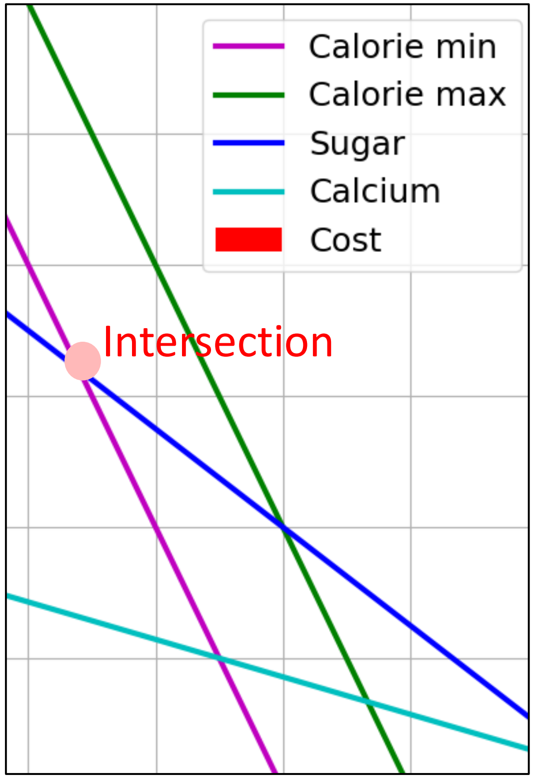


Piazza Poll 5

Simplex Algorithm is most similar to which search algorithm?

- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found



A: Depth First Search
B: Random Walk
C: Hill Climbing
D: Beam Search

Solving an LP

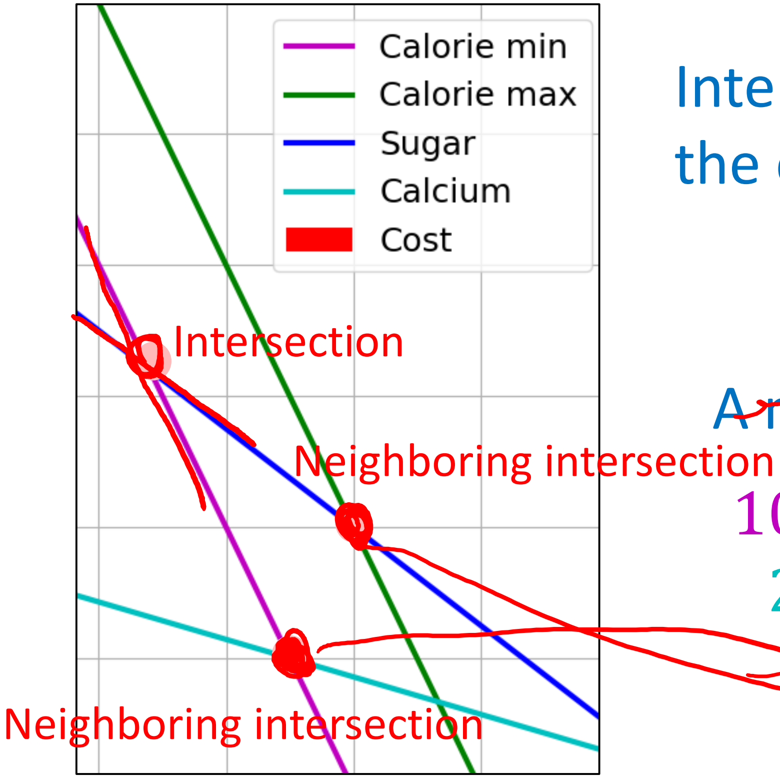
$A[(1,3),:]\, x = b[(1,3)]$

But, how do we find a neighboring intersection?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



Intersection determined by: two rows in A matrix time $\boldsymbol{x}$ equals the corresponding rows in $\boldsymbol{b}$

$$100\,x_1 + 50\,x_2 = 2000 \quad ✗$$
$$3\,x_1 + 4\,x_2 = 100$$

A neighboring intersection only have one different row

$$100\,x_1 + 50\,x_2 = 2000$$
$$20\,x_1 + 70\,x_2 = 700$$

$$100\,x_1 + 50\,x_2 = 2500 \quad ✓$$
$$3\,x_1 + 4\,x_2 = 100$$

Focus of Today: (Linear) Optimization Problem

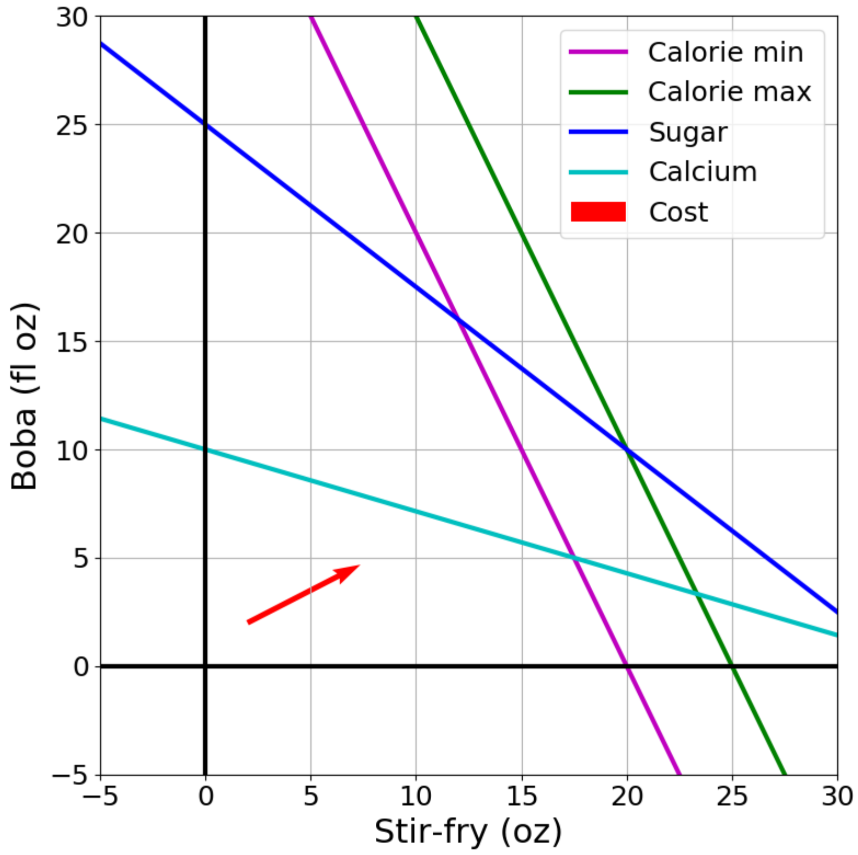
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



“Marty, you’re not thinking fourth-dimensionally”

Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$a_1 x_1 + a_2 x_2 + a_3 x_3 = b_1$

	2-D	3-D	N-D
$a_1\, x_1 + a_2\, x_2 = b_1$	Line	Plane	hyperplane
$a_1\, x_1 + a_2\, x_2 \leq b_1$	half plane	half space	halfspace
$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$ $a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$ $a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$ $a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$	polygon	polyhedron	polytope

What are intersections in higher dimensions?

How do these linear shapes extend to 3-D, N-D?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

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Calorie min
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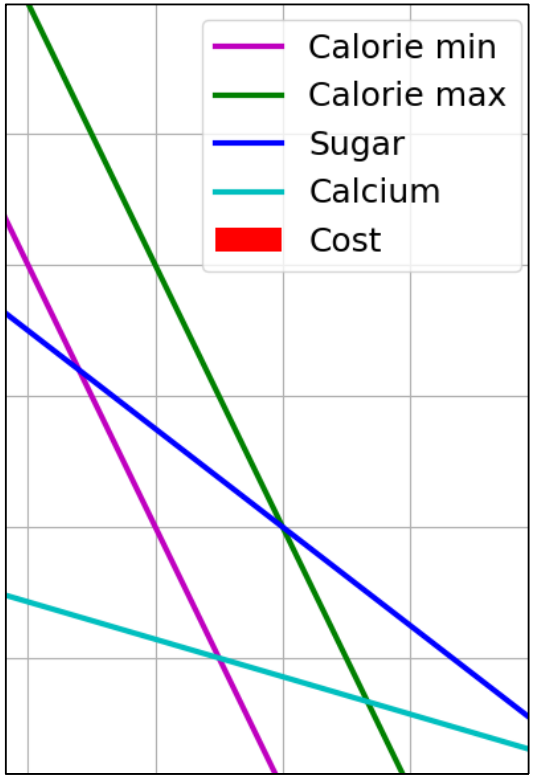


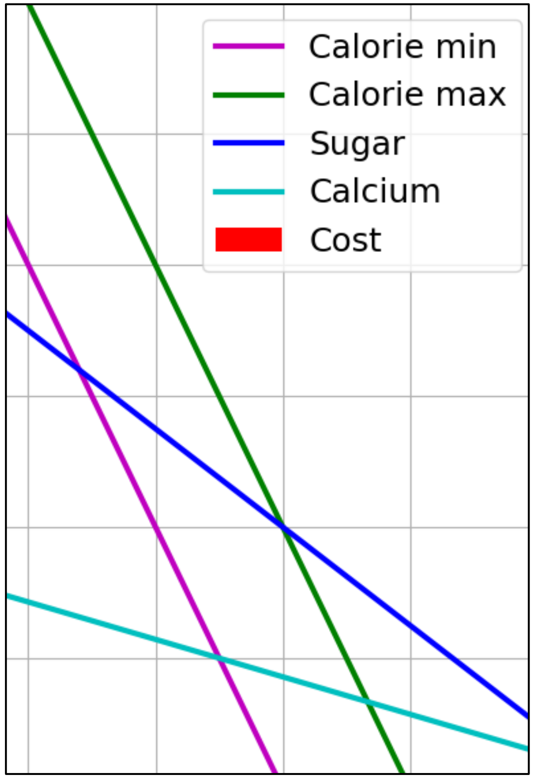
How do we find intersections in higher dimensions?

Still looking at subsets of rows in the A matrix

$$\min_{x} \quad c^T x$$
$$\text{s.t.} \quad Ax \preceq b$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



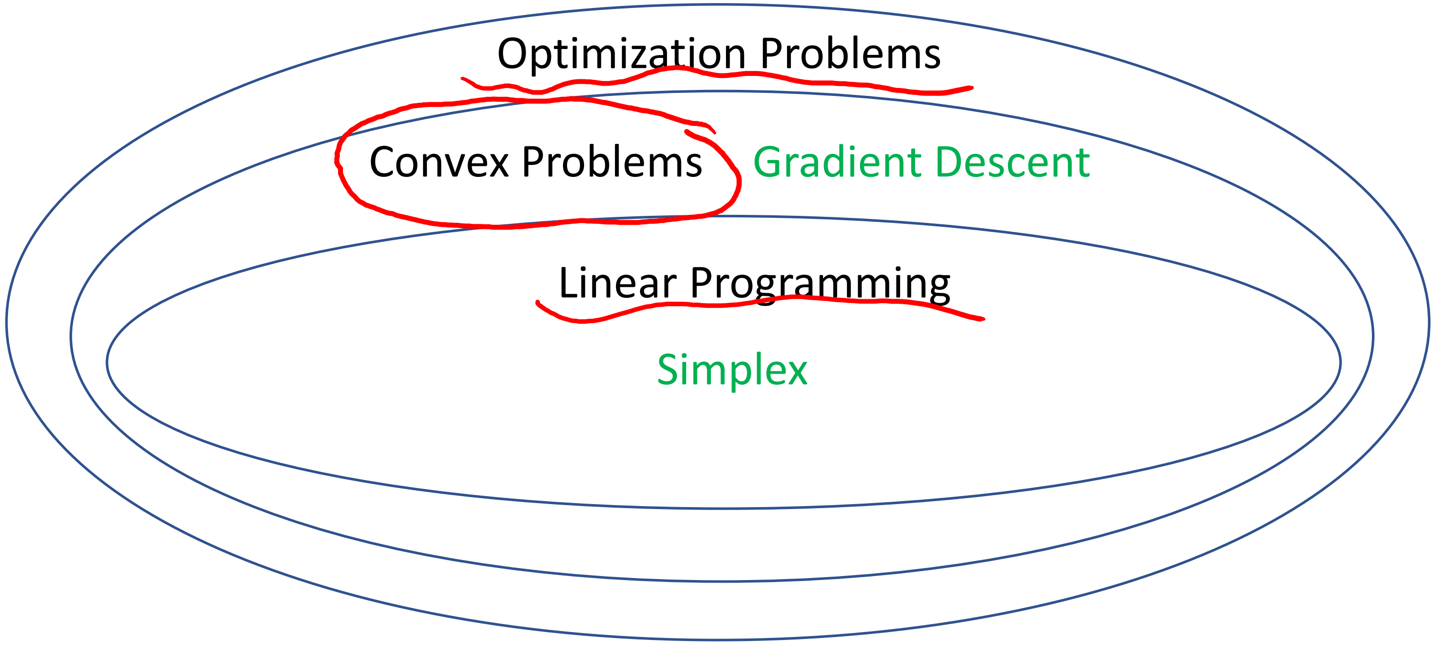
Rank(A[1:2,:])=? 1
Rank(A[(1,3),:])=? 2

x_3 $\quad x \in \mathbb{R}^3$

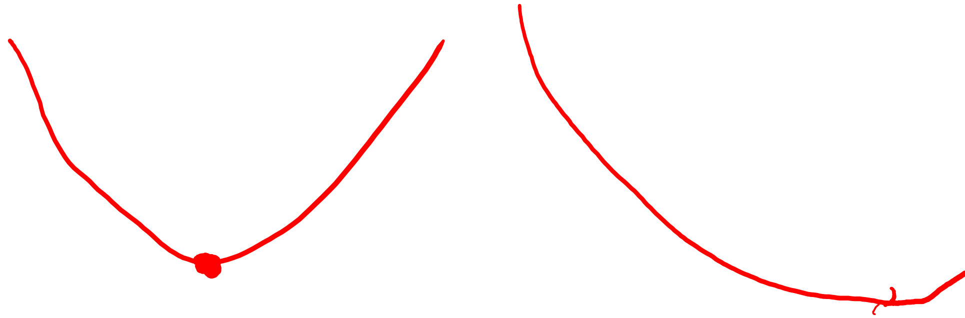
$A[(1,3,4),:]\,x = b$

Summary

- Linear optimization problem handles continuous space but is closely connected to discrete space (only need to consider vertices)
- For a problem with two variables, graphical representation is helpful
- LP is a special class of optimization problems

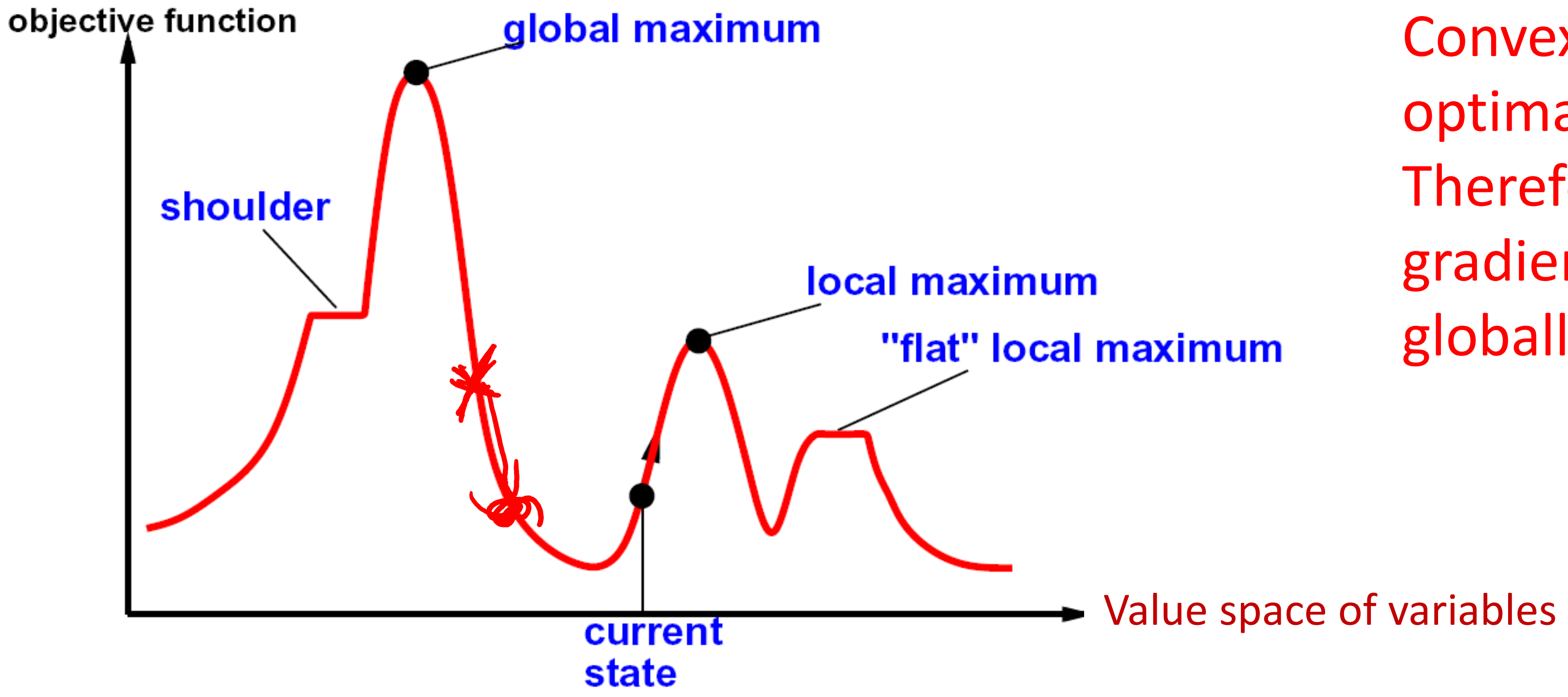


Gradient Descent/Ascent



Find $x \in [a, b]$ that minimize / maximizes $f(x)$?

- Gradient descent / ascent
  - Use gradient to find best direction
  - Use the magnitude of the gradient to determine how big a step you move



Convex Problems: Any local optima is global optima. Therefore, for convex problems, gradient descent/ascent leads to globally optimal solution.