Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

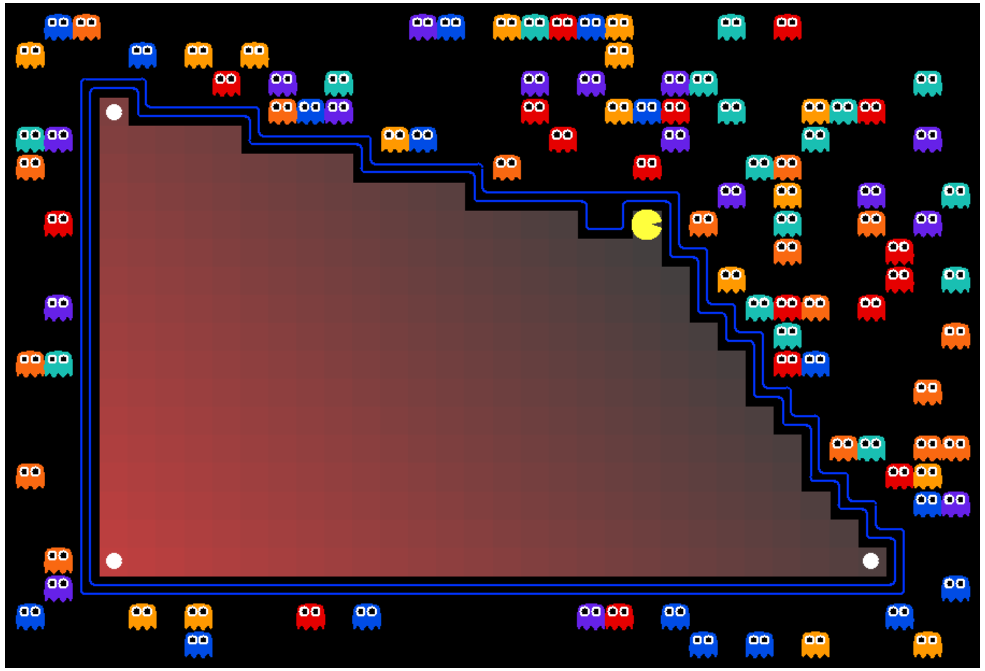
Announcements

Assignments:

- P2
  - Sat 10/5, 10 pm
  - Due ~~Thu 10/3, 10 pm~~
- P3
  - Will be released later today
  - Due Thu 10/17, 10 pm
- HW5 (written)
  - Released Tue 10/1
  - Due Tue 10/8, 10 pm

AI: Representation and Problem Solving

Optimization & Linear Programming



Instructors: Fei Fang & Pat Virtue

Slide credits: CMU AI, http://ai.berkeley.edu

Learning Objectives

- Formulate a problem as a Linear Program (LP)
- Convert a LP into a required form, e.g., inequality form
- Plot the graphical representation of a linear optimization problem with two variables and find the optimal solution
- Understand the relationship between optimal solution of an LP and the intersections of constraints
- Describe and implement a LP solver based on vertex enumeration
- Describe the high-level idea of Simplex algorithm

Next Lecture

Recap

What have we learned so far?

What do they have in common?

This lecture:

(1) Continuous space

(2) General formulation

Recap

What have we learned so far?

- Search: Depth/Breadth-first search, A* search, local search
- Constraint Satisfaction Problem: 8-queen, graph coloring
- Logic and Planning: Propositional Logic, SAT, First-order Logic

What do they have in common?

  - Variables/Symbols, Finite options

This lecture:

(1) Move to continuous space (with connections to the discrete space)

(2) Provide a general formulation that can be used to represent many of the previously seen problems

Focus of Today: (Linear) Optimization Problem

Problem Description

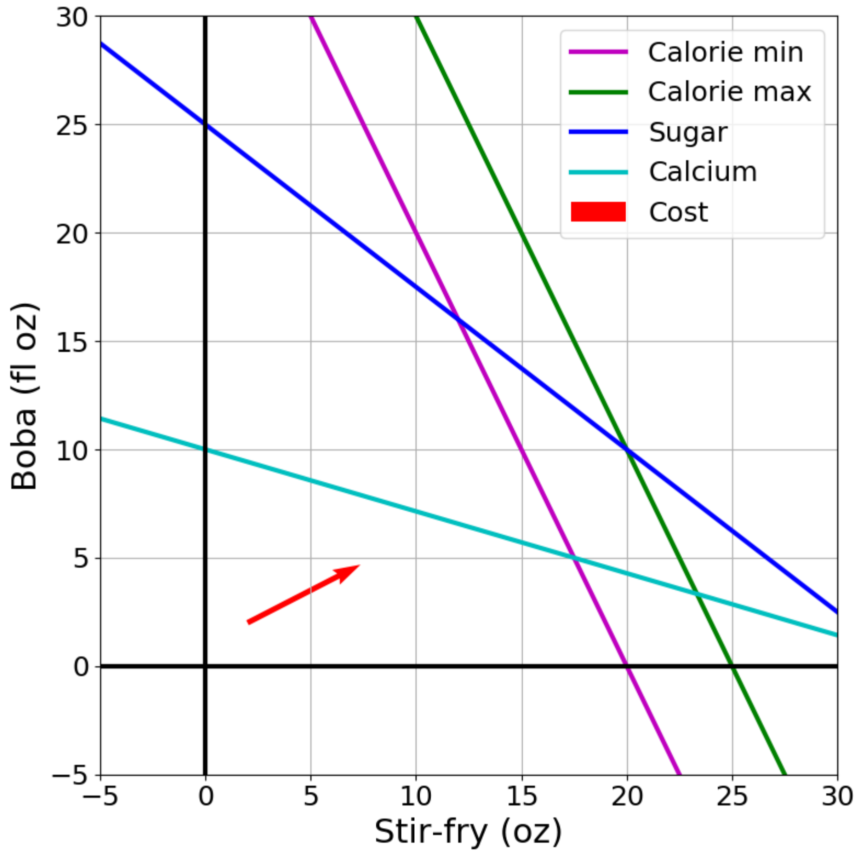
Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Notation Alert!

Graphical Representation



Diet Problem: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
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What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

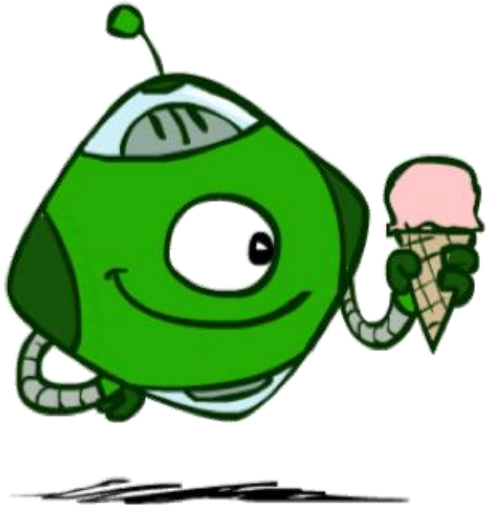
Problem Formulation

Can we formulate it as a Constraint Satisfaction Problem?

Variable:

Domain:

Constraint:



What are the issues with this CSP formulation?

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation

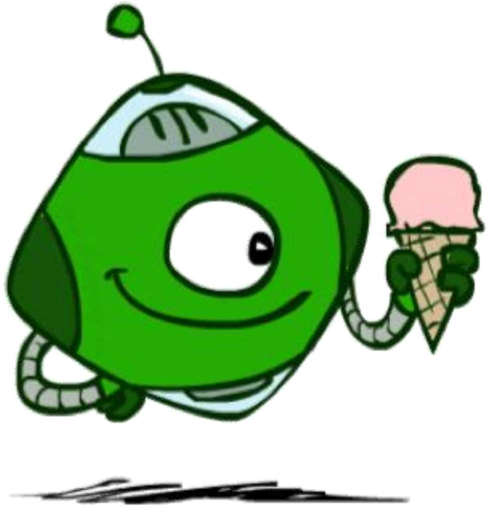
Can we formulate it as a Constraint Satisfaction Problem?

Variable: x_1 (ounces for stir-fry), x_2 (ounces for boba)

Domain: $[0, +\infty)$

Constraint: Implicit: $100\ x_1 + 50\ x_2 \in [2000{,}2500]$

$$3\ x_1 + 4\ x_2 \leq 100,$$
$$20\ x_1 + 70\ x_2 \geq 700$$



What are the issues with this CSP formulation?

Healthy Squad Goals

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What is the cheapest way to stay “healthy” with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Problem Formulation

Optimization problem: Finding the best solution from all feasible solutions

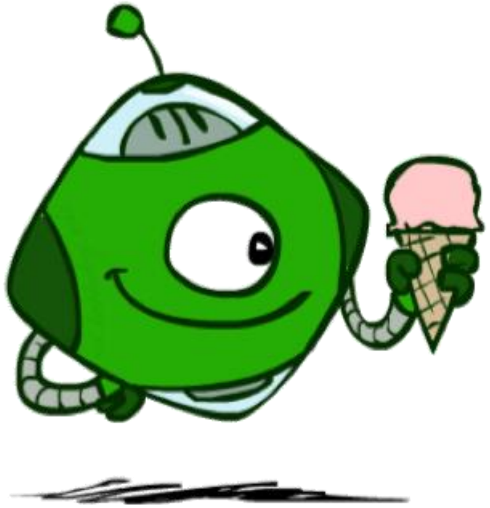
From CSP to Optimization Problem

Variable
Domain
Constraint

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \qquad \text{Objective}$$

$$\text{s.t.} \quad \boldsymbol{x} \;\; satisfies \;\; constraints$$

Notation Alert!



Healthy Squad Goals

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- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Optimization problem: Finding the best solution from all feasible solutions

From CSP to Optimization Problem

(Optimization) Variable

~~Domain~~ Can be represented as constraints

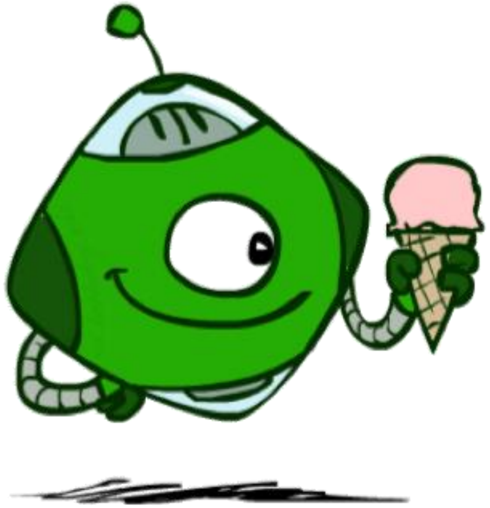
Constraint

Optimization Objective

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \qquad \text{Objective}$$

$$\text{s.t.} \quad \boldsymbol{x} \;\; satisfies \;\; constraints$$

Notation Alert!



Healthy Squad Goals

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- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x})$$

$$\text{s.t.} \quad \boldsymbol{x} \;\; satisfies \;\; constraints$$

Variable:

Objective:

Constraints:

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x})$$

$$\text{s.t.} \quad \boldsymbol{x} \ \ satisfies \ \ constraints$$

Variable: $\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

Objective: $\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$

Constraints:

$$\text{calories}(\boldsymbol{x}) \ in\ required\ range$$
$$\text{sugar}(\boldsymbol{x}) \leq limit$$
$$\text{calcium}(\boldsymbol{x}) \geq limit$$
$$x_1 \geq 0, x_2 \geq 0$$

Can be ignored in this problem. Why?

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$$
$$\text{s.t.} \quad \text{calories}(\boldsymbol{x}) \; \textit{in required range}$$
$$\text{sugar}(\boldsymbol{x}) \leq \textit{limit}$$
$$\text{calcium}(\boldsymbol{x}) \geq \textit{limit}$$

$\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

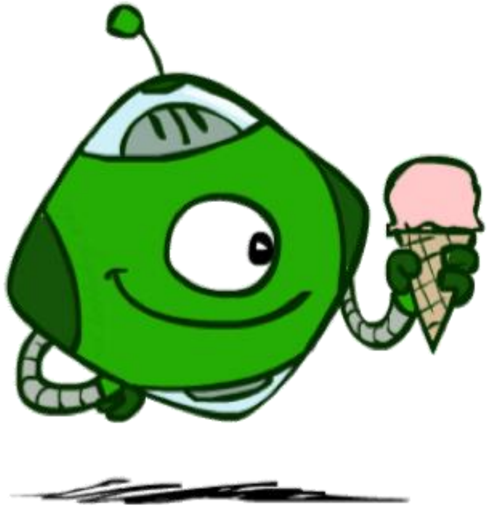
What is the expression of

$\text{cost}(\boldsymbol{x})$

$\text{calories}(\boldsymbol{x})$

$\text{sugar}(\boldsymbol{x})$

$\text{calcium}(\boldsymbol{x})$



Healthy Squad Goals

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Problem Formulation

Formulate Diet Problem as an optimization problem

$$\min_{\boldsymbol{x}} \quad \text{cost}(\boldsymbol{x})$$
$$\text{s.t.} \quad \text{calories}(\boldsymbol{x}) \ \textit{in required range}$$
$$\text{sugar}(\boldsymbol{x}) \leq \textit{limit}$$
$$\text{calcium}(\boldsymbol{x}) \geq \textit{limit}$$

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

$\boldsymbol{x} = [x_1, x_2]^T$. x_1: ounces for stir-fry, x_2: ounces for boba

What is the expression of

$$\text{cost}(\boldsymbol{x}) = x_1 + 0.5x_2$$

$$\text{calories}(\boldsymbol{x}) = 100x_1 + 50x_2$$

$$\text{sugar}(\boldsymbol{x}) = 3x_1 + 4x_2$$

$$\text{calcium}(\boldsymbol{x}) = 20x_1 + 70x_2$$

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Problem Formulation

Formulate Diet Problem as an optimization problem

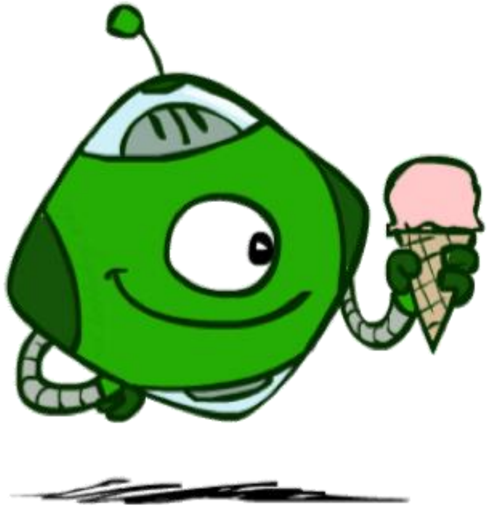
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$

$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700$$



Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Piazza Poll 1

Is $x_1 = 10, x_2 = 30$ a feasible solution of the following optimization problem?

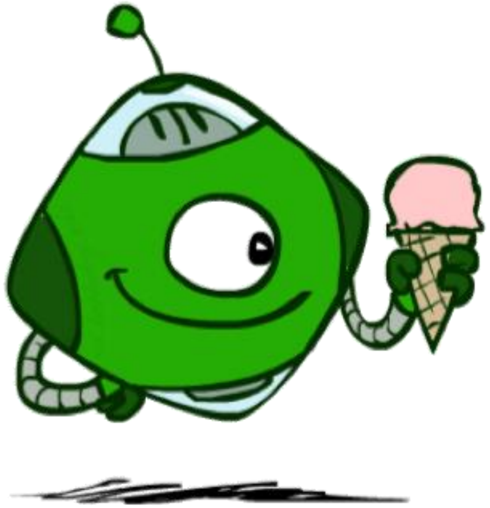
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$

$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700$$



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Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Linear Programming

Linear Programming:

- Technique for the optimization of a linear objective function subject to linear equality and linear inequality constraints

Example Linear program

$$\begin{aligned} \min_{x_1, x_2} \quad & 1\,x_1 + 0.5\,x_2 \\ \text{s.t.} \quad & 100\,x_1 + 50\,x_2 \geq 2000 \\ & 100\,x_1 + 50\,x_2 \leq 2500 \\ & 3\,x_1 + \ \ 4\,x_2 \leq 100 \\ & 20\,x_1 + 70\,x_2 \geq 700 \end{aligned}$$

Linear program in Inequality Form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a} + \boldsymbol{b}$? $\boldsymbol{a} - \boldsymbol{b}$?

What does $\boldsymbol{a} \preceq 0$ mean?

(Note: It is fine if you directly write $\boldsymbol{a} \leq 0$)

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a} + \boldsymbol{b}$? $\boldsymbol{a} - \boldsymbol{b}$?

$$\boldsymbol{a} \pm \boldsymbol{b} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

What does $\boldsymbol{a} \preceq 0$ mean?

$$\forall i, a_i \leq 0$$

(Note: It is fine if you directly write $\boldsymbol{a} \leq 0$)

Recap of Linear Algebra

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

What is $\boldsymbol{a}^T \boldsymbol{b}$?

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $A\boldsymbol{b}$?

Recap of Linear Algebra

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

What is $\boldsymbol{a}^T\boldsymbol{b}$?

$$\boldsymbol{a}^T\boldsymbol{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

What is $A\boldsymbol{b}$?

$$A\boldsymbol{b} = \begin{bmatrix} a_{1,1}b_1 + a_{1,2}b_2 + \cdots a_{1,m}b_m \\ \vdots \\ a_{n,1}b_1 + a_{n,2}b_2 + \cdots a_{n,m}b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1,i}b_i \\ \vdots \\ \sum_{i=1}^{m} a_{n,i}b_i \end{bmatrix}$$

Problem Formulation

$$\begin{aligned} \min_{x_1, x_2} \quad & 1\, x_1 + 0.5\, x_2 \\ \text{s.t.} \quad & 100\, x_1 + 50\, x_2 \geq 2000 \\ & 100\, x_1 + 50\, x_2 \leq 2500 \\ & 3\, x_1 + 4\, x_2 \leq 100 \\ & 20\, x_1 + 70\, x_2 \geq 700 \end{aligned}$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Problem Formulation

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2 \qquad \boldsymbol{c}^T\boldsymbol{x} \text{ with } \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000 \qquad -100\,x_1 - 50\,x_2 \leq -2000$$

$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700 \qquad -20\,x_1 - 70\,x_2 \leq -700$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Problem Formulation

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \min_{x_1, x_2} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & -100\, x_1 - 50\, x_2 \le -2000 \\ & 100\, x_1 + 50\, x_2 \le 2500 \\ & 3\, x_1 + \quad 4\, x_2 \le 100 \\ & -20\, x_1 - 70\, x_2 \le -700 \end{aligned}$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Problem Formulation

$$\min_{x_1, x_2} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad -100\, x_1 - 50\, x_2 \leq -2000$$

$$100\, x_1 + 50\, x_2 \leq 2500$$

$$3\, x_1 + \; 4\, x_2 \leq 100$$

$$-20\, x_1 - 70\, x_2 \leq -700$$

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Convert it into the following inequality form, what should $\boldsymbol{c}$, A, and $\boldsymbol{b}$ be?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Piazza Poll 2

What has to increase to add more nutrition constraints?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) height A

D) width A

E) length $\boldsymbol{b}$

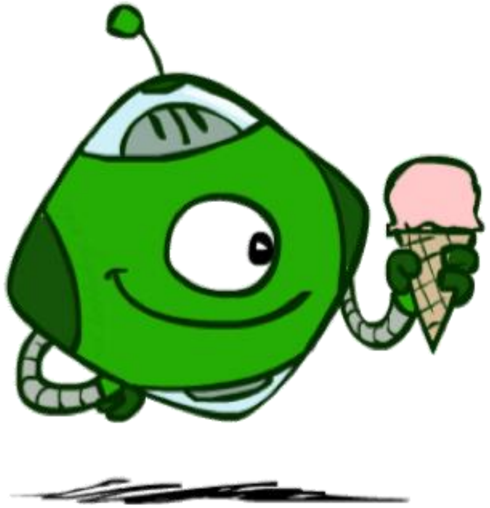
Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- …(More Constraints)

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Piazza Poll 2

What has to increase to add more nutrition constraints?



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
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$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

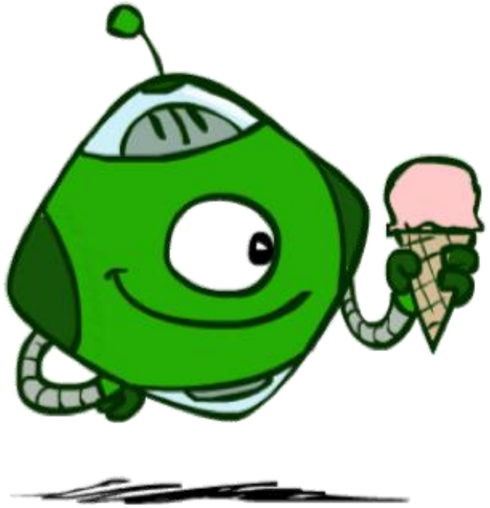
Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- Sugar ≥ 20 g

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Piazza Poll 2

What has to increase to add more nutrition constraints?



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

$$3x_1 + 4x_2 \geq 20$$

$$-3x_1 - 4x_2 \leq -20$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \\ -3 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \\ -20 \end{bmatrix}$$

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg
- Sugar ≥ 20 g

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Question

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Select all that apply

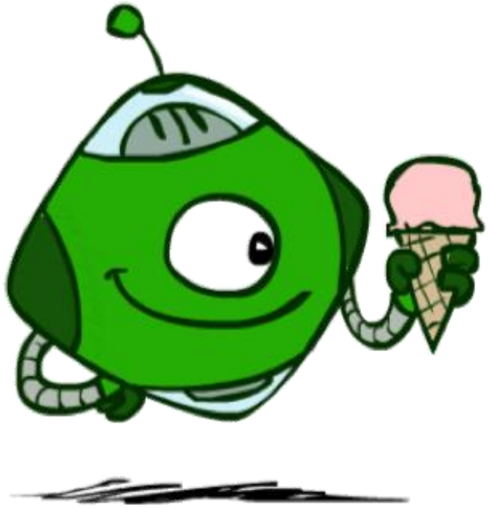
A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) height A

D) width A

E) length $\boldsymbol{b}$



Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
New Item	…	…	…	…

Question

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
Beef (per oz)	2	80	1	30

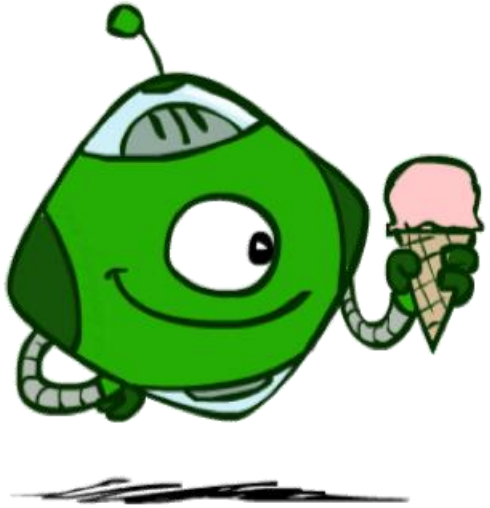
Question

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 & -80 \\ 100 & 50 & 80 \\ 3 & 4 & 1 \\ -20 & -70 & -30 \end{bmatrix} \quad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70
Beef (per oz)	2	80	1	30

Question

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

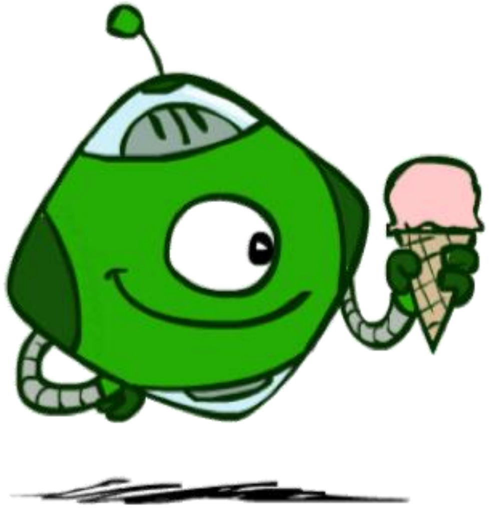
$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$



Linear Programming

Different forms

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Important to pay attention to form!

Can switch between formulations!

Note: Different books have different definitions. We will refer to the forms in this slide (also consistent with B&V book).

Focus of Today: (Linear) Optimization Problem

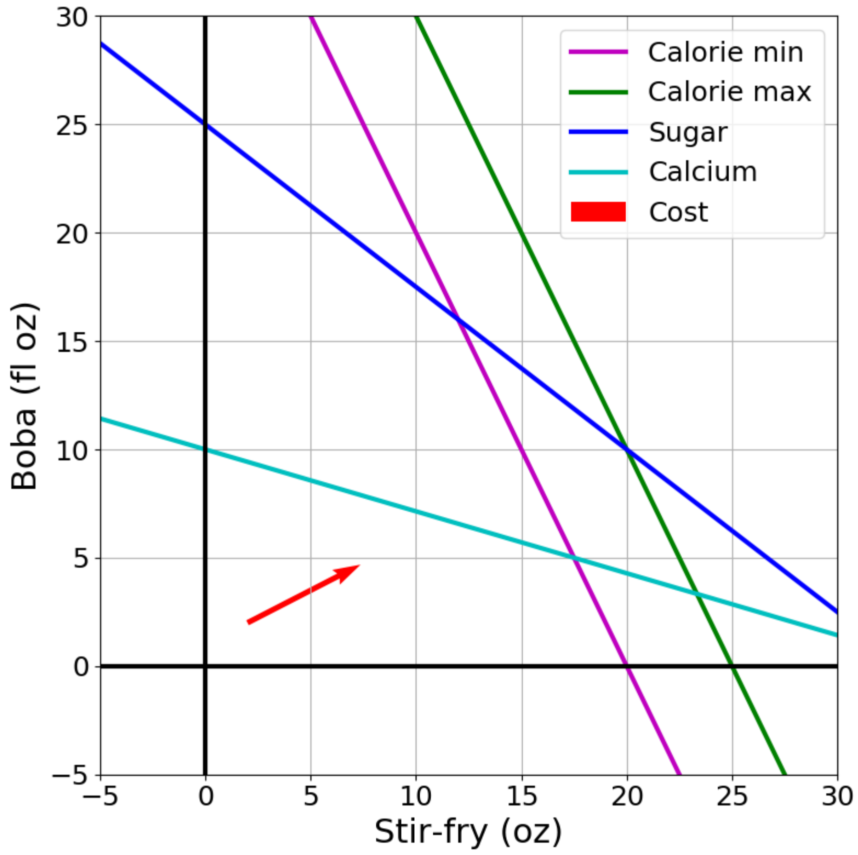
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

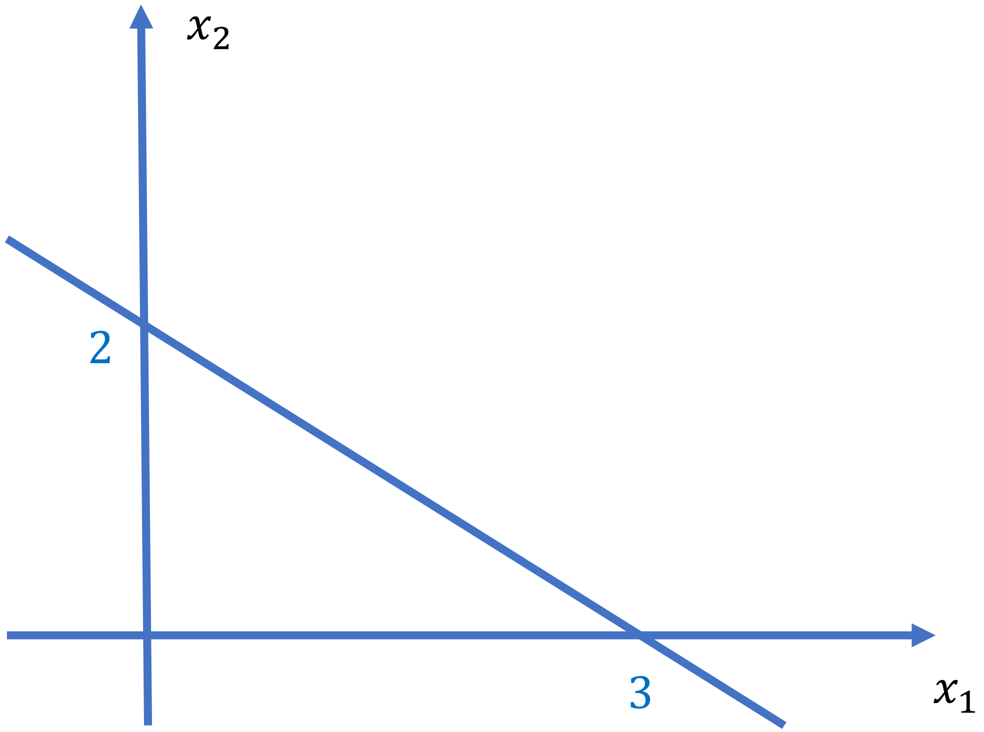
Graphical Representation



Shape of Linear Equality

Geometry / Algebra Question

What is the equality present this line?



Shape of Linear Equality

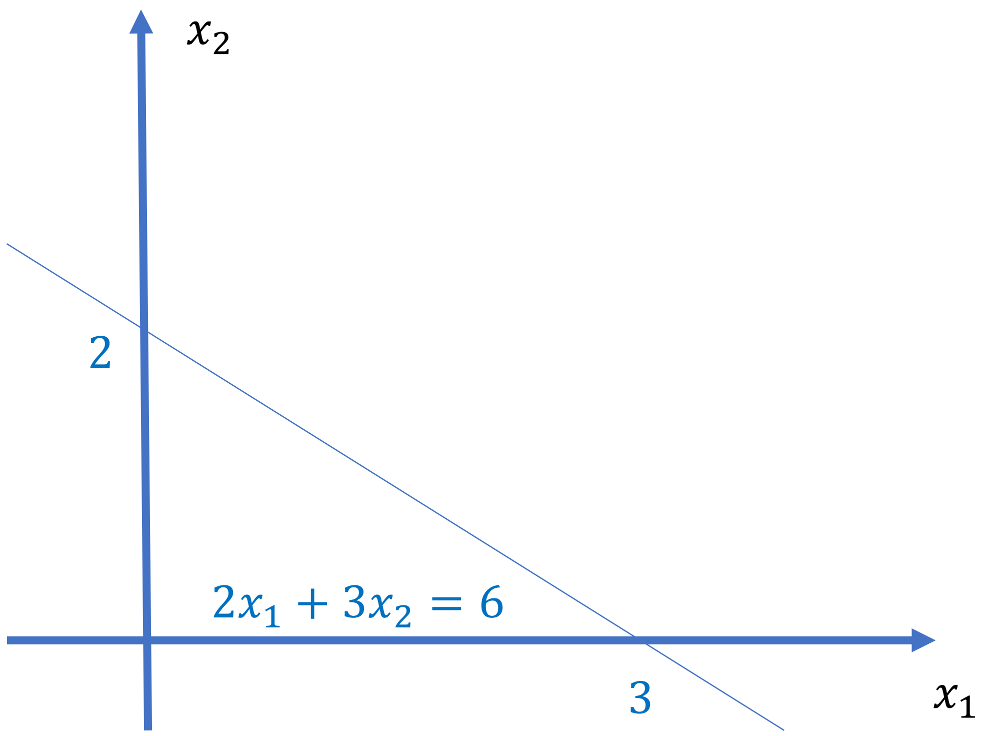
Geometry / Algebra Question

What is the equality present this line?

$$x_2 = -\frac{2}{3}x_1 + 2$$

$$3x_2 + 2x_1 = 6$$

To make sure the line intersects with the axes at (3,0) and (0,2)



Shape of Linear Inequality

Geometry / Algebra Question

What shape does this inequality represent?

$$a_1\,x_1 + a_2\,x_2 \leq b_1$$

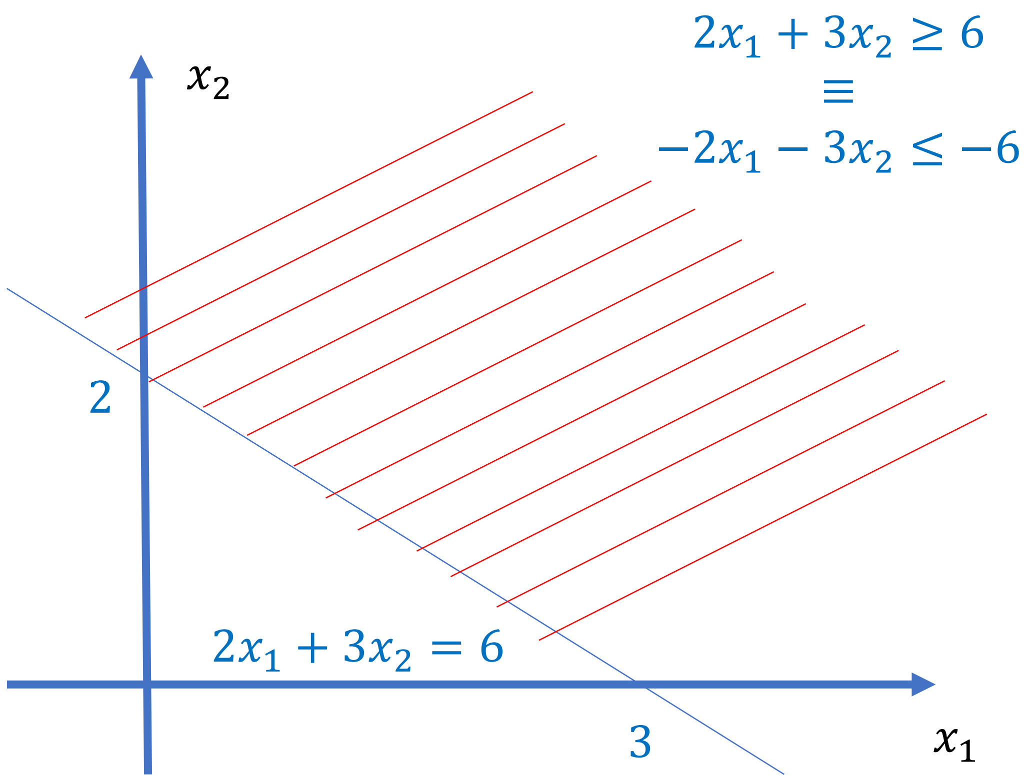
Shape of Linear Inequality

Geometry / Algebra Question

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \le b_1$$ Half Plane

$$a_1\, x_1 + a_2\, x_2 = b_1$$ Line



Shape of Linear Inequality

Geometry / Algebra Question

What shape does these inequalities jointly represent?

$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$

Shape of Linear Inequality

Geometry / Algebra Question

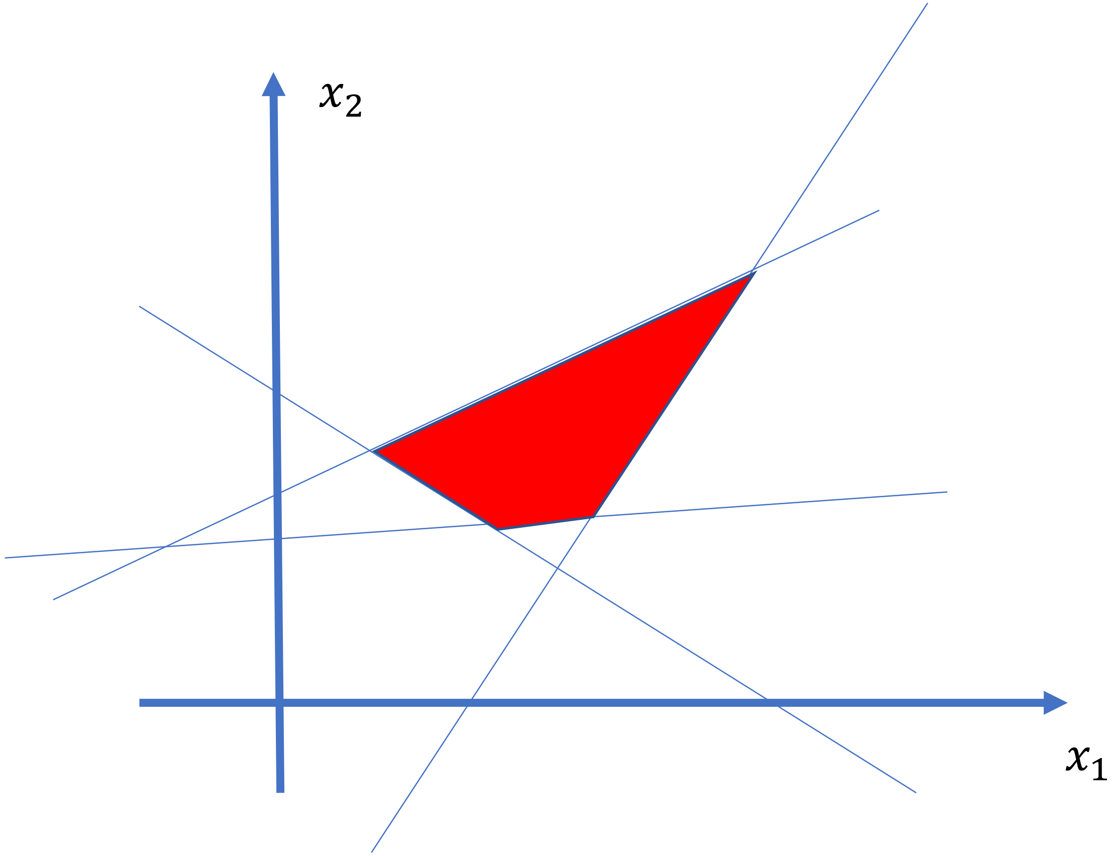
What shape does these inequalities jointly represent?

$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \le b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \le b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \le b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \le b_4$$

Intersection of half planes

Could be polyhedron

Could be empty



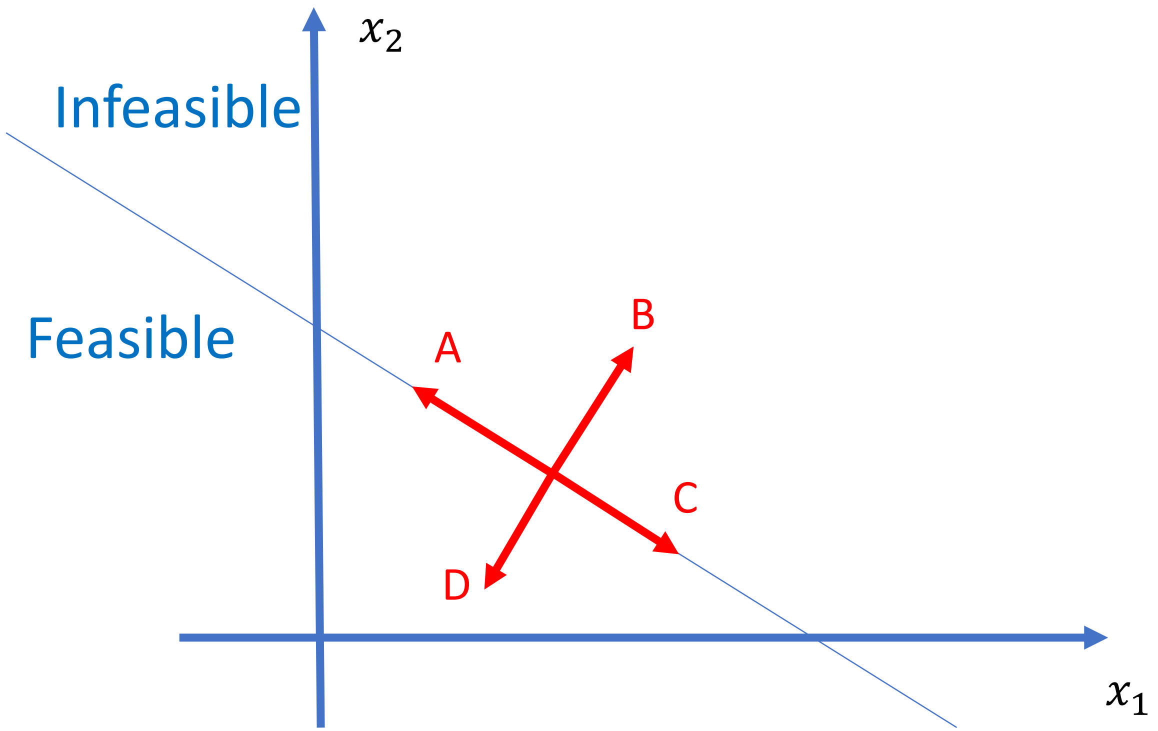
Piazza Poll 3

What is the relationship between the half plane:

$$a_1 x_1 + a_2 x_2 \leq b_1$$

and the vector:

$$[a_1, a_2]^T$$



Lowering Cost

Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle\, \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle\, \boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?

Lowering Cost

Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle\, \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle\, \boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?

$$\boldsymbol{c}^T \boldsymbol{x}^{(1)} = \boldsymbol{c}^T\left(\boldsymbol{x}^{(0)} + \triangle\, \boldsymbol{x}\right) = \boldsymbol{c}^T \boldsymbol{x}^{(0)} + \boldsymbol{c}^T \triangle\, \boldsymbol{x}$$

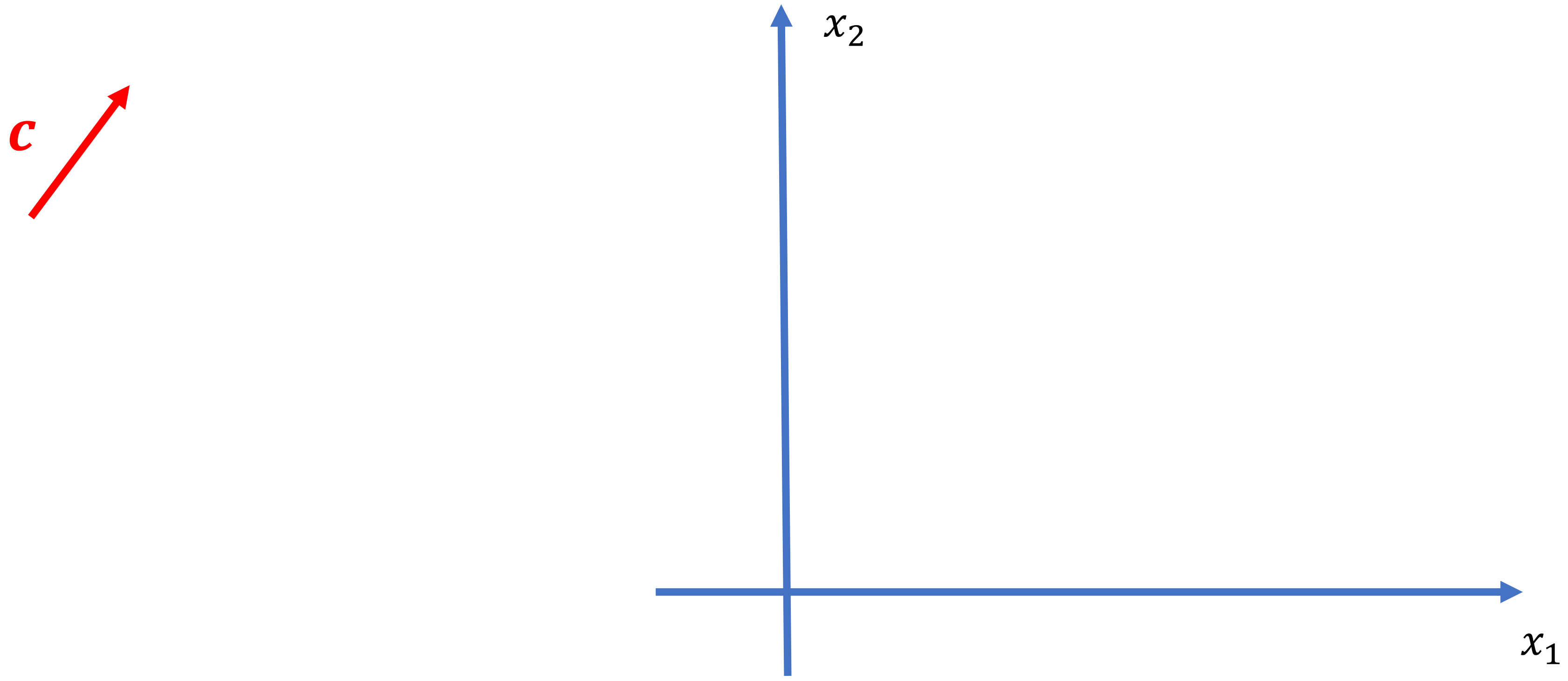
$$\boldsymbol{c}^T \triangle\, \boldsymbol{x} = |\boldsymbol{c}||\triangle\, \boldsymbol{x}| \cos(\boldsymbol{c}, \triangle\, \boldsymbol{x})$$

$\cos(\boldsymbol{c}, \triangle\, \boldsymbol{x}) = -\mathbf{1}$ when $\boldsymbol{c}$ and $\triangle\, \boldsymbol{x}$ have opposite directions

Cost Contours

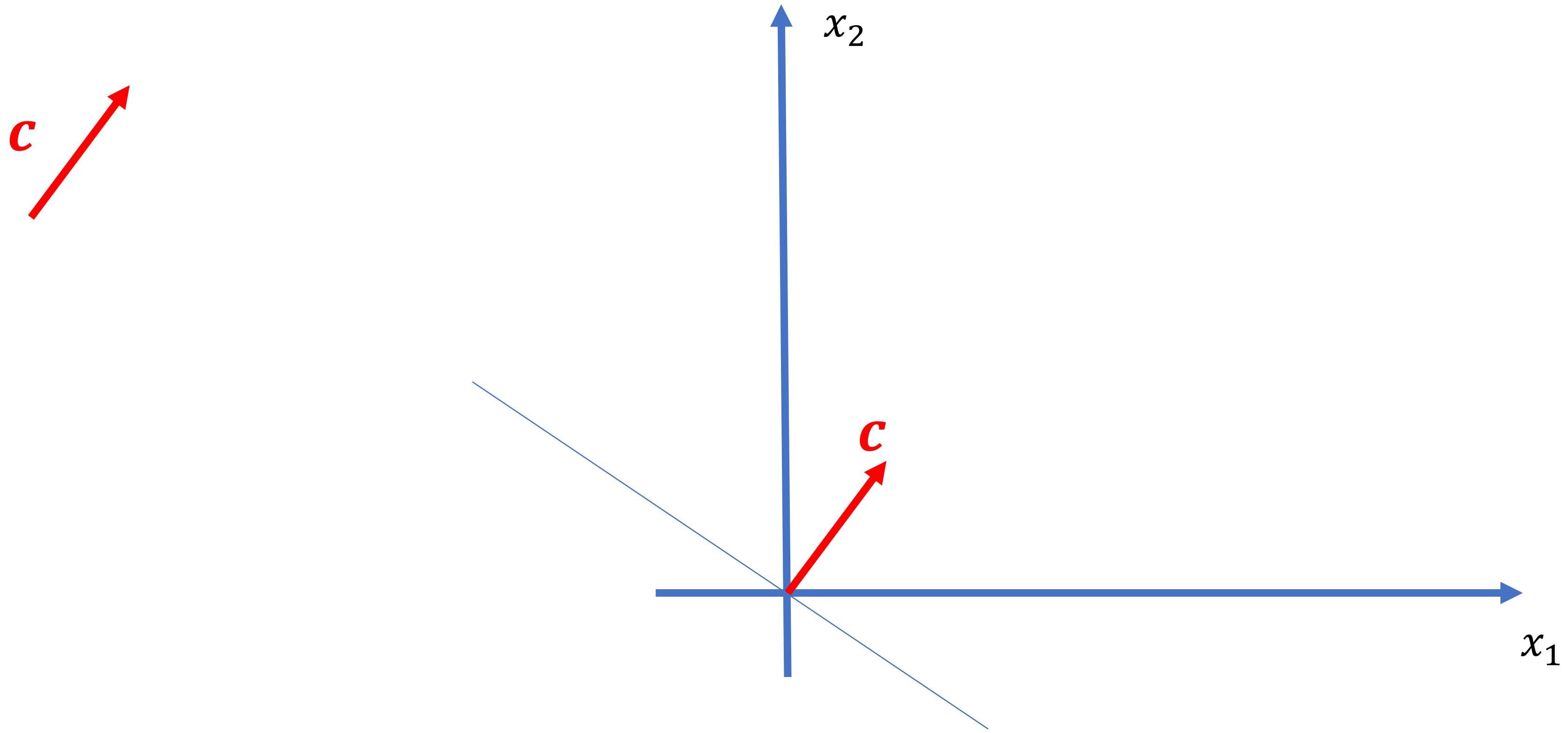
Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T \boldsymbol{x} = 0$?

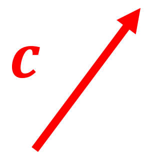


Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will $\boldsymbol{c}^T\boldsymbol{x} = 0$?



Cost Contours



Given the cost vector $[c_1, c_2]^T$ where will

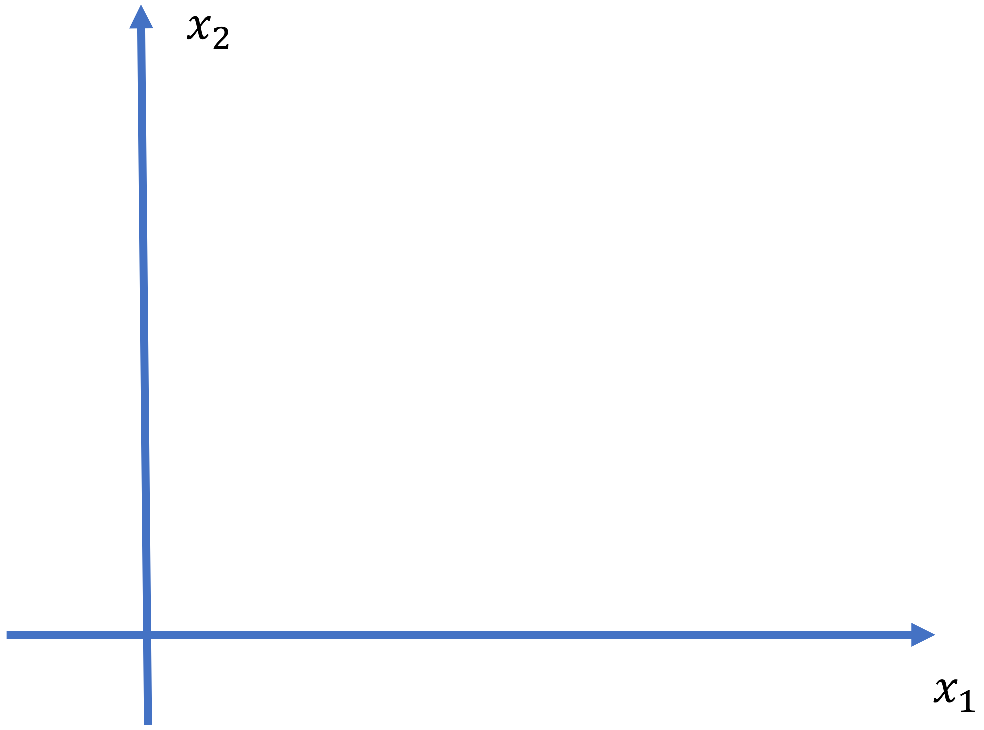
$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -2 ?



Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

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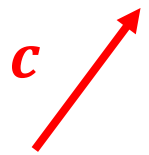
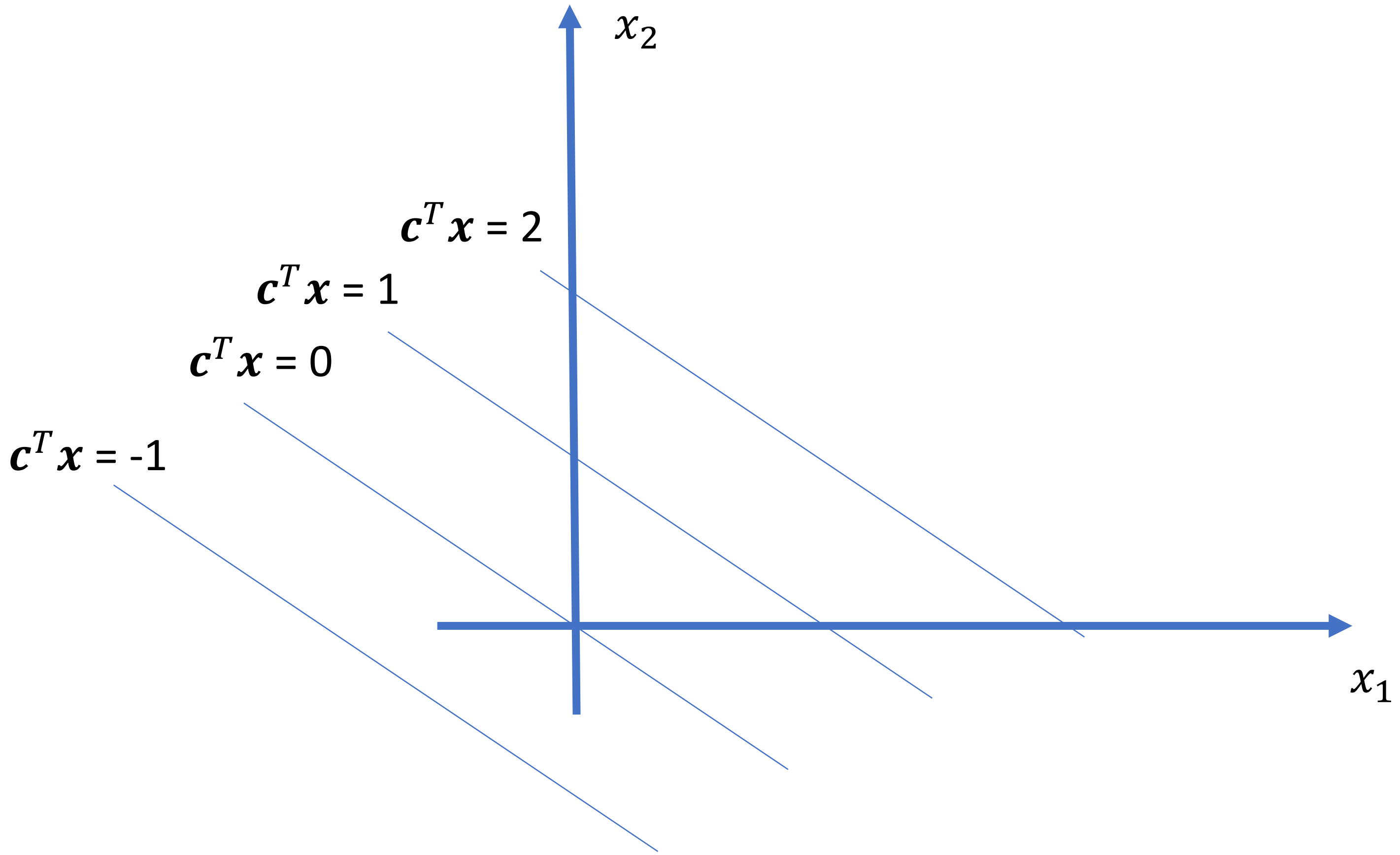
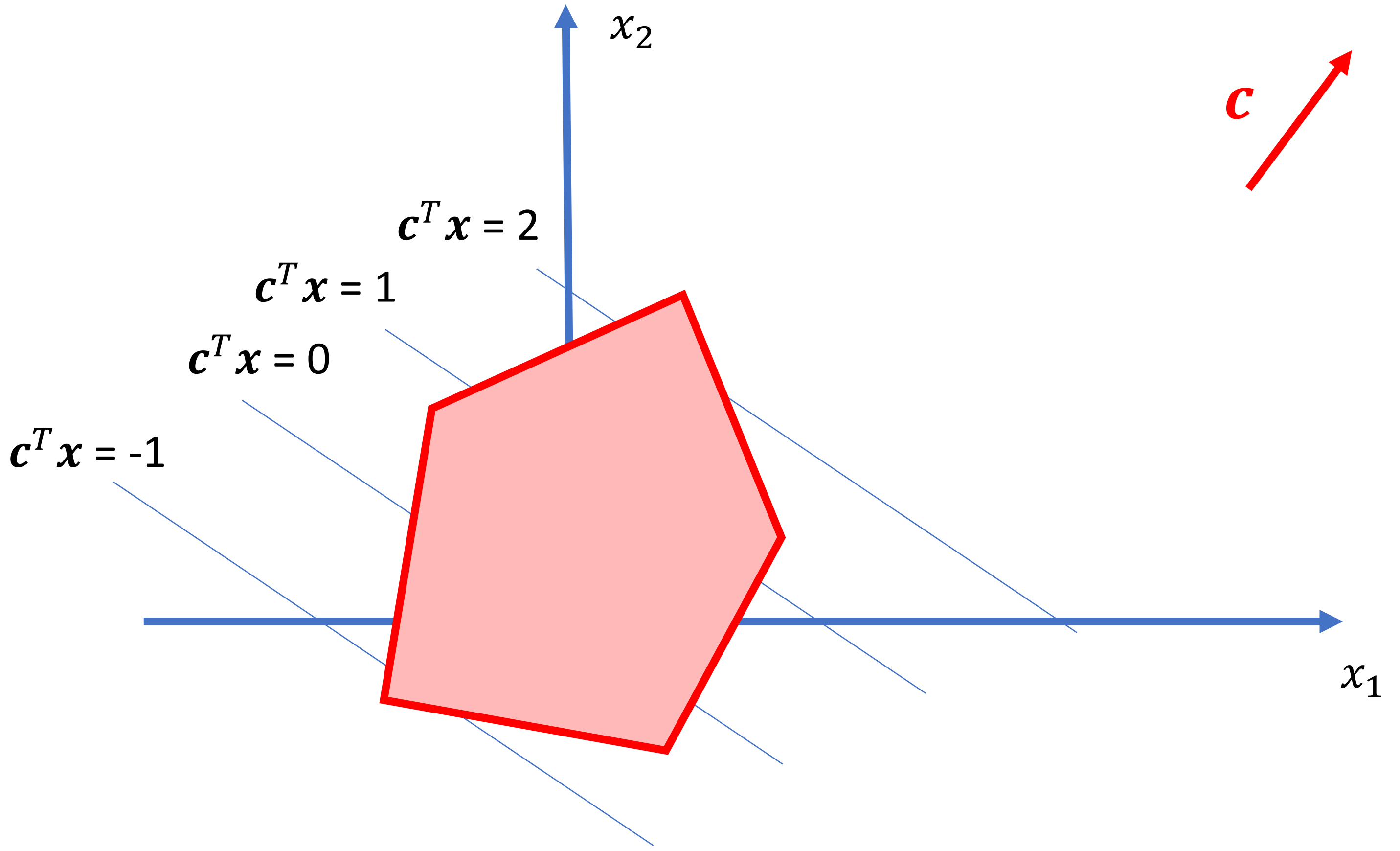




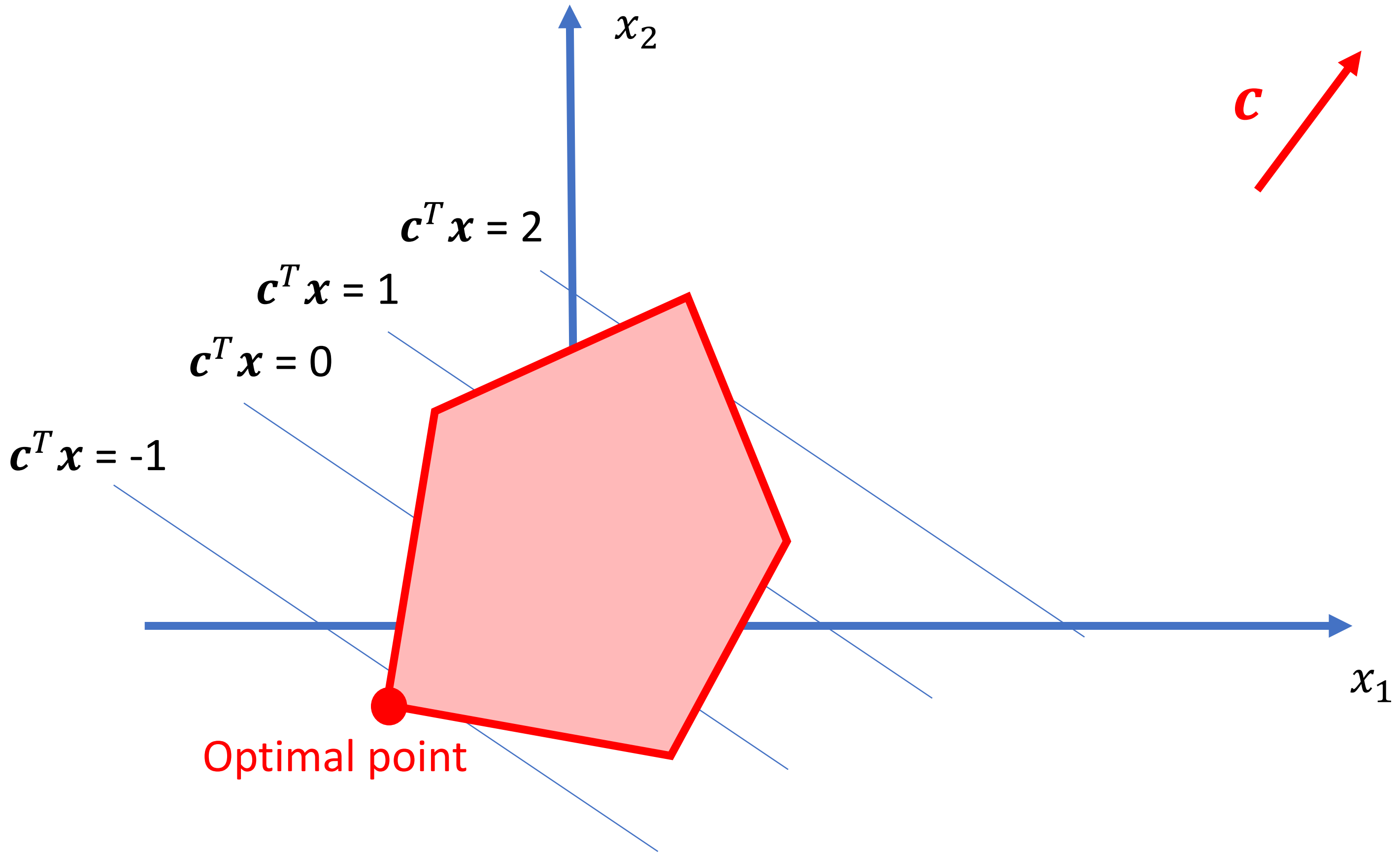
Optimizing Cost

Given the cost vector $[c_1, c_2]^T$, which point in the red polyhedron below can minimize $\boldsymbol{c}^T\boldsymbol{x}$?



Optimizing Cost

Given the cost vector $[c_1, c_2]^T$, which point in the red polyhedron below can minimize $\boldsymbol{c}^T\boldsymbol{x}$?



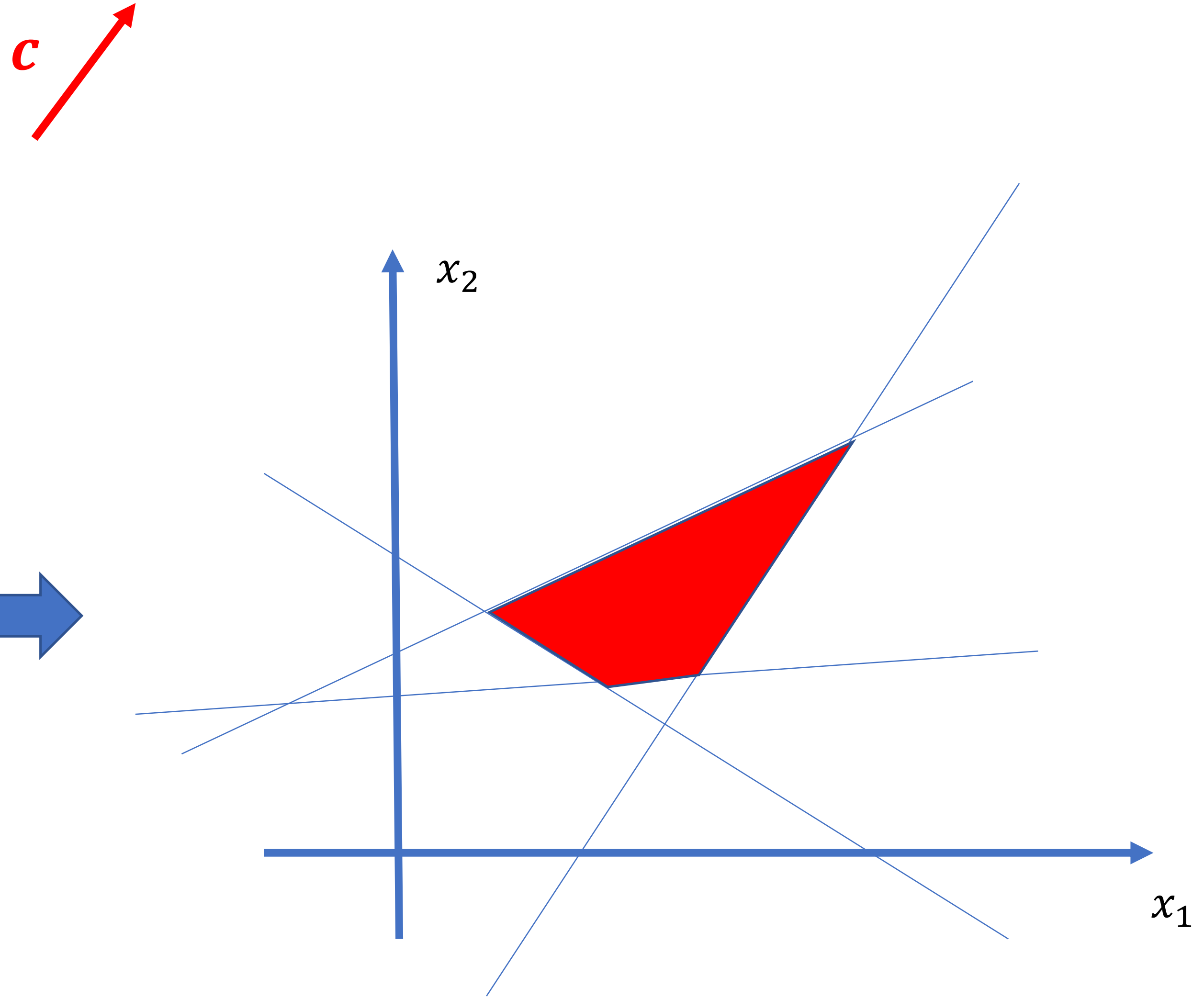
LP Graphical Representation

Inequality form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x} \quad \text{Objective Function}$$

$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b} \quad \text{Feasible Region}$$

$$a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



Piazza Poll 4

True or False: An minimizing LP with exactly one constraint, will always have a minimum objective value of $-\infty$.

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

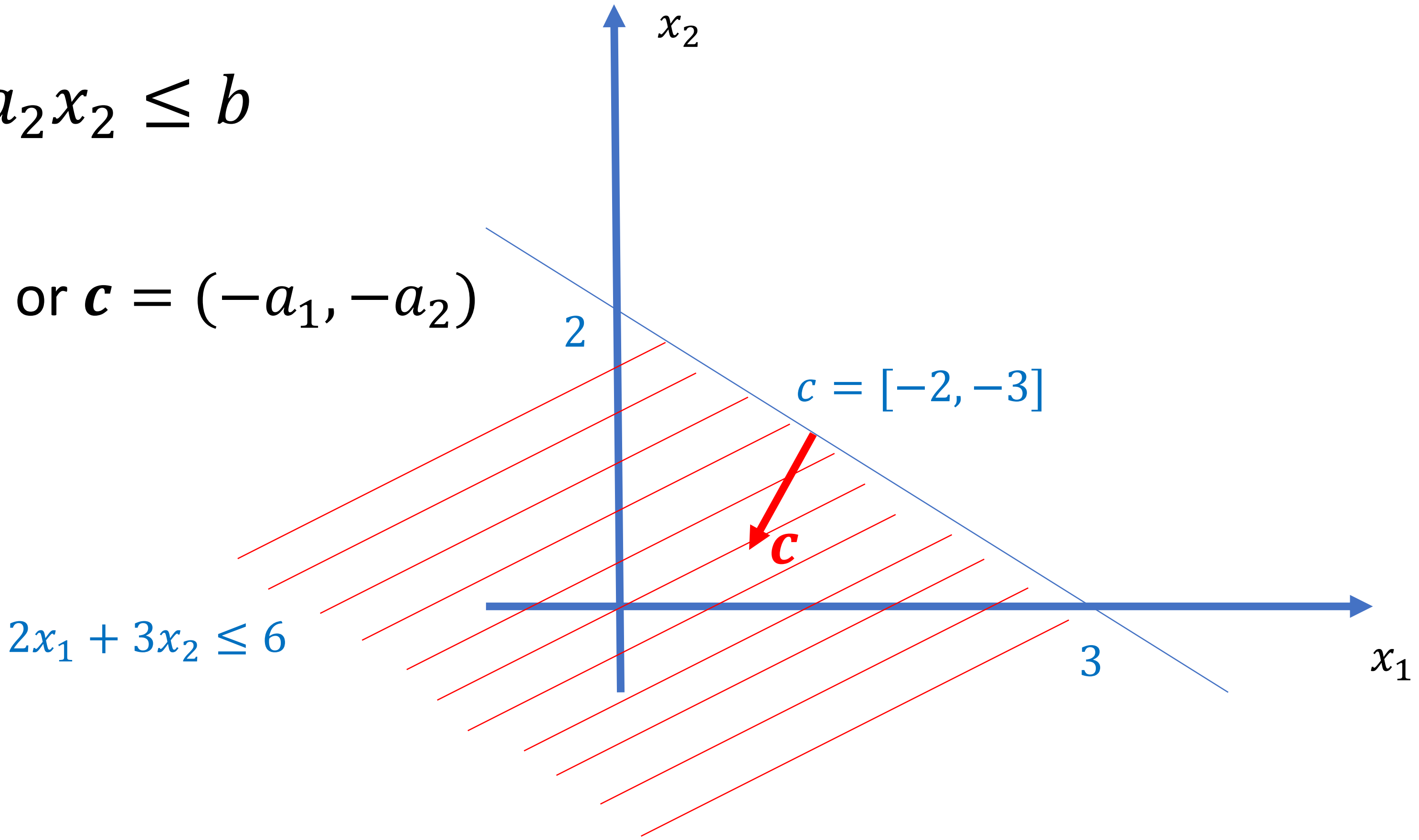
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

Piazza Poll 4

True or False: An minimizing LP with exactly one constraint, will always have a minimum objective value of $-\infty$.

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

False: $\boldsymbol{c} = (0,0)$ or $\boldsymbol{c} = (-a_1, -a_2)$



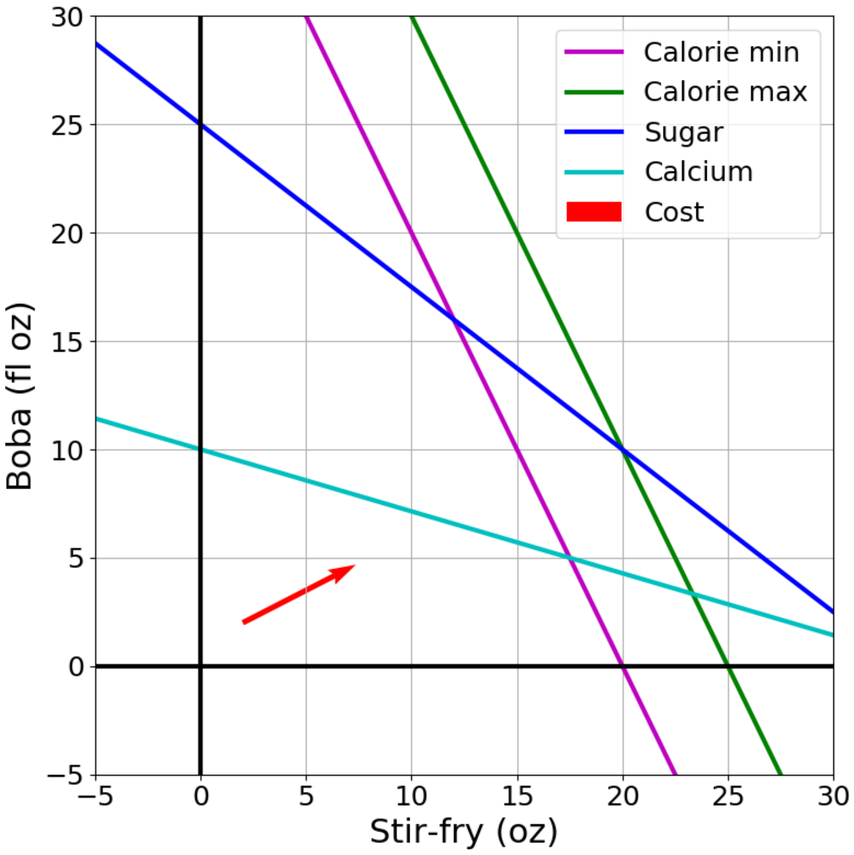
Focus of Today: (Linear) Optimization Problem

Problem Description

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$

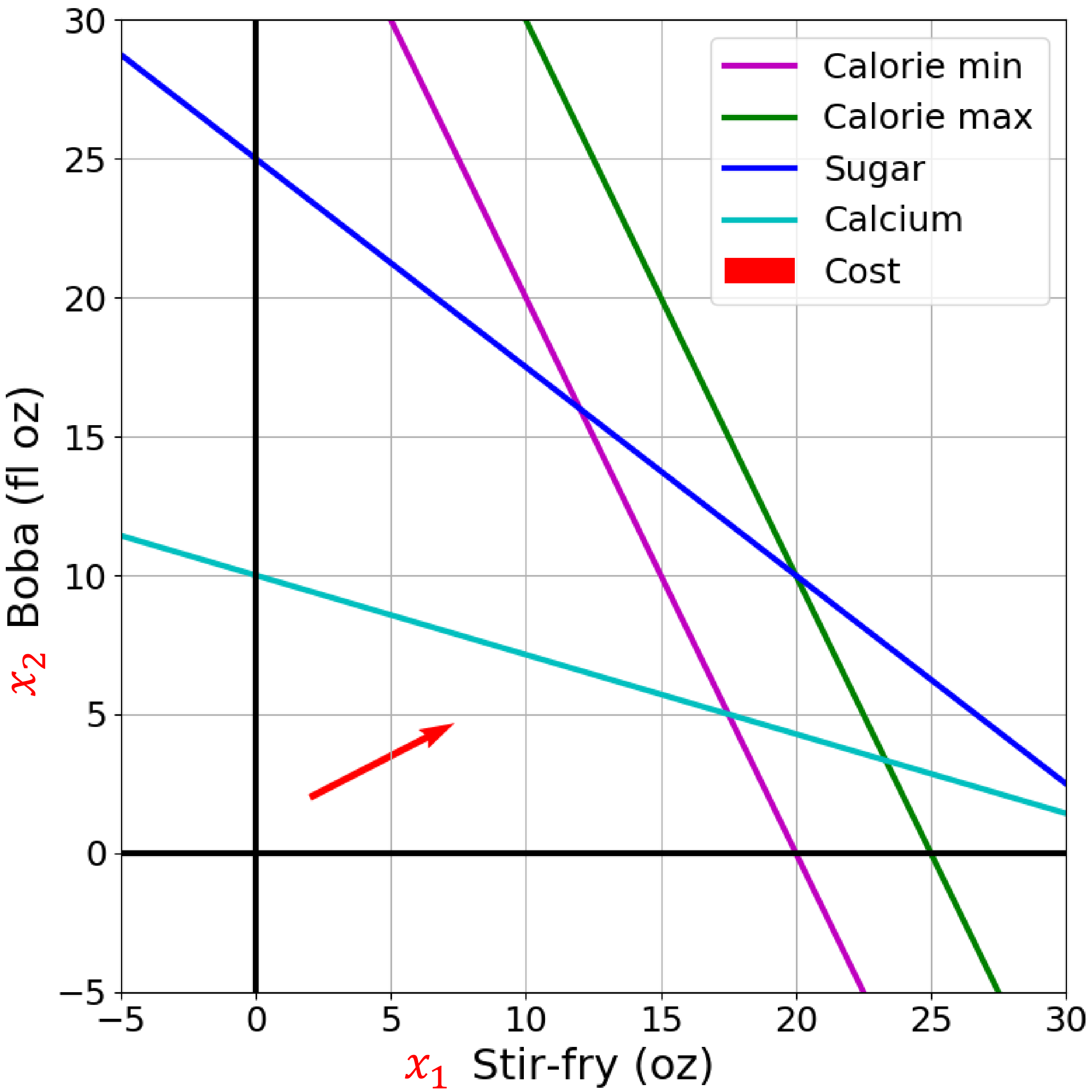
$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

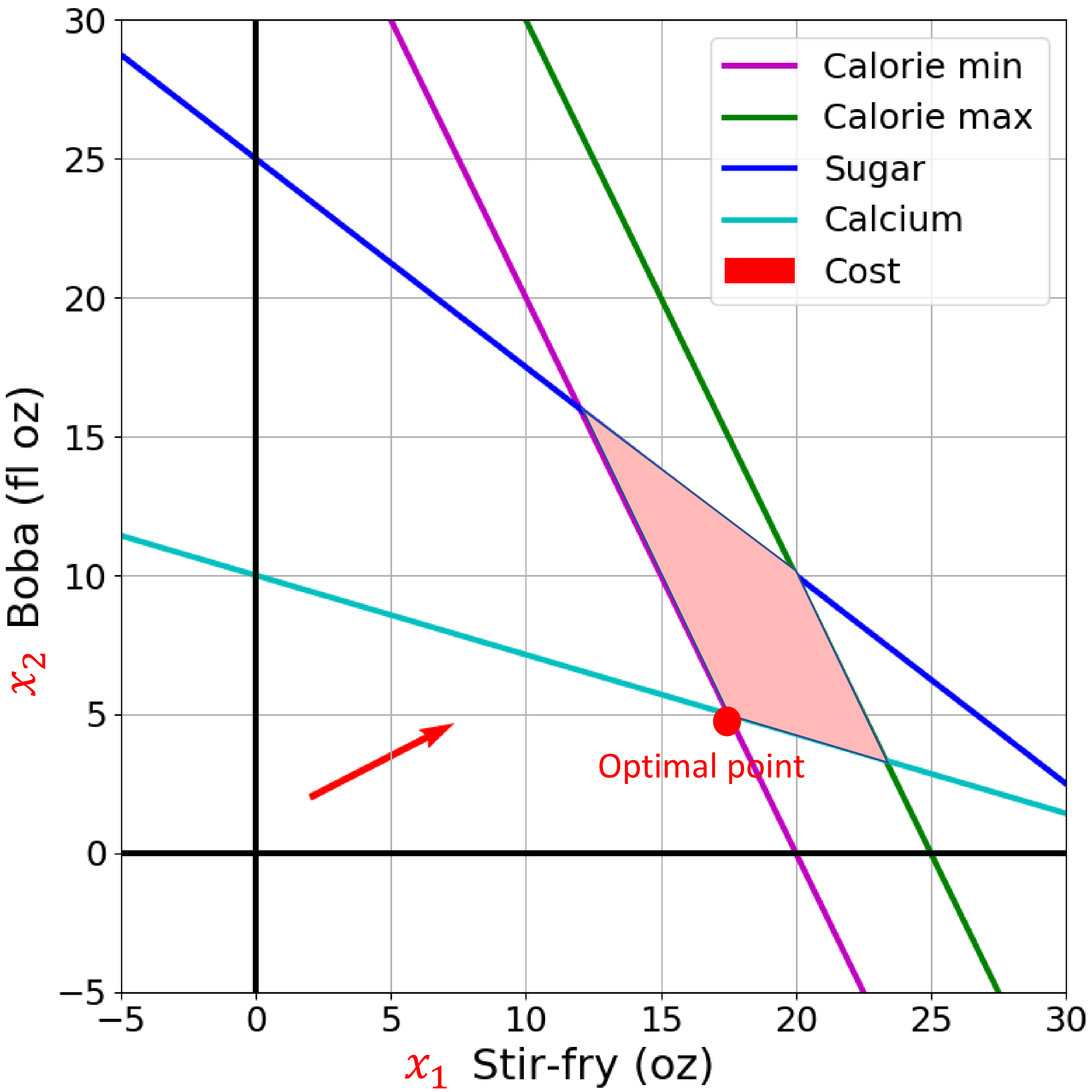
- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
$$3\,x_1 + 4\,x_2 \leq 100$$
$$20\,x_1 + 70\,x_2 \geq 700$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- Calories ≥ 2500
- Sugar ≥ 100 g
- Calcium ≥ 700 mg

$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

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$$3\, x_1 + 4\, x_2 \geq 100$$

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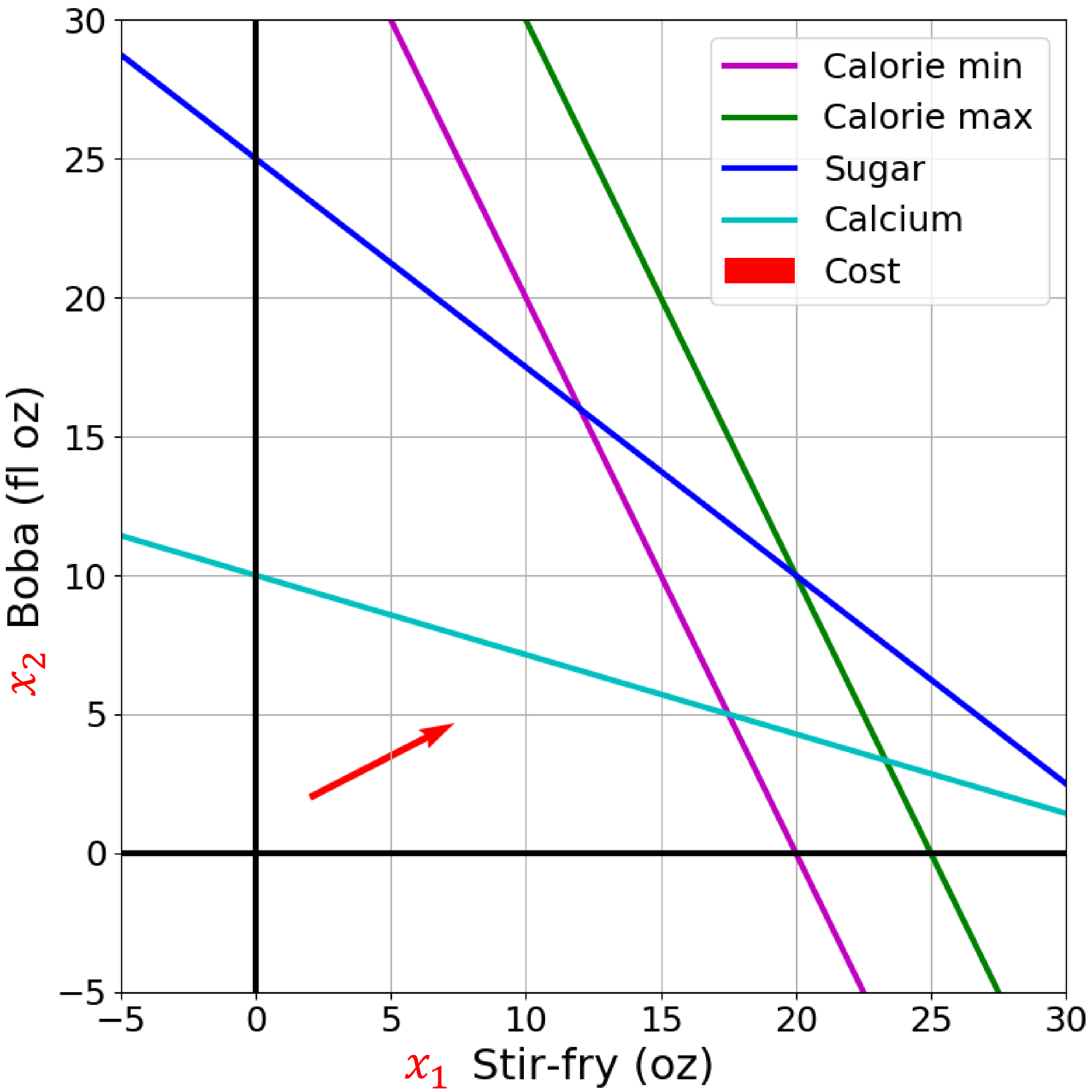
What is the optimal solution?



Healthy Squad Goals

- Calories ≥ 2500
- Sugar ≥ 100 g
- Calcium ≥ 700 mg

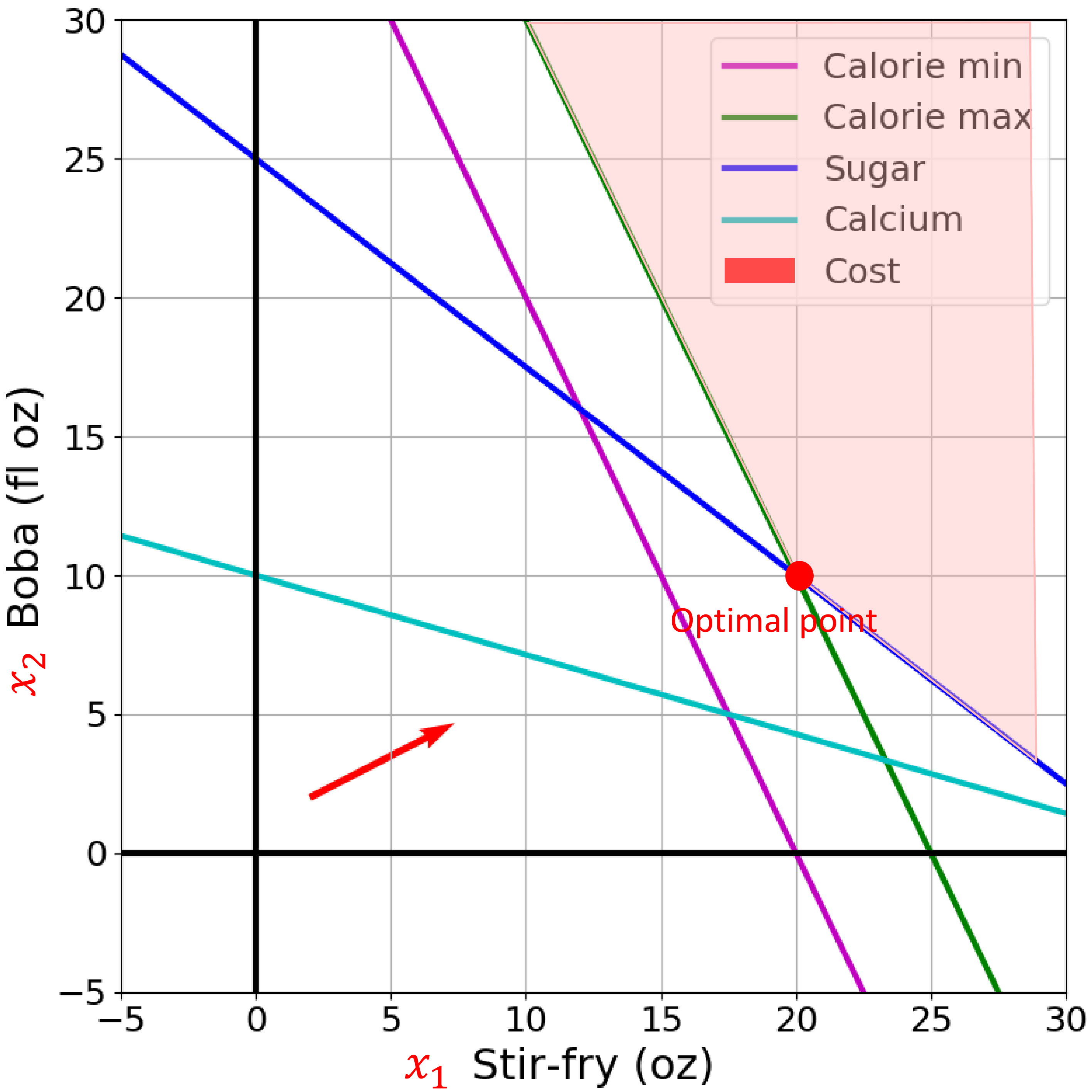
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2500$$

$$3\,x_1 + 4\,x_2 \geq 100$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- Calories $\leq$ 2000
- Sugar $\leq$ 100 g

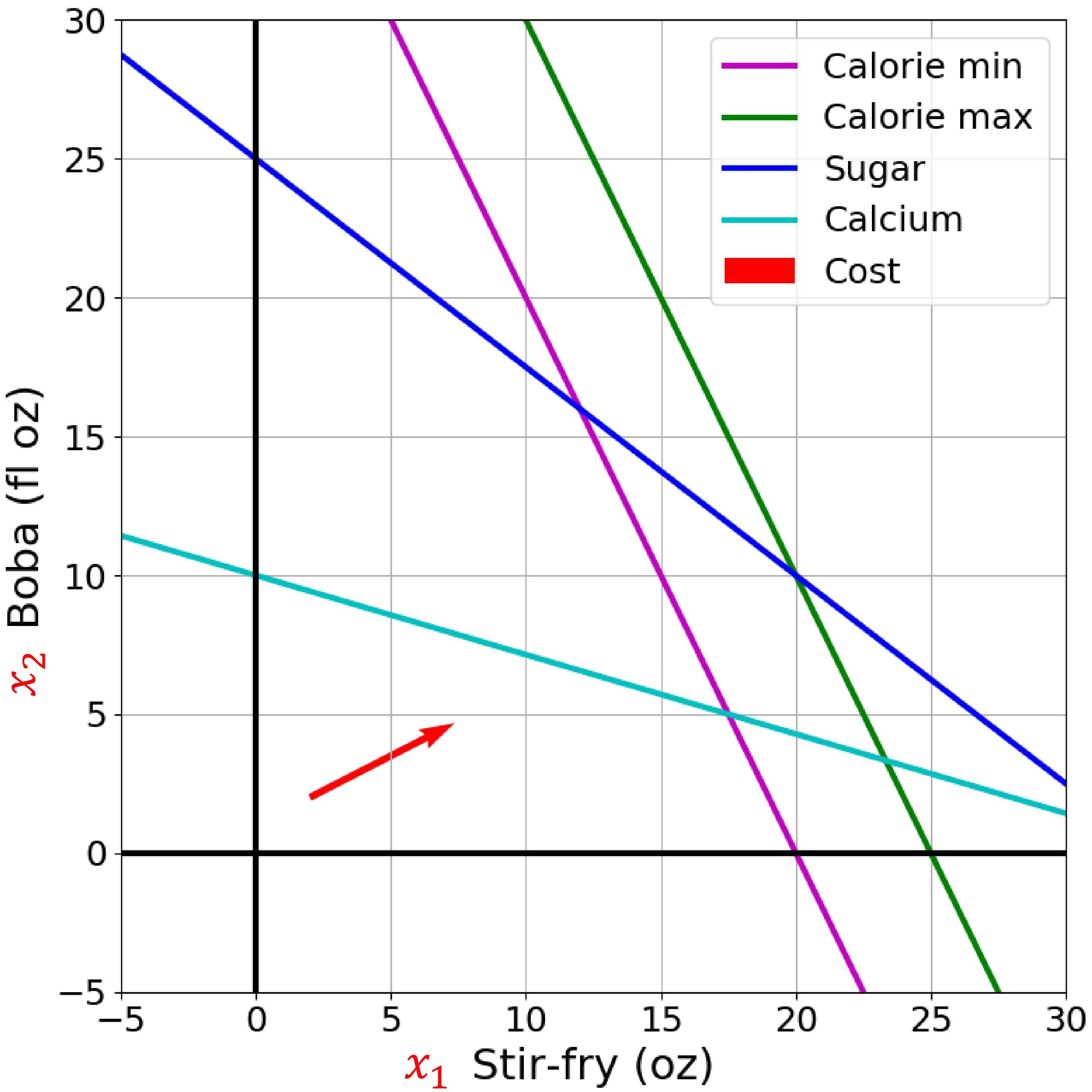
$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \leq 2000$$

$$3\,x_1 + 4\,x_2 \leq 100$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

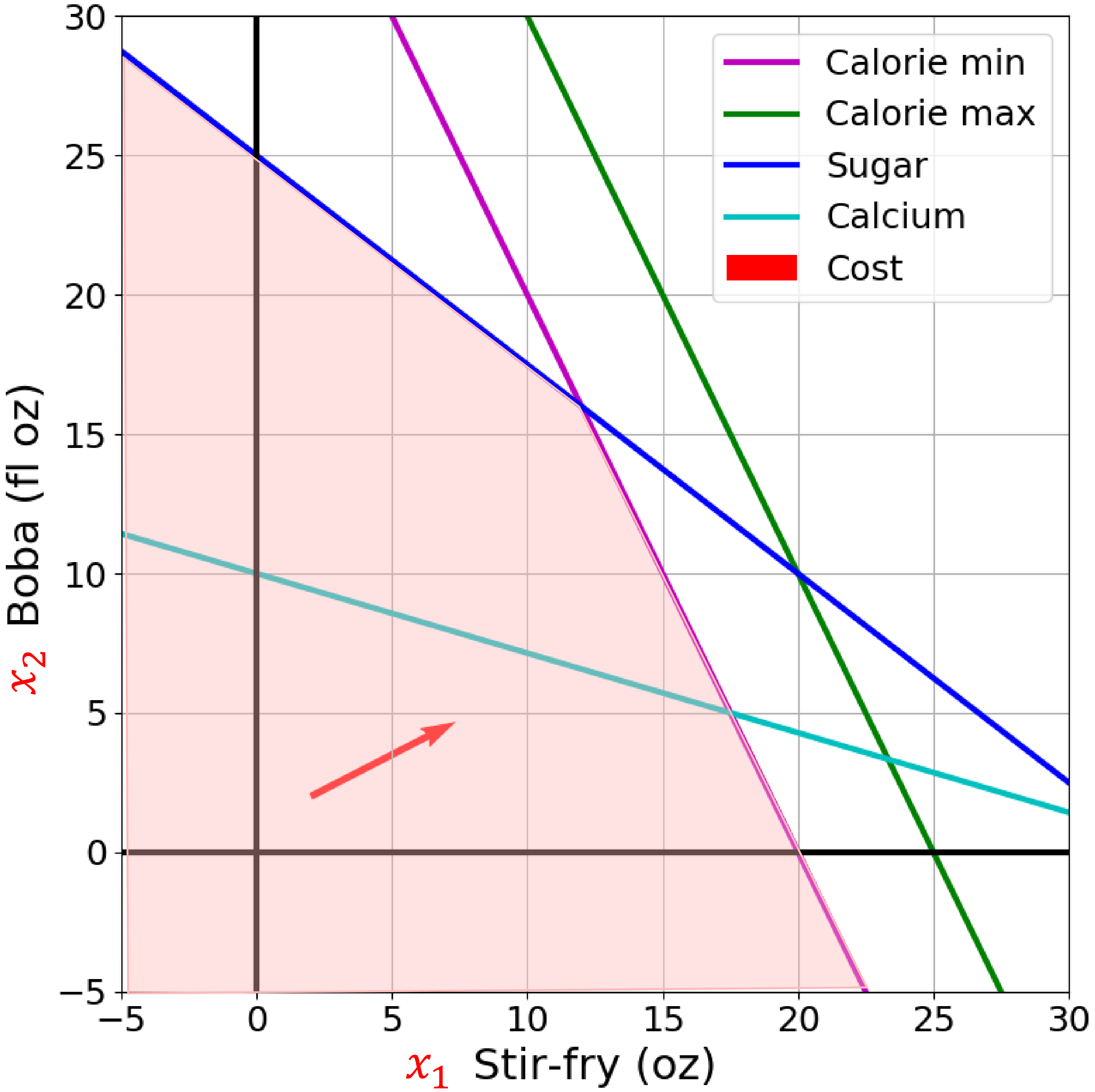
- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$
$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \le 2000$$
$$3\,x_1 + 4\,x_2 \le 100$$

What is the feasible region?

What is the optimal solution?

Problem unbounded!



Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \leq 2000$$

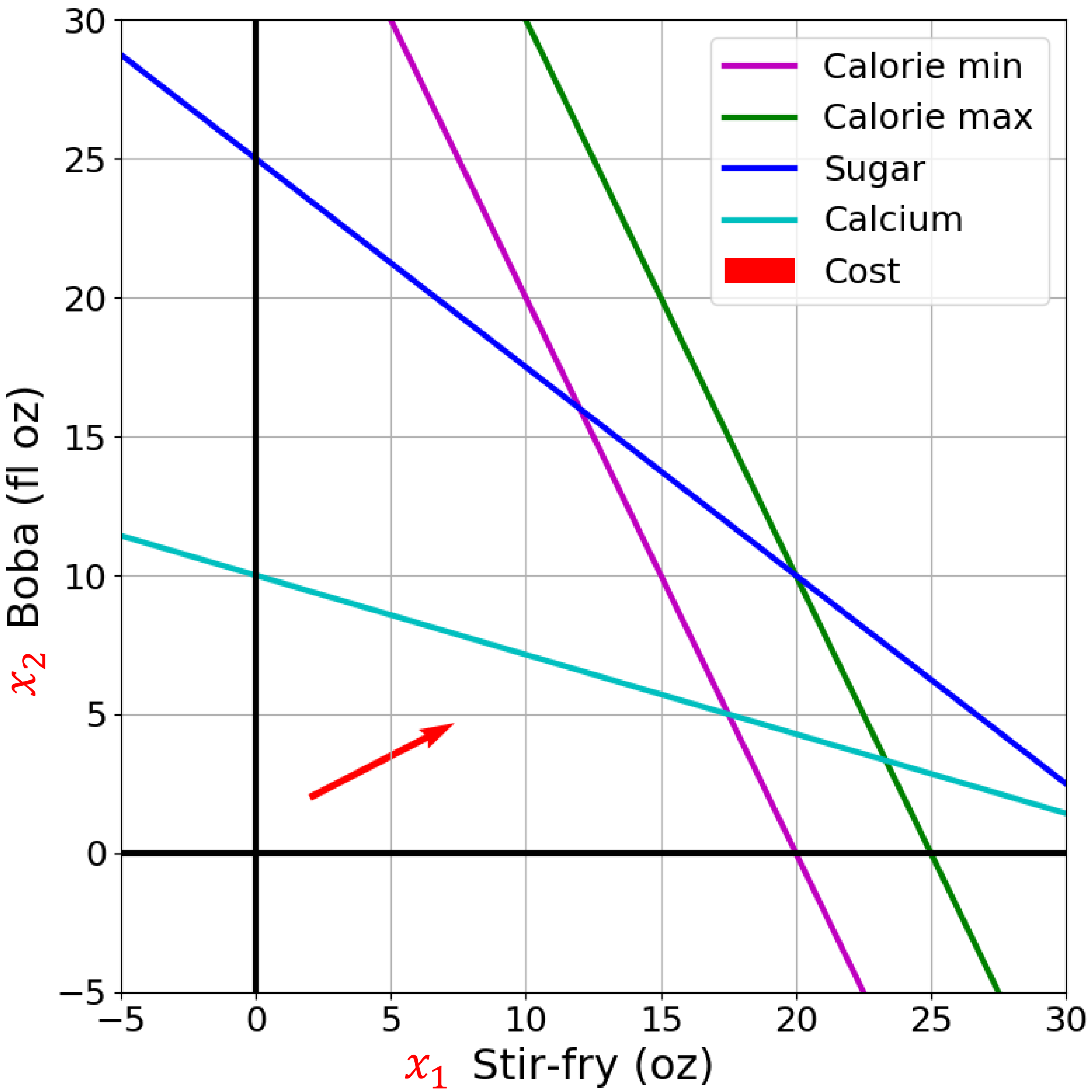
$$3\,x_1 + 4\,x_2 \leq 100$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

What is the feasible region?

What is the optimal solution?



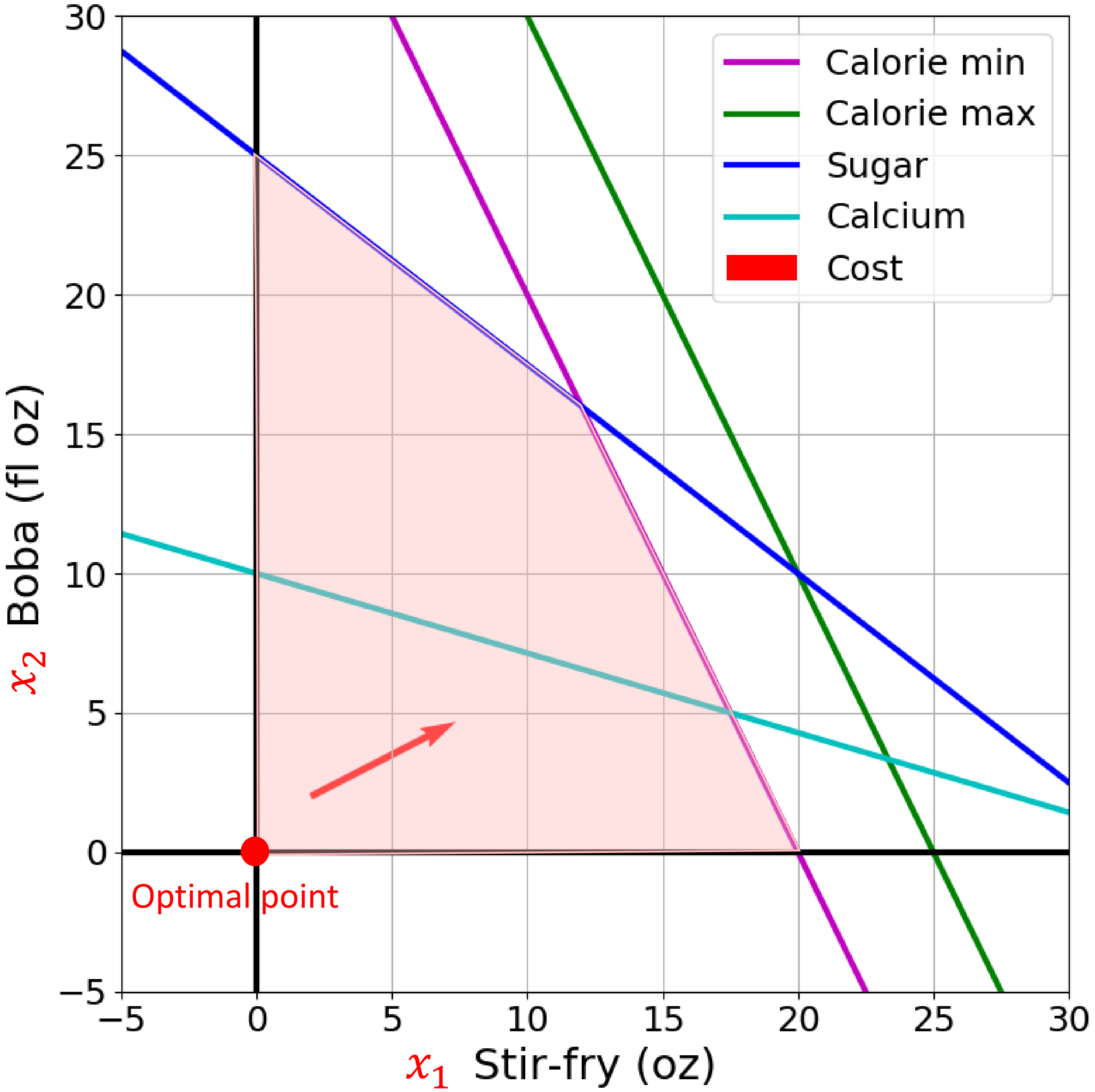
Healthy Squad Goals

- Calories ≤ 2000
- Sugar ≤ 100 g

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$
$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \le 2000$$
$$3\,x_1 + 4\,x_2 \le 100$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- 2000 ≤ Calories ≤ 2500
- Sugar ≥ 100 g
- Calcium ≤ 700 mg

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$
$$100\,x_1 + 50\,x_2 \leq 2500$$
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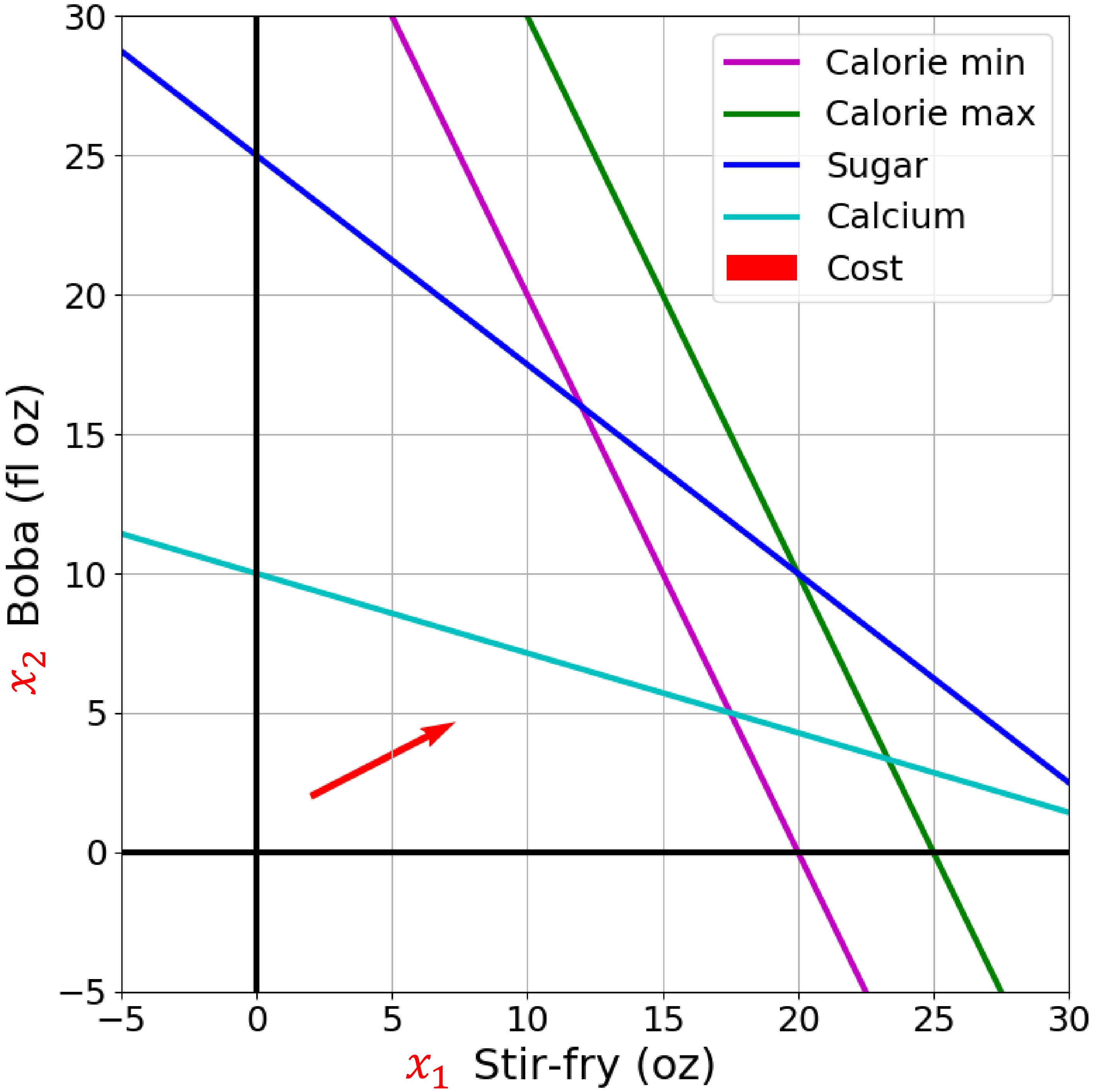
What is the feasible region?

What is the optimal solution?



Healthy Squad Goals

- $2000 \le$ Calories ≤ 2500
- Sugar $\ge$ 100 g
- Calcium $\le$ 700 mg

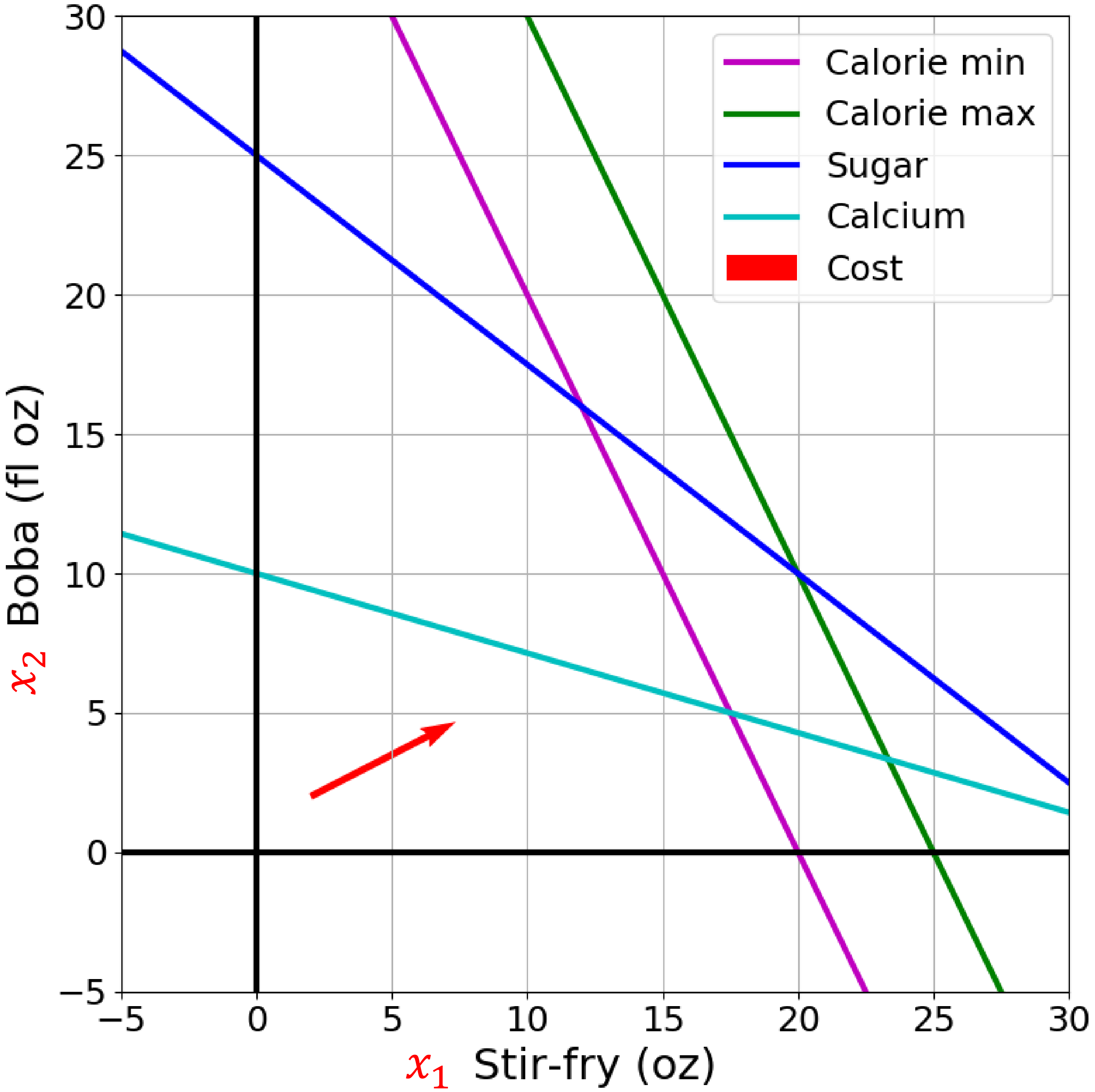
$$\min_{x_1, x_2} \quad 1\, x_1 + 0.5\, x_2$$

$$\text{s.t.} \quad \begin{aligned} 100\, x_1 + 50\, x_2 &\ge 2000 \\ 100\, x_1 + 50\, x_2 &\le 2500 \\ 3\, x_1 + 4\, x_2 &\ge 100 \\ 20\, x_1 + 70\, x_2 &\le 700 \end{aligned}$$

What is the feasible region?

What is the optimal solution?

Infeasible!

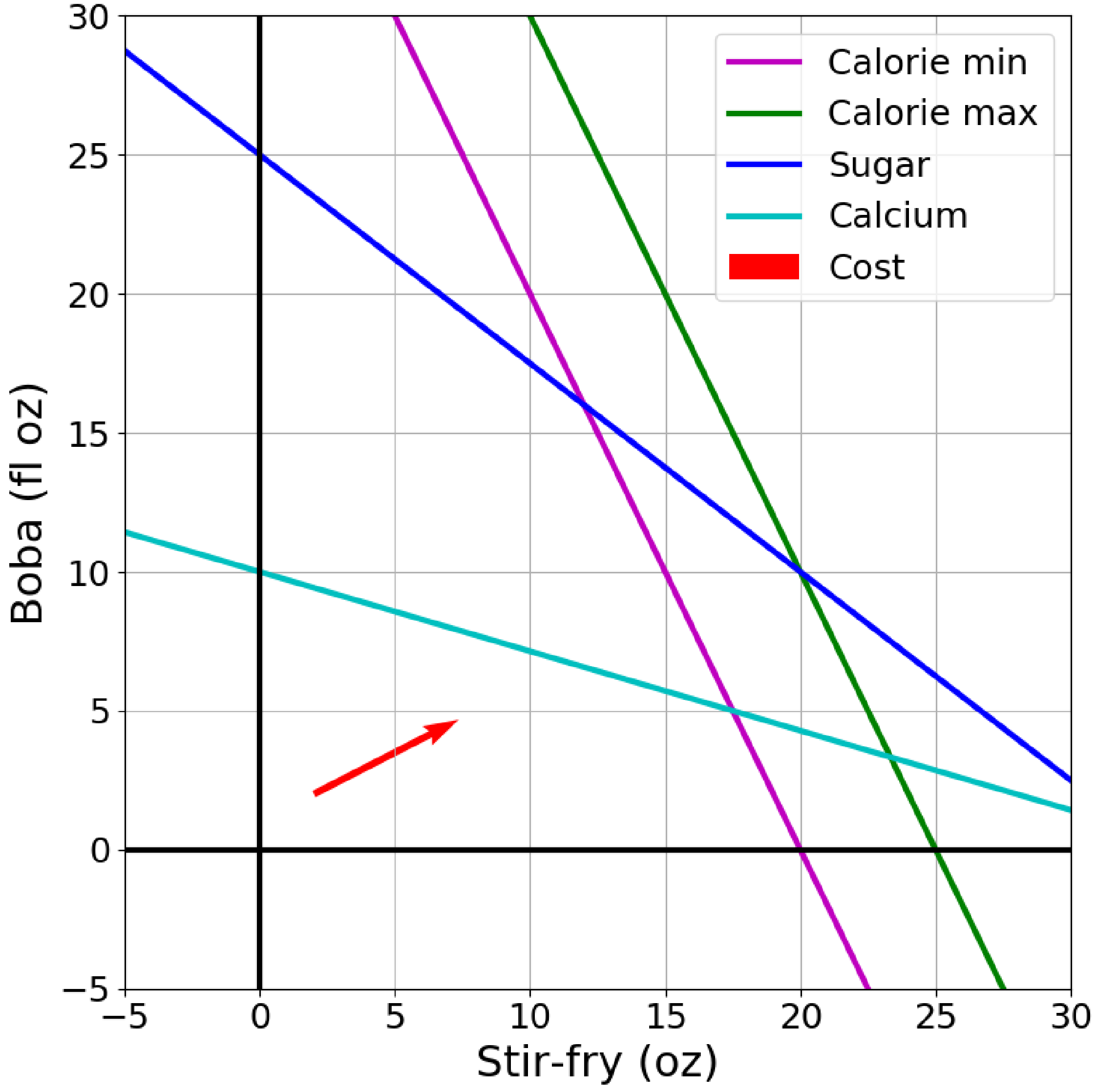


Solving an LP

If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value



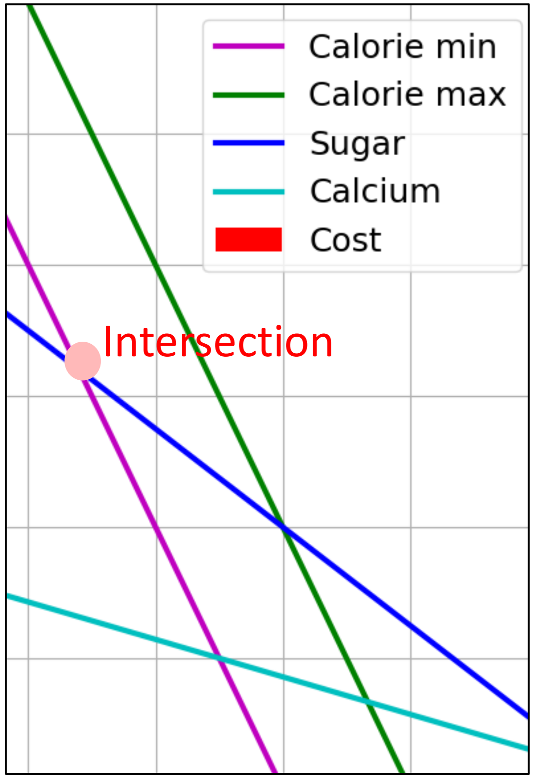
Solving an LP

But, how do we find the intersections?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



A point is the intersection of two lines

Pick two lines: two rows in A matrix time $\boldsymbol{x}$ equals the corresponding rows in $\boldsymbol{b}$

$$100\, x_1 + 50\, x_2 = 2000$$
$$3\, x_1 + \quad 4\, x_2 = 100$$

Not any pair of rows can lead to an intersection

$$100\, x_1 + 50\, x_2 = 2000$$
$$100\, x_1 + 50\, x_2 = 2500$$

Solving an LP

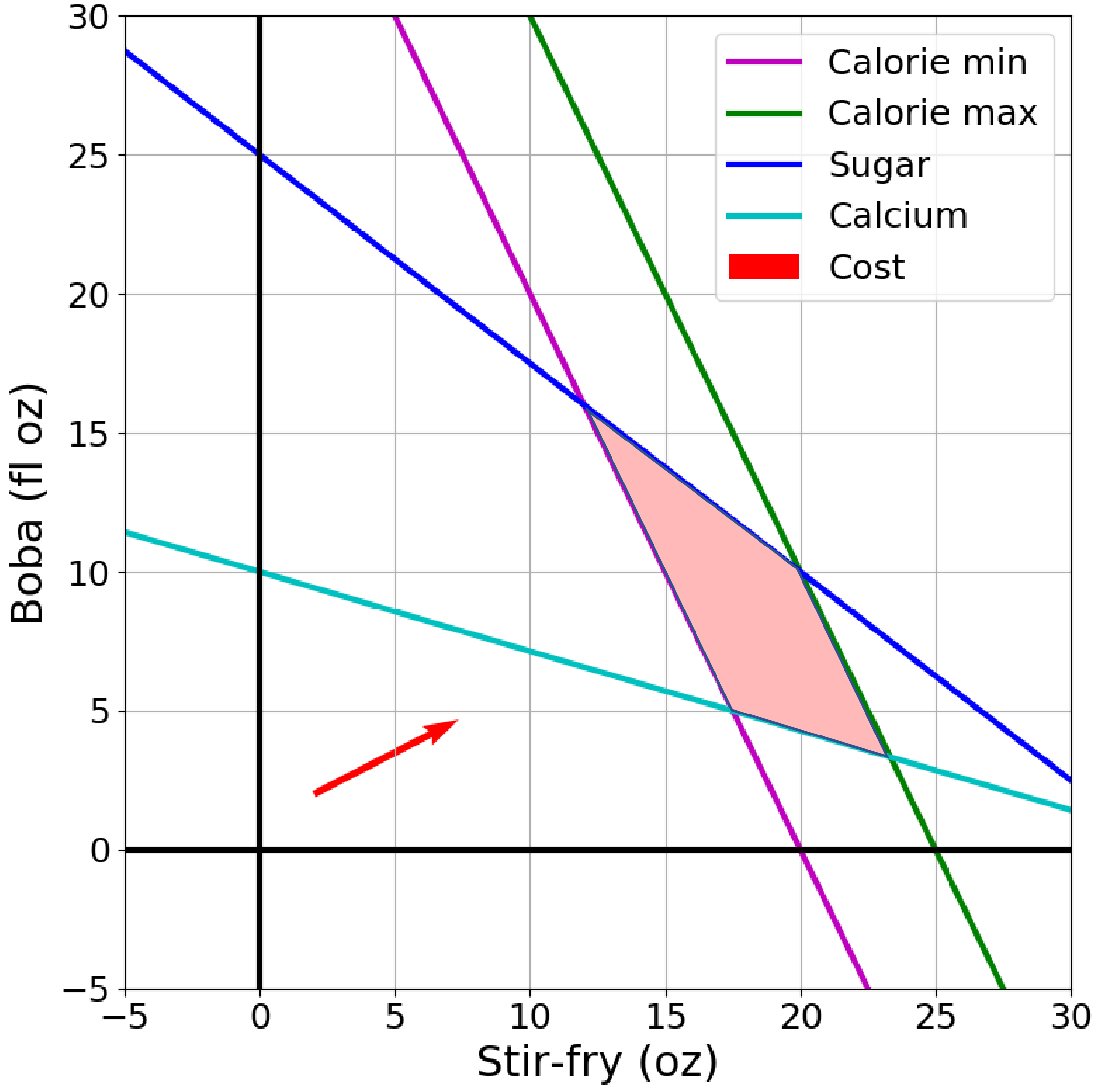
If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found

Simplex is most similar to which Local Search algorithm?

Hill Climbing!

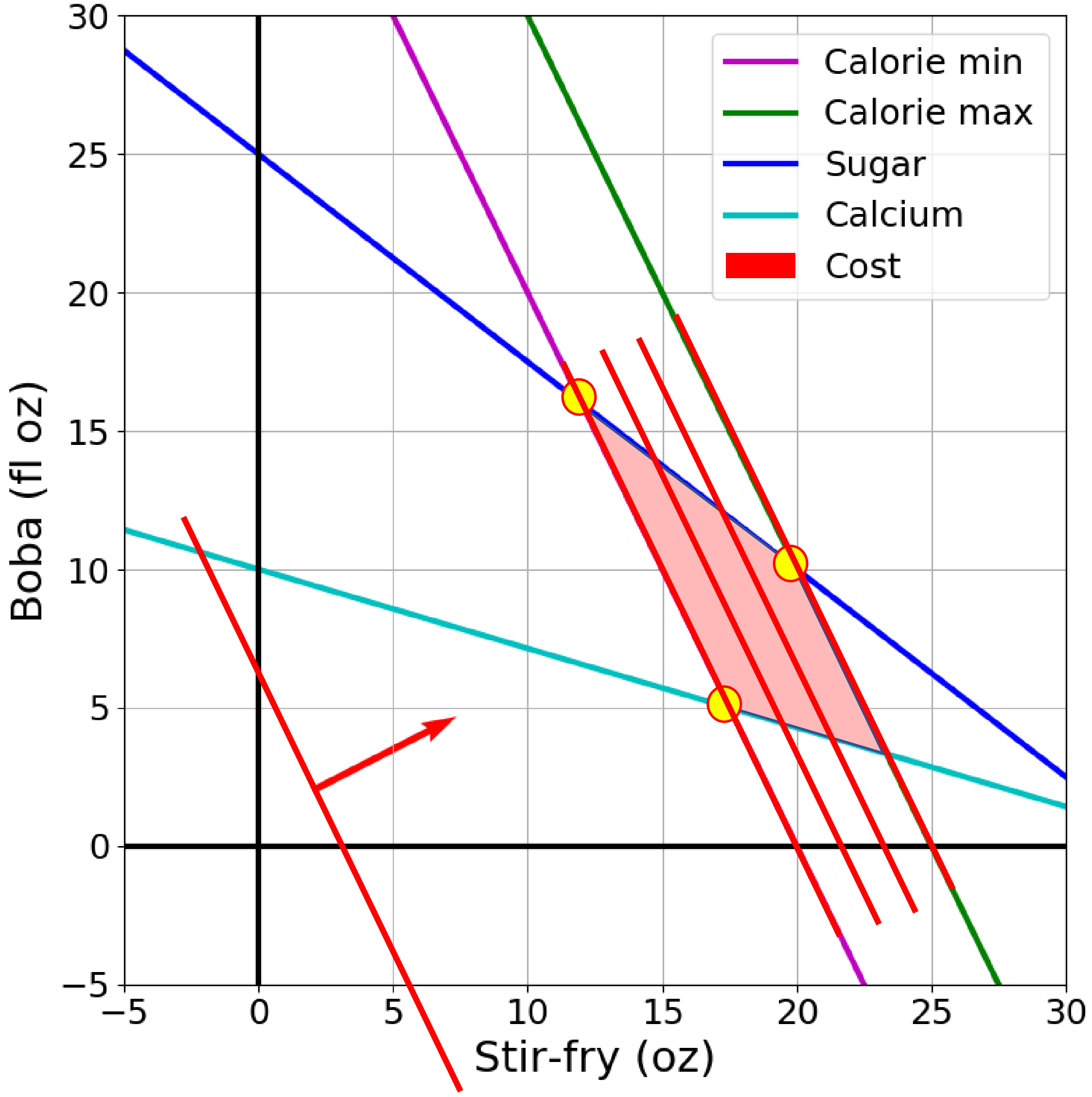


Solving an LP

If LP is feasible and bounded, at least one solution is at feasible intersections of constraint boundaries!

Algorithms

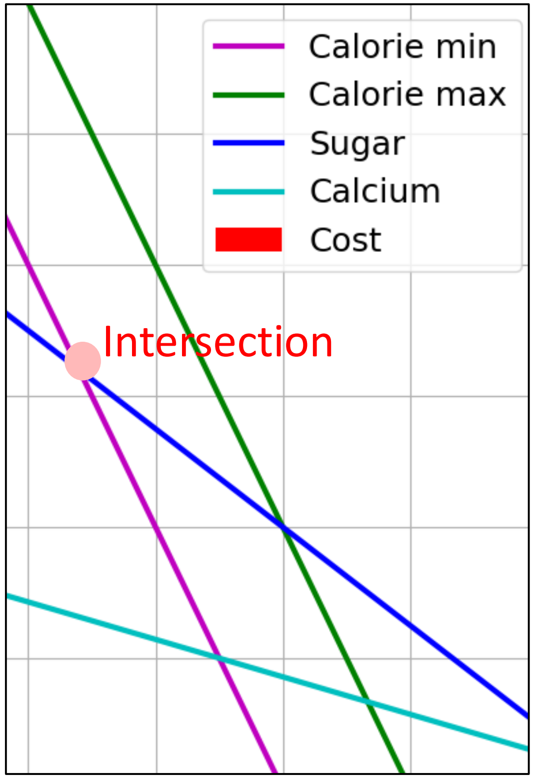
- Vertex enumeration: Find all vertices of feasible region (feasible intersections), check objective value
- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found



Piazza Poll 5

Simplex Algorithm is most similar to which search algorithm?

- Simplex: Start with an arbitrary vertex. Iteratively move to a best neighboring vertex until no better neighbor found



A: Depth First Search
B: Random Walk
C: Hill Climbing
D: Beam Search

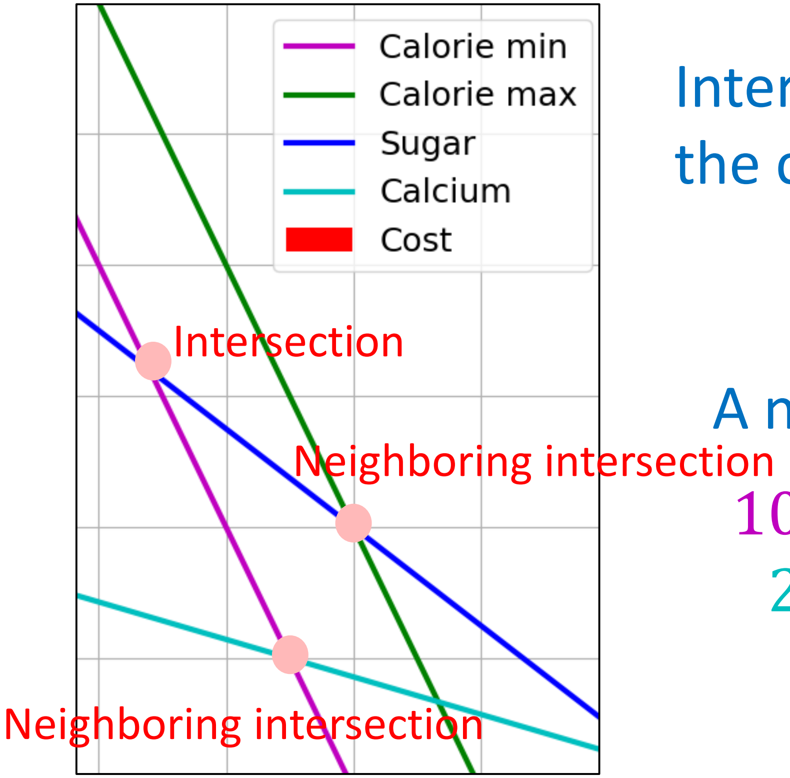
Solving an LP

But, how do we find a neighboring intersection?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



Intersection determined by: two rows in A matrix time $\boldsymbol{x}$ equals the corresponding rows in $\boldsymbol{b}$

$$100\,x_1 + 50\,x_2 = 2000$$
$$3\,x_1 + \ \ 4\,x_2 = 100$$

A neighboring intersection only have one different row

$$100\,x_1 + 50\,x_2 = 2000 \qquad 100\,x_1 + 50\,x_2 = 2500$$
$$20\,x_1 + 70\,x_2 = 700 \qquad 3\,x_1 + \ \ 4\,x_2 = 100$$

Focus of Today: (Linear) Optimization Problem

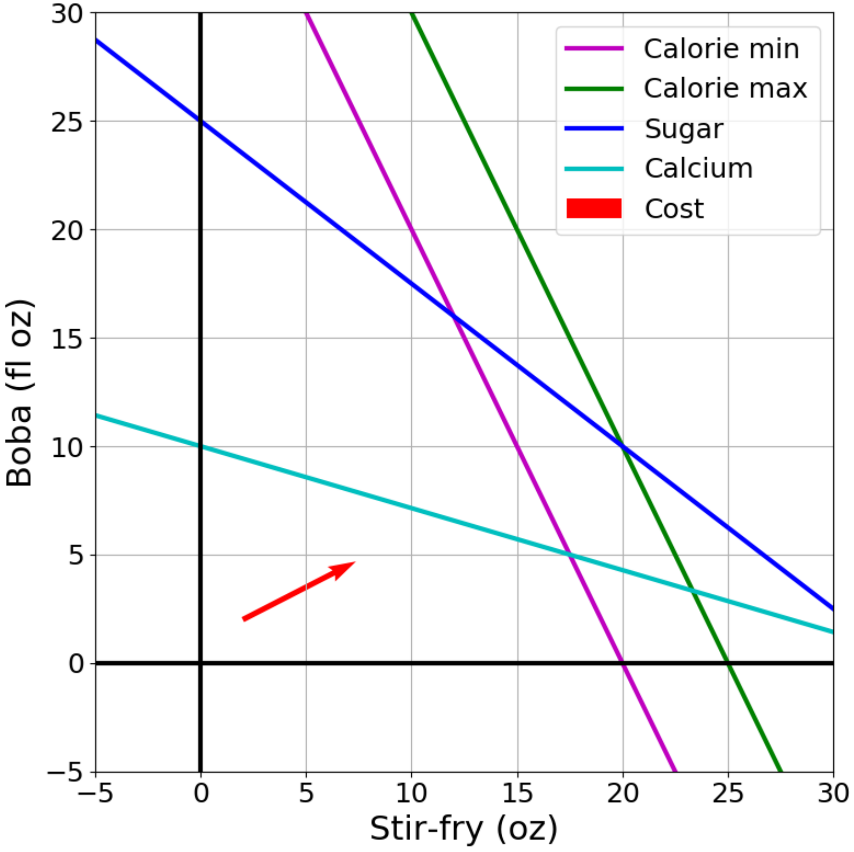
Problem Description

Linear Programming

Linear Program Formulation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



“Marty, you’re not thinking fourth-dimensionally”

Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$$a_1\, x_1 + a_2\, x_2 = b_1$$

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

$$\begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\leq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\leq b_4 \end{aligned}$$

What are intersections in higher dimensions?

How do these linear shapes extend to 3-D, N-D?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

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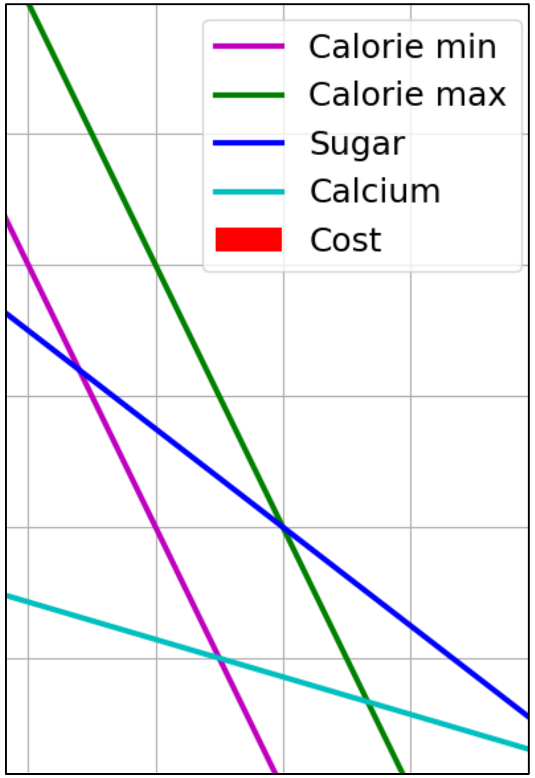

How do we find intersections in higher dimensions?

Still looking at subsets of rows in the A matrix

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
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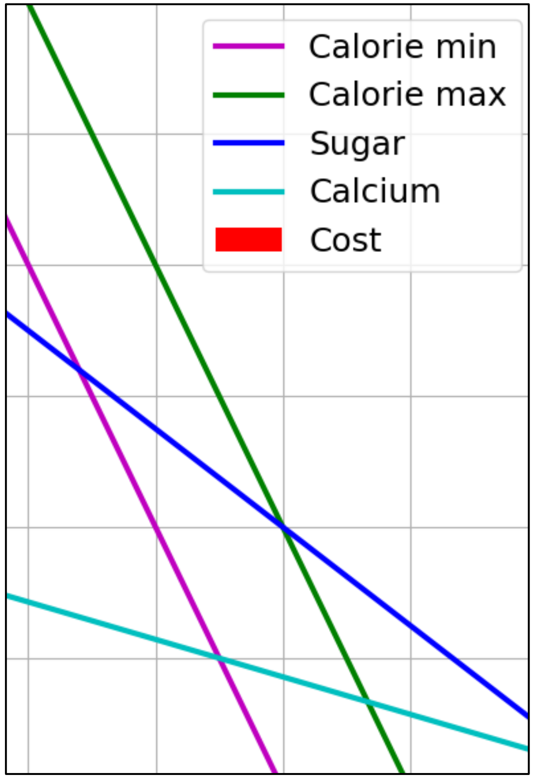
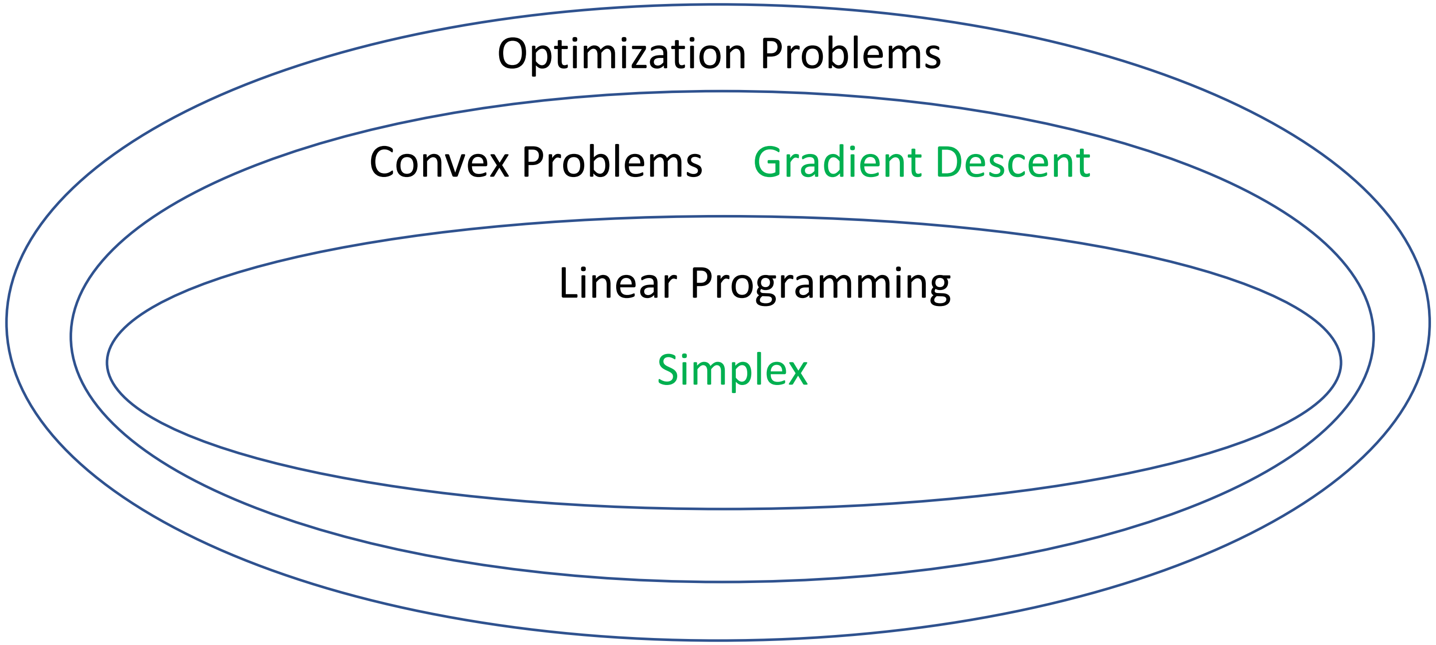


Summary

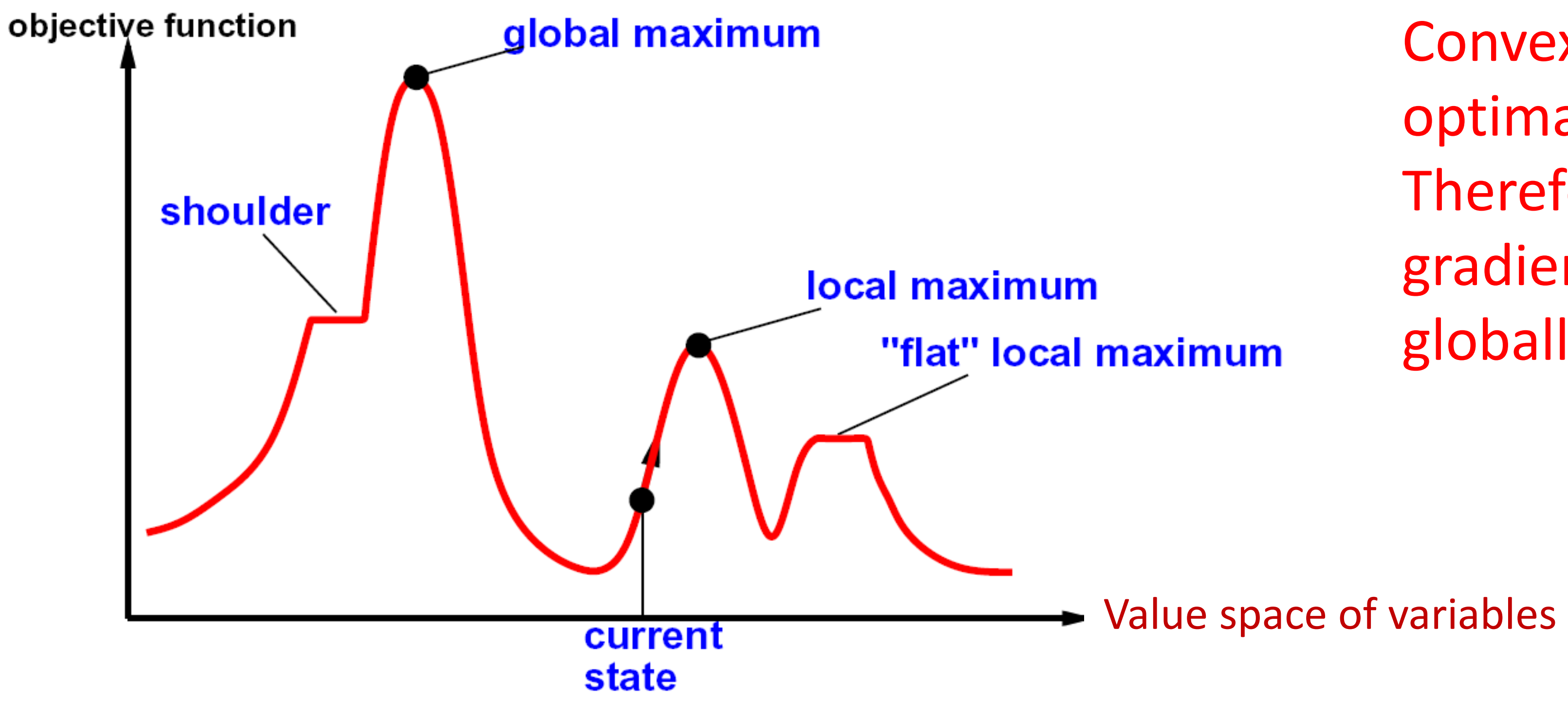
- Linear optimization problem handles continuous space but is closely connected to discrete space (only need to consider vertices)
- For a problem with two variables, graphical representation is helpful
- LP is a special class of optimization problems



Gradient Descent/Ascent

Find $x \in [a, b]$ that minimize / maximizes $f(x)$?

- Gradient descent / ascent
  - Use gradient to find best direction
  - Use the magnitude of the gradient to determine how big a step you move



Convex Problems: Any local optima is global optima. Therefore, for convex problems, gradient descent/ascent leads to globally optimal solution.