Johnson's Algorithm:

Solves the all pairs shortest path (APSP) problem. Works with negative weight edges (but not if there are negative cycles).

Could solve APSP problem by running Bellman Ford from each vertex:

\[ \text{Work} = O(mn) \times n = O(mn^2) \]
This is not efficient.

Idea: Two Phasos

1) Run Bellman Ford and use result to convert graph to have no negative weight edges
2) Run Dijkstra from each vertex

<table>
<thead>
<tr>
<th></th>
<th>Work</th>
<th>Span</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bellman Ford</td>
<td>O(mn)</td>
<td>O(n log n)</td>
</tr>
<tr>
<td>Dijkstra x n</td>
<td>O(m log n) x n</td>
<td>O(m log n)</td>
</tr>
<tr>
<td>Total</td>
<td>O(m n log n)</td>
<td>O(m log n)</td>
</tr>
</tbody>
</table>
Observation:
Assign a "potential" $p(v)$ to each vertex $v$.

Then adjust edges:

$$w'(u,v) = w(u,v) + p(u) - p(v)$$

Claim: original weights

Claim: adjusted weights

Proof by picture:

$$\delta_e(a,d) = w(a,b) + p(a) - p(b) + w(b,c) + p(b) - p(c)$$

$$+ w(c,d) + u(c,d) - u(d)$$

$$= w(a,b) + w(b,c) + w(c,d) + p(a) - p(d)$$

$$= \delta_e'(a,d) + p(a) - p(d)$$

Therefore,

$$\delta_e(a,d) = \delta_e'(a,d) - p(a) + p(d)$$
Johnson's algorithm

1) Add a dummy vertex $s$ to $G$ with edge to every $v \in V$ with weight 0.
   (Note: instead could use any $s \in V$ that can reach all other vertices)

2) Run Bellman Ford on new Graph starting with source $s$.

3) use $d(s, v) = p(v)$ to adjust weights.

4) Run Dijkstra to find all $d_G(u, v)$ return $d_G(u, v) = d_G(u, v) + d(u) - d(v)$.

Claim: no negative weight edges in

\[ w' \left( u, v \right) = w(u, v) + p(u) - p(v) \]

Why: General property of shortest paths

\[ d(s, v) \leq d(s, u) + w(u, v) \]

Since the shortest path cannot be larger than the shortest path through $u$.

Hence \[ 0 \leq d(s, u) + w(u, v) - d(s, v) \]

\[ = w(u, v) + d(s, u) - d(s, v) \]

Q. E. D.
Example:

\[ G' = a \]

Bellman Ford

\[ \begin{align*}
0 &-4
\end{align*} \]

\[ \text{adjusted weights} \]

\[ 0 - 0 + (-4) = -4 \]

\( a, b, c, d \)

\[ \begin{align*}
\sigma_{a,d} = 0 \\
\sigma_{c,a} = 0 \\
\sigma_{b,a} = 1 \\
\sigma_{c,b} = 1 - (-3) + 0 = 4
\end{align*} \]