Chapter 16

Recurrences

This chapter covers recurrences and presents three methods for solving recurrences: the "Tree Method" the "Brick Method" , and the "Substitution Method" .

1 The Basics

Recurrences are simply recursive functions for which the argument(s) and result are numbers. As is normal with recursive functions, recurrences have a recursive case along with one or more base cases. Although recurrences have many applications, in this book we mostly use them to represent the cost of algorithms, and in particular their work and span. They are typically derived directly from recursive algorithms by abstracting the arguments of the algorithm based on their sizes, and using the cost model described in Cost Models Chapter . Although the recurrence is itself a function similar to the algorithm it abstracts, the goal is not to run it, but instead the goal is to determine a closed form solution to it using other methods. Often we satisfy ourselves with finding a closed form that specifies an upper or lower bound on the function, or even just an asymptotic bound.

Example 16.1 (Fibonacci)**.** Here is a recurrence written in SPARC that you should recognize:

$$
\begin{aligned}
F(n) = \quad & \texttt{case}\ n\ \texttt{of} \\
& \quad 0\ \texttt{=>}\ 0 \\
& \mid 1\ \texttt{=>}\ 1 \\
& \mid \_\ \texttt{=>}\ F(n-1) + F(n-2)\ .
\end{aligned}
$$

It has an exact closed form solution:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\ ,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. We can write this in asymptotic notation as

$$F(n) = \Theta(\varphi^n)$$

since the first term dominates asymptotically. Solving this recurrence exactly is more that we will ask in this course, but the substitution method described in this chapter will allow you to prove it correct.

Example 16.2 (Mergesort Recurrence)**.** Assuming that the input length is a power of 2, we can write the code for parallel mergesort algorithm as follows.

$$
\begin{array}{l}
\mathit{msort}(A) = \\
\quad \texttt{if } |A| \le 1 \texttt{ then } A \\
\quad \texttt{else} \\
\quad\quad \texttt{let } (L, R) = \ \mathit{msort}(A[0 \ldots |A|/2]) \ || \ \mathit{msort}(A[|A|/2 \ldots |A|]) \\
\quad\quad \texttt{in } \mathit{merge}(L, R) \texttt{ end}
\end{array}
$$

By abstracting based on the length of A, and using the cost model described in Cost Models Chapter , we can write a recurrence for the work of mergesort as:

$$
\begin{array}{l}
W_{msort}(n) = \\
\quad \texttt{if } n \le 1 \texttt{ then } c_1 \\
\quad \texttt{else} \\
\quad\quad \texttt{let } (W_L, W_R) = \ (W_{msort}(n/2), W_{msort}(n/2)) \\
\quad\quad \texttt{in } W_L + W_R + W_{merge}(n) + c_2 \texttt{ end}
\end{array}
$$

where the c_i are constants. Assuming $W_{merge}(n) = c_3 n + c_4$ this can be simplified to

$$
W_{msort}(n) = \quad \begin{array}{l} \texttt{if } n \le 1 \texttt{ then } c_1 \\ \texttt{else } 2W_{msort}(n/2) + c_3 n + c_5 \end{array}
$$

where $c_5 = c_2 + c_4$. We will show in this chapter that this recurrence solves to

$$
W_{msort}(n) = O(n \lg n)
$$

using all three of our methods.

2 Some conventions

To reduce notation we use several conventions when writing recurrences.

Syntax. We typically write recurrences as mathematical relations of the form

$$
W_f(n) = \begin{cases} c_1 & \text{base case 1} \\ c_2 & \text{base case 2} \\ \cdots & \cdots \\ \text{recursive definition} & \text{otherwise.} \end{cases}
$$

Dropping the subscript. We often drop the subscript on the cost W or S (span) when obvious from the context.

Base case. Often base cases are trivial—i.e., some constant if $n \leq 1$. In such cases, we usually leave them out.

Big-O inside a recurrence. Technically using big-O notation in a recurrence as in:

$$W(n) = 2W(n/2) + O(n)$$

is not well defined. This is because $2W(n/2)+O(n)$ indicates a set of functions, not a single function. In this book when we use $O(f(n))$ in a recurrences it is meant as shorthand for $c_1 f(n) + c_2$, for some constants c_1 and c_2. Furthermore, when solving the recurrence the $O(f(n))$ should always be replaced by $c_1 f(n) + c_2$.

Inequality. Because we are mostly concerned with upper bounds, we can be sloppy and add (positive) constants on the right-hand side of an equation. In such cases, we typically use an inequality. For example, we may write for some constants c_1, c_2,

$$W(n) \leq 2W(n/2) + c_1 n + c_2.$$

Input size inprecision. A technical issue concerns rounding of input sizes. Going back to the mergesort example , note that we assumed that the size of the input to merge sort, n, is a power of 2. If we did not make this assumption, i.e., for general n, we would partition the input into two parts, whose sizes may differ by up to one element. In such a case, we could write the work recurrence as

$$W(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ W(\lceil n/2 \rceil) + W(\lfloor n/2 \rfloor) + O(n) & \text{otherwise.} \end{cases}$$

When working with recurrences, we typically ignore floors and ceiling because they change the size of the input by at most one, which does not usually affect the closed form by more than a constant factor.

Example 16.3 (Mergesort recurrence revisited)**.** Using our conventions we can write our recurrence for the work of mergesort as:

$$W(n) \leq 2W(n/2) + O(n)\ .$$

However, when solving it is best to write it as:

$$W(n) \leq \begin{cases} c_b & \text{if } n \leq 1 \\ 2W(n/2) + c_1 n + c_2 & \text{otherwise .} \end{cases}$$

Assuming *merge* has logarithmic span, we can similarly write a recurrence for the span of the parallel mergesort as:

$$S(n) \leq S(n/2) + O(\lg n)\ .$$

3 The Tree Method

Definition 16.1 (Tree Method)**.** The ***tree method*** is a technique for solving recurrences. Given a recurrence, the idea is to derive a closed form solution of the recurrence by first unfolding the recurrence as a tree and then deriving a bound by considering the cost at each level of the tree. To apply the technique, we start by replacing the asymptotic notations in the recurrence, if any. We then draw a tree where each recurrence instance is represented by a subtree and the root is annotated with the cost that occurs at this level, that is beside the recurring costs.

After we determine the tree, we ask several questions.

- How many levels are there in the tree?
- What is the problem size on level i?
- What is the cost of each node on level i?
- How many nodes are there on level i?
- What is the total cost across the level i?

Based on the answers to these questions, we can write the cost as a sum and calculate it.

Example 16.4. Consider the recurrence

$$W(n) = 2W(n/2) + O(n).$$

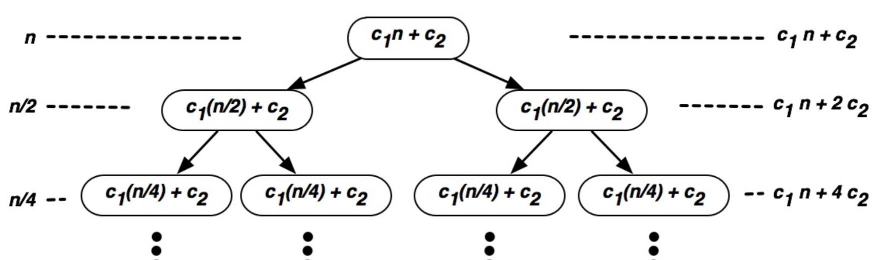
By the definition of asymptotic complexity, we can establish that

$$W(n) \quad \leq \quad 2W(n/2) + c_1 \cdot n + c_2,$$

where c_1 and c_2 are constants.

We now draw a tree to represent the recursion. Since there are two recursive calls, the tree is a binary tree, whose input is half the size of the size of the parent node. We then annotate each node in the tree with its cost noting that if the problem has size m, then the cost, excluding that of the recursive calls, is at most $c_1 \cdot m + c_2$.

The drawing below illustrates the resulting tree; each level is annotated with the problem size (left) and the cost at that level (right).



We observe that:

- level i (the root is level $i = 0$) contains 2^i nodes,
- a node at level i costs at most $c_1(n/2^i) + c_2$.

Thus, the total cost on level i is at most

$$2^i \cdot \left(c_1 \frac{n}{2^i} + c_2\right) \quad = \quad c_1 \cdot n + 2^i \cdot c_2.$$

Because we keep halving the input size, the number of levels $i \leq \lg n$. Hence, we have

$$\begin{aligned} W(n) &\leq \sum_{i=0}^{\lg n} \left(c_1 \cdot n + 2^i \cdot c_2\right) \\ &= c_1 n(1 + \lg n) + c_2(n + \tfrac{n}{2} + \tfrac{n}{4} + \cdots + 1) \\ &= c_1 n(1 + \lg n) + c_2(2n - 1) \\ &\in O(n \lg n), \end{aligned}$$

where in the second to last step, we apply the fact that for $a > 1$,

$$1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1} \leq a^{n+1}.$$

4 The Brick Method

The ***brick method*** is a special case of the tree method, aimed at recurrences that grow or decay geometrically across levels of the recursion tree. A sequence of numbers has ***geometric growth*** if it grows by at least a constant factor (> 1) from element to element, and has ***geometric decay*** if it decreases by at least a constant factor. The beauty of a geometric sequence is that its sum is bounded by a constant times the last element (for geometric growth), or the first element (for geometric decay).

Exercise 16.1 (Sums of geometric series.)**.** Consider the sum of the sequence $S = \langle 1, \alpha, \alpha^2, \ldots, \alpha^n \rangle$. Show that

1. for $\alpha > 1$ (geometric growth), the sum of S is at most $\left(\frac{\alpha}{\alpha - 1}\right) \cdot \alpha^n$, and
2. for $\alpha < 1$ (geometric decay), the sum of S is at most $\left(\frac{1}{1 - \alpha}\right) \cdot 1$.

Hint: for the first let s be the sum, and consider $\alpha s - s$, cancelling terms as needed.

Solution. Let

$$s = \sum_{i=0}^{n} \alpha^i \, .$$

To solve the first case we use

$$\begin{aligned}
\alpha s - s &= \left(\alpha \sum_{i=0}^{n} \alpha^i\right) - \sum_{i=0}^{n} \alpha^i \\
&= \left(\sum_{i=0}^{n} \alpha^{i+1}\right) - \sum_{i=0}^{n} \alpha^i \\
&= \alpha^{n+1} - 1 \\
&< \alpha^{n+1} .
\end{aligned}$$

Now by dividing through by $\alpha - 1$ we get

$$s < \frac{\alpha^{n+1}}{\alpha - 1} = \left(\frac{\alpha}{\alpha - 1}\right) \cdot \alpha^n ,$$

which is we wanted to show.

The second case is similar but using $s - \alpha s$.

In the tree method, if the costs grow or decay geometrically across levels (often the case), then for analyzing asymptotic costs we need only consider the cost of the root (decay) , or the total cost of the leaves (growth). If there is no geometric growth or decay then it often suffices to calculate the cost of the worst level (often either the root or leaves) and multiply it by the number of levels. This leads to three cases which we refer to as root dominated, leaf dominated and balanced. Conveniently, to distinguish these three cases we need only consider the cost of each node in the tree and how it relates to the cost of its children.

Definition 16.2 (Brick Method)**.** Consider each node v of the recursion tree, and let $N(v)$ denote its input size, $C(v)$ denote its cost, and $D(v)$ denote the set of its children. There exists constants $a \geq 1$ (base size), $\alpha > 1$ (grown/decay rate) such that:

Root Dominated For all nodes v such that $N(v) > a$,

$$C(v) \geq \alpha \sum_{u \in D(v)} C(u),$$

i.e., the cost of the parent is at least a constant factor greater than the sum of the costs of the children. In this case, the total cost is dominated by the root, and is upper bounded by $\frac{\alpha}{\alpha-1}$ times the cost of the root.

Leaves Dominated For all v such that $N(v) > a$,

$$C(v) \leq \frac{1}{\alpha} \sum_{u \in D(v)} C(u),$$

i.e., the cost of the parent is at least a constant factor less than the sum of the costs of the children. In this case, the total cost is dominated by the cost of the leaves, and is upper bounded by $\frac{\alpha}{\alpha-1}$ times the sum of the cost of the leaves. Most often all leaves have constant cost so we just have to count the number of leaves.

Balanced When neither of the two above cases is true. In this case the cost is upper bounded by the number of levels times the maximum cost of a level.

Proof. We first consider the root dominated case. For this case if the root has cost $C(r)$, level i (the root is level 0) will have total cost at most $(1/\alpha)^i C(r)$. This is because the cost of the children of every node on a level decrease by at least a factor of α to the next level. The total cost is therefore upper bounded by

$$\sum_{i=0}^{\infty} \left(\frac{1}{\alpha}\right)^i C(r).$$

This is a decaying geometric sequence and therefore is upper bounded by $\frac{\alpha}{\alpha-1}C(r)$, as claimed.

For the leaf dominated case, if all leaves are on the same level and have the same cost, we can make a similar argument as above but in the other direction—i.e. the levels increase geometrically down to the leaves. The cost is therefore dominated by the leaf level. In general, however, not all leaves are at the same level.

For the general leaf-dominated case, let L be the set of leaves. Consider the cost $C(l)$ for $l \in L$, and account a charge of $(1/\alpha)^i C(l)$ to its i-th ancestor in the tree (its parent is its first ancestor). Adding up the contributions from every leaf to the internal nodes of the tree gives the maximum possible cost for all internal nodes. This is because for this charging every internal node will have a cost that is exactly $(1/\alpha)$ the sum of the cost of the children, and this is the most each node can have by our assumption of leaf-dominated recurrences. Now summing the contributions across leaves, including the cost of the leaves themselves ($i = 0$), we have as an upper bound on the total cost across the tree:

$$\sum_{l \in L} \sum_{i=0}^{\infty} \left(\frac{1}{\alpha}\right)^i C(l) .$$

This is a sum of sums of decaying geometric sequences, giving an upper bound on the total cost across all nodes of $\frac{\alpha}{\alpha-1} \sum_{l \in L} C(l)$, as claimed.

The balanced case follows directly from the fact that the total cost is the sum of the cost of the levels, and hence at most the number of levels times the level with maximum cost. □

Remark. The term "brick" comes from thinking of each node of the tree as a brick and the width of a brick being its cost. The bricks can be thought of as being stacked up by level. A recurrence is leaf dominated if the pile of bricks gets narrower as you go up to the root. It is root dominated if it gets wider going up to the root. It is balanced if it stays about the same width.

Example 16.5 (Root dominated)**.** Lets consider the recurrence

$$W(n) = 2W(n/2) + n^2.$$

For a node in the recursion tree of size n we have that the cost of the node is n^2 and the sum of the cost of its children is $(n/2)^2 + (n/2)^2 = n^2/2$. In this case the cost has **decreased** by a factor of two going down the tree, and hence the recurrence is root dominated. Therefore for asymptotic analysis we need only consider the cost of the root, and we have that $W(n) = O(n^2)$.

In the leaf dominated case the cost is proportional to the number of leaves, but we have to calculate how many leaves there are. In the common case that all leaves are at the same level (i.e. all recursive calls are the same size), then it is relatively easy. In particular, one can calculate the number of recursive calls at each level, and take it to the power of the depth of the tree, i.e., (branching factor)$^{\text{depth}}$.

Example 16.6 (Leaf dominated)**.** Lets consider the recurrence

$$W(n) = 2W(n/2) + \sqrt{n}\ .$$

For a node of size n we have that the cost of the node is $\sqrt{n}$ and the sum of the cost of its two children is $\sqrt{n/2} + \sqrt{n/2} = \sqrt{2}\sqrt{n}$. In this case the cost has **increased** by a factor of $\sqrt{2}$ going down the tree, and hence the recurrence is leaf dominated. Each leaf corresponds to the base case, which has cost 1.

Now we need to determine how many leaves there are. Since each recursive call halves the input size, the depth of recursion is going to be $\lg n$ (the number of times one needs to half n before getting to size 1). Now on each level the recursion is making two recursive calls, so the number of leaves will be $2^{\lg n} = n$. We therefore have that $W(n) = O(n)$.

Example 16.7 (Balanced)**.** Lets consider the same recurrence we considered for the tree method, i.e.,

$$W(n) = 2W(n/2) + c_1 n + c_2.$$

For all nodes we have that the cost of the node is $c_1 n + c_2$ and the sum of the cost of the two children is $(c_1 n/2 + c_2) + (c_1 n/2 + c_2) = c_1 n + 2c_2$. In this case the cost is about the same for the parent and children, and certainly not growing or decaying geometrically. It is therefore a balanced recurrence. The maximum cost of any level is upper bounded by $(c_1 + c_2)n$, since there are at most n total elements across any level (for the $c_1 n$ term) and at most n nodes (for the $c_2 n$ term). There are $1 + \lg n$ levels, so the total cost is upper bounded by $(c_1 + c_2)n(1 + \lg n)$. This is slightly larger than our earlier bound of $c_1 n \lg n + c_2(2n - 1)$, but it makes no difference asymptotically—they are both $O(n \lg n)$.

Remark. Once you are used to using the brick method, solving recurrences can often be done very quickly. Furthermore the brick method can give a strong intuition of what part of the program dominates the cost—either the root or the leaves (or both if balanced). This can help a programmer decide how to best optimize the performance of recursive code. If it is leaf dominated then it is important to optimize the base case, while if it is root dominated it is important to optimize the calls to other functions used in conjunction with the recursive calls. If it is balanced, then, unfortunately, both need to be optimized.

Exercise 16.2. For each of the following recurrences state whether it is leaf dominated, root dominated or balanced, and then solve the recurrence

$$\begin{aligned}
W(n) &= 3W(n/2) + n \\
W(n) &= 2W(n/3) + n \\
W(n) &= 3W(n/3) + n \\
W(n) &= W(n-1) + n \\
W(n) &= \sqrt{n}W(\sqrt{n}) + n^2 \\
W(n) &= W(\sqrt{n}) + W(n/2) + n
\end{aligned}$$

Solution. The recurrence $W(n) = 3W(n/2) + n$ is leaf dominated since $n \leq 3(n/2) = \frac{3}{2}n$. It has $3^{\lg n} = n^{\lg 3}$ leaves so $W(n) = O(n^{\lg 3})$.

The recurrence $W(n) = 2W(n/3) + n$ is root dominated since $n \geq 2(n/3) = \frac{2}{3}n$. Therefore $W(n) = O(n)$, i.e., the cost of the root.

The recurrence $W(n) = 3W(n/3) + n$ is balanced since $n = 3(n/3)$. The depth of recursion is $\log_3 n$, so the overall cost is n per level for $\log_3 n$ levels, which gives $W(n) = O(n \log n)$.

The recurrence $W(n) = W(n-1) + n$ is balanced since each level only decreases by 1 instead of by a constant fraction. The largest level is n (at the root) and there are n levels, which gives $W(n) = O(n \cdot n) = O(n^2)$.

The recurrence $W(n) = \sqrt{n}W(\sqrt{n}) + n^2$ is root dominated since $n^2 \geq \sqrt{n} \cdot (\sqrt{n})^2 = n^{3/2}$. In this case the decay is even faster than geometric. Certainly for any $n \geq 2$, it satisfies our root dominated condition for $\alpha = \sqrt{2}$. Therefore $W(n) = O(n^2)$.

The recurrence $W(n) = W(\sqrt{n}) + W(n/2) + n$ is root dominated since for $n > 16$, $n \geq \frac{4}{3}(\sqrt{n} + n/2)$. Note that here we are using the property that a leaf can be any problem size greater than some constant a. Therefore $W(n) = O(n)$, i.e., the cost of the root.

Advanced. In some leaf-dominated recurrences not all leaves are at the same level. An example is $W(n) = W(n/2) + W(n/3) + 1$. Let $L(n)$ be the number of leaves as a function of n. We can solve for $L(n)$ using yet another recurrence. In particular the number of leaves for an internal node is simply the sum of the number of leaves of each of its children. In the example this will give the recurrence $L(n) = L(n/2) + L(n/3)$. Hence, we need to find a function $L(n)$ that satisfies this equation. If we guess that it has the form $L(n) = n^\beta$ for some β, we can plug it into the equation and try to solve for β:

$$
\begin{aligned}
n^\beta &= \left(\tfrac{n}{2}\right)^\beta + \left(\tfrac{n}{3}\right)^\beta \\
&= n^\beta \left(\left(\tfrac{1}{2}\right)^\beta + \left(\tfrac{1}{3}\right)^\beta\right)
\end{aligned}
$$

Now dividing through by n^β gives

$$
\left(\frac{1}{2}\right)^\beta + \left(\frac{1}{3}\right)^\beta = 1 .
$$

This gives $\beta \approx .788$ (actually a tiny bit less). Hence $L(n) < n^{.788}$, and because the original recurrence is leaf dominated: $W(n) \in O(n^{.788})$.

This idea of guessing a form of a solution and solving for it is key in our next method for solving recurrences, the substitution method.

5 Substitution Method

The tree method can be used to find the closed form solution to many recurrences but in some cases, we need a more powerful techniques that allows us to make a guess and then verify our guess via mathematical induction. The substitution method allows us to do that exactly.

Important. This technique can be tricky to use: it is easy to start on the wrong foot with a poor guess and then derive an incorrect proof, by for example, making a small mistake. To minimize errors, you can follow the following tips:

1. Spell out the constants—do not use asymptotic notation such as big-O. The problem with asymptotic notation is that it makes it super easy to overlook constant factors, which need to be carefully accounted for.
2. Be careful that the induction goes in the right direction.
3. Add additional lower-order terms, if necessary, to make the induction work.

Example 16.8. Consider the recurrence

$$W(n) = 2W(n/2) + O(n).$$

By the definition of asymptotic complexity, we can establish that

$$W(n) \quad \leq \quad 2W(n/2) + c_1 \cdot n + c_2,$$

where c_1 and c_2 are constants.

We will prove the following theorem using strong induction on n.

Theorem. Let a constant $k > 0$ be given. If $W(n) \leq 2W(n/2) + k \cdot n$ for $n > 1$ and $W(n) \leq k$ for $n \leq 1$, then we can find constants κ_1 and κ_2 such that

$$W(n) \;\leq\; \kappa_1 \cdot n \lg n + \kappa_2.$$

Proof. Let $\kappa_1 = 2k$ and $\kappa_2 = k$. For the base case ($n = 1$), we check that $W(1) \leq k \leq \kappa_2$. For the inductive step ($n > 1$), we assume that

$$W(n/2) \leq \kappa_1 \cdot \tfrac{n}{2} \lg(\tfrac{n}{2}) + \kappa_2,$$

And we'll show that $W(n) \leq \kappa_1 \cdot n \lg n + \kappa_2$. To show this, we substitute an upper bound for $W(n/2)$ from our assumption into the recurrence, yielding

$$\begin{aligned} W(n) \;&\leq\; 2W(n/2) + k \cdot n \\ &\leq\; 2(\kappa_1 \cdot \tfrac{n}{2} \lg(\tfrac{n}{2}) + \kappa_2) + k \cdot n \\ &=\; \kappa_1 n(\lg n - 1) + 2\kappa_2 + k \cdot n \\ &=\; \kappa_1 n \lg n + \kappa_2 + (k \cdot n + \kappa_2 - \kappa_1 \cdot n) \\ &\leq\; \kappa_1 n \lg n + \kappa_2, \end{aligned}$$

where the final step follows because $k \cdot n + \kappa_2 - \kappa_1 \cdot n \leq 0$ as long as $n > 1$.

Variants of the recurrence considered in our last example arise commonly in algorithms. Next, we establish a theorem that shows that the same bound holds for a more general class of recurrences.

Theorem 16.1 (Superlinear Recurrence)**.** Let $\varepsilon > 0$ be a constant and consider the recurrence

$$W(n) = 2W(n/2) + k \cdot n^{1+\varepsilon}.$$

If $W(n) \leq 2W(n/2) + k \cdot n^{1+\varepsilon}$ for $n > 1$ and $W(n) \leq k$ for $n \leq 1$, then for some constant κ,

$$W(n) \ \leq \ \kappa \cdot n^{1+\varepsilon}.$$

Proof. Let $\kappa = \frac{1}{1-1/2^\varepsilon} \cdot k$. The base case is easy: $W(1) = k \leq \kappa_1$ as $\frac{1}{1-1/2^\varepsilon} \geq 1$. For the inductive step, we substitute the inductive hypothesis into the recurrence and obtain

$$\begin{aligned} W(n) &\leq 2W(n/2) + k \cdot n^{1+\varepsilon} \\ &\leq 2\kappa \left(\frac{n}{2}\right)^{1+\varepsilon} + k \cdot n^{1+\varepsilon} \\ &= \kappa \cdot n^{1+\varepsilon} + \left(2\kappa \left(\frac{n}{2}\right)^{1+\varepsilon} + k \cdot n^{1+\varepsilon} - \kappa \cdot n^{1+\varepsilon}\right) \\ &\leq \kappa \cdot n^{1+\varepsilon}, \end{aligned}$$

where in the final step, we use the fact that for any $\delta > 1$:

$$\begin{aligned} 2\kappa \left(\frac{n}{2}\right)^{\delta} + k \cdot n^{\delta} - \kappa \cdot n^{\delta} &= \kappa \cdot 2^{-\varepsilon} \cdot n^{\delta} + k \cdot n^{\delta} - \kappa \cdot n^{\delta} \\ &= \kappa \cdot 2^{-\varepsilon} \cdot n^{\delta} + (1 - 2^{-\varepsilon})\kappa \cdot n^{\delta} - \kappa \cdot n^{\delta} \\ &\leq 0. \end{aligned}$$

An alternative way to prove the same theorem is to use the tree method and evaluate the sum directly. The recursion tree here has depth $\lg n$ and at level i (again, the root is at level 0), we have 2^i nodes, each costing $k \cdot (n/2^i)^{1+\varepsilon}$. Thus, the total cost is

$$\begin{aligned} \sum_{i=0}^{\lg n} k \cdot 2^i \cdot \left(\frac{n}{2^i}\right)^{1+\varepsilon} &= k \cdot n^{1+\varepsilon} \cdot \sum_{i=0}^{\lg n} 2^{-i \cdot \varepsilon} \\ &\leq k \cdot n^{1+\varepsilon} \cdot \sum_{i=0}^{\infty} 2^{-i \cdot \varepsilon}. \end{aligned}$$

But the infinite sum $\sum_{i=0}^{\infty} 2^{-i \cdot \varepsilon}$ is at most $\frac{1}{1-1/2^\varepsilon}$. Hence, we conclude $W(n) \in O(n^{1+\varepsilon})$.

□

6 Master Method

You might have learned in a previous course about the ***master method*** for solving recurrences. We do not like to use it, because it only works for special cases and does not help develop intuition. It requires that all recursive calls are the same size and are some constant factor smaller than n. It doesn't work for recurrences such as:

$$\begin{aligned} W(n) &= W(n-1)+1 \\ W(n) &= W(2n/3)+W(n/3)+n^3 \\ W(n) &= \sqrt{n}\,W(\sqrt{n})+1 \end{aligned}$$

all for which the tree, brick, and substitution method work. We note, however, that the three cases of the master method correspond to limited cases of leaves dominated, balanced, and root dominated of the brick method.