Modules I

15-150

Lecture 16: October 27, 2026

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Motivation

An important idea in computer science

An important idea in computer science

Abstraction

An important idea in computer science

Abstraction

What is abstraction? What does it entail?

An important idea in computer science

Abstraction

- What is abstraction? What does it entail?
- Separation of **specification** from **implementation**.

An important idea in computer science

Abstraction

What is abstraction? What does it entail?

Separation of **specification** from **implementation**.

Specification: externally visible promise to deliver

An important idea in computer science

Abstraction

What is abstraction? What does it entail?

Separation of **specification** from **implementation**.

Specification: externally visible promise to deliver

Implementation: internal choice of how to deliver promise

An important idea in computer science

Abstraction

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Abstraction

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Abstraction

An important idea in computer science



What?

Abstraction

An important idea in computer science

Abstraction



What?

Implementation

An important idea in computer science

Abstraction



What?

Implementation

How?

An important idea in computer science

Client 

Specification

What?

Abstraction

Implementation

How?

An important idea in computer science

Client 

Specification

What?

Abstraction

information hiding

Implementation

How?

An important idea in computer science

Client 

Specification

What?

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Implementation

How?

An important idea in computer science

Client 

Specification

What?

Abstraction

information hiding

Implementation

How?

→ Implementations can be replaced as long as satisfy specification!

An important idea in computer science

Client 

Specification

What?

Abstraction

information hiding

representation independence

Implementation

How?

→ Implementations can be replaced as long as satisfy specification!

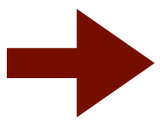
An important idea in computer science

An important idea in computer science

Benefits of abstraction:

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Code evolution without disturbing client code.

An important idea in computer science

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- Code evolution without disturbing client code.
- Reasoning: client only needs to consider specification.

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- Separate development: specifications as blueprint.

An important idea in computer science

Benefits of abstraction:

- Code evolution without disturbing client code.
- Reasoning: client only needs to consider specification.
- Facilitates verification: implementation against specification.
- Separate development: specifications as blueprint.
- The only means by which programs become scalable.

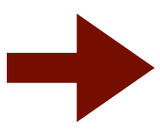
Abstractions in SML

Abstractions in SML

Abstraction at small: functions

Abstractions in SML

Abstraction at small: functions



specification: function type with pre-/post-condition

Abstractions in SML

Abstraction at small: functions

- specification: function type with pre-/post-condition
- implementation: function body

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Abstraction at large: modules

- Offer a way to combine what belongs together in one unit.

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- Offer a way to combine what belongs together in one unit.
- Specification: **signature**.

Abstractions in SML

Abstraction at small: functions

- specification: function type with pre-/post-condition
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Abstraction at large: modules

- Offer a way to combine what belongs together in one unit.
- Specification: **signature**.
- Implementation: **structure**.

Abstractions in SML

Abstraction at small: functions

- specification: function type with pre-/post-condition
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Abstraction at large: modules

- Offer a way to combine what belongs together in one unit.
- Specification: **signature**.
- Implementation: **structure**.
- Composing structures using **functors**.

Abstractions in SML

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SML Basis Library makes use of modules. E.g.,

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```
Int.toString
```

Abstractions in SML

Abstraction at large: modules

- Offer a way to combine what belongs together in one unit.
- Specification: **signature**.
- Implementation: **structure**.
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SML Basis Library makes use of modules. E.g.,

```
Int.toString
```

is a function inside a **structure** called `Int`. The structure `Int` has the **signature** called `INTEGER` ascribed.

Abstractions in SML

Abstraction at large: modules

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- Specification: **signature**.
- Implementation: **structure**.
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SML Basis Library makes use of modules. E.g.,

```
Int.toString
```

is a function inside a **structure** called `Int`. The structure `Int` has the **signature** called `INTEGER` ascribed.

- Let's unravel…

Queue module

Modules at an example: queue

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Queue, a **first-in first-out** (FIFO) data structure.

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Modules at an example: queue

Queue, a **first-in first-out** (FIFO) data structure.



dequeue,
yields a queue with
this element removed

Modules at an example: queue

Queue, a **first-in first-out** (FIFO) data structure.



dequeue,
yields a queue with
this element removed

enqueue v_7,
yields a queue with
element added at end

Modules at an example: queue

Queue, a **first-in first-out** (FIFO) data structure.



We can describe queue abstractly by specifying a **queue** type, along with operations on that type.

Modules at an example: queue

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That's a **signature**.

Modules at an example: queue

Queue, a **first-in first-out** (FIFO) data structure.



- We can describe queue abstractly by specifying a **queue** type, along with operations on that type.
  - That's a **signature**.
- Then we implement it in a **structure**.

Queue signature

Queue signature

```
signature QUEUE =
sig

end
```

Queue signature

```
signature QUEUE =
sig

end
```

Queue signature

```
signature QUEUE =
sig
```




```
end
```

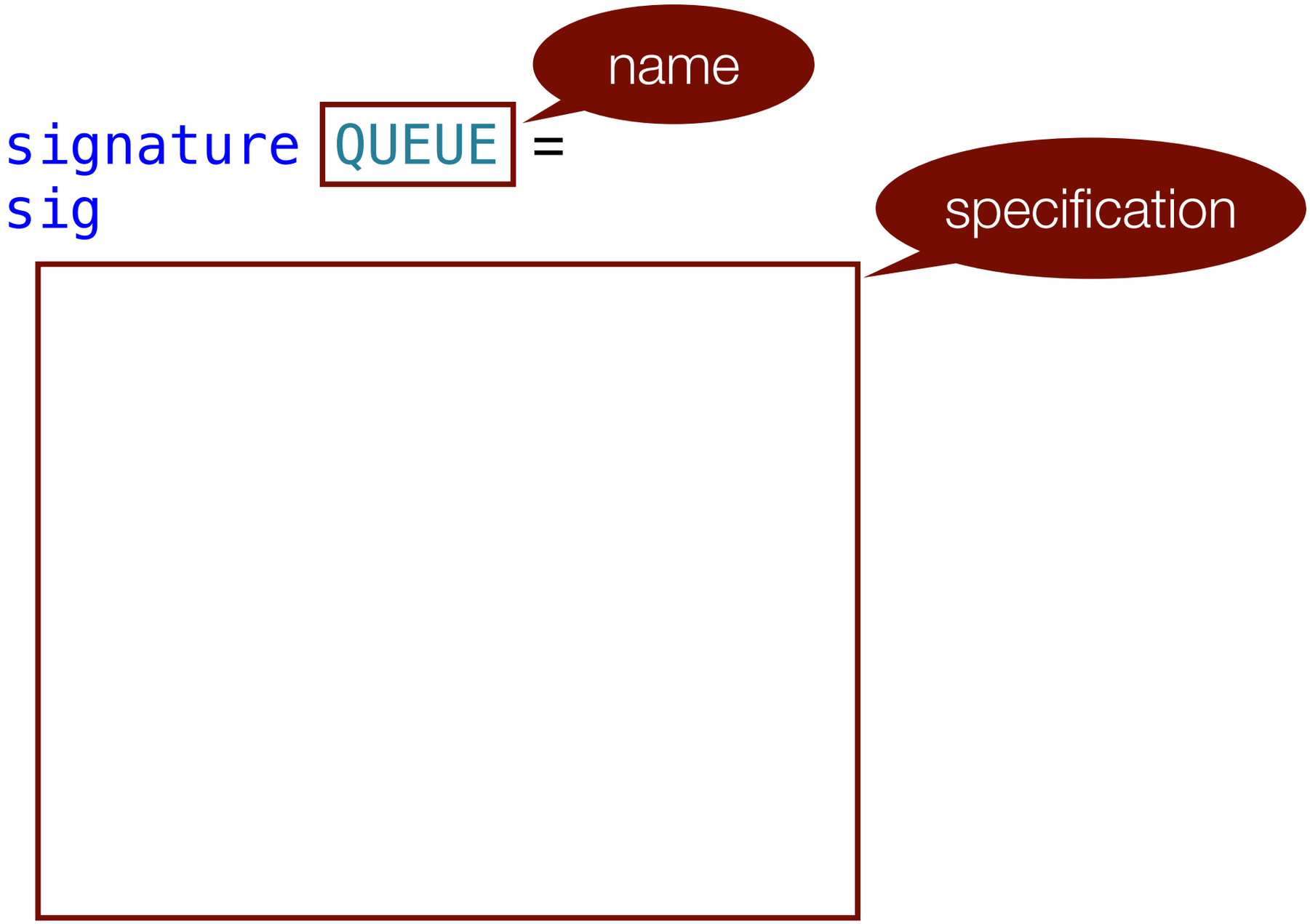
Queue signature

```
signature QUEUE =
sig
```

name

```
end
```

Queue signature



Queue signature

```
signature QUEUE =
sig

end
```

Queue signature

```
signature QUEUE =
sig

  type 'a queue          (* abstract type *)

end
```

Queue signature

```
signature QUEUE =
sig
  type 'a queue              (* abstract type *)
```

end

Queue signature

```
signature QUEUE =
sig

  type 'a queue                (* abstract type *)
```

type name

```
end
```

Queue signature

```
signature QUEUE =
sig
  type 'a queue                (* abstract type *)
```

type name

```
end
```

Queue signature

```
signature QUEUE =
sig

  type 'a queue                    (* abstract type *)

end
```

polymorphic

type name

Queue signature

```
signature QUEUE =
sig

  type 'a queue                    (* abstract type *)

end
```

Postulate existence of an alpha queue

polymorphic

type name

Queue signature

```
signature QUEUE =
sig

  type 'a queue                (* abstract type *)

end
```

Postulate existence of an alpha queue

polymorphic

type name

No details on how represented.

Queue signature

```
signature QUEUE =
sig

  type 'a queue          (* abstract type *)
```

Postulate existence of an alpha queue

```
end
```

Queue signature

```
signature QUEUE =
sig

  type 'a queue                (* abstract type *)

  (* Operations on an alpha queue *)

end
```

Postulate existence of an alpha queue

Queue signature

```
signature QUEUE =
sig

  type 'a queue          (* abstract type *)
```

end

Queue signature

```
signature QUEUE =
sig
  type 'a queue          (* abstract type *)
  val empty : 'a queue
```

end

Queue signature

```
signature QUEUE =
sig

  type 'a queue              (* abstract type *)

  val empty : 'a queue

  val enq : 'a queue * 'a -> 'a queue




end
```

Queue signature

```
signature QUEUE =
sig
  type 'a queue              (* abstract type *)
  val empty : 'a queue
  val enq : 'a queue * 'a -> 'a queue
  val null : 'a queue -> bool


end
```

Queue signature

```
signature QUEUE =
sig
  type 'a queue              (* abstract type *)
  val empty : 'a queue
  val enq : 'a queue * 'a -> 'a queue
  val null : 'a queue -> bool
  exception Empty
  (* deq (q) raises Empty if q is empty *)
  val deq : 'a queue -> 'a * 'a queue
end
```

Let’s implement QUEUE in a structure

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Implementation I: **represent** alpha queue as a single list.

Let's implement QUEUE in a structure

Implementation I: **represent** alpha queue as a single list.



Let's implement QUEUE in a structure

Implementation I: **represent** alpha queue as a single list.



List represents alpha queue elements in arrival order.

Single list representation of alpha queue

Single list representation of alpha queue

```
signature QUEUE =
sig
 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Single list representation of alpha queue

```
signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
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 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
struct



end
```

Single list representation of alpha queue

```
signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
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end
```

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
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 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
struct
```

name

```
end
```

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
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```

```
structure Queue1 : QUEUE =
struct
```

name

```
end
```

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
struct


end
```

Ascribe signature **QUEUE** to structure **Queue1**

name

Single list representation of alpha queue

```
signature QUEUE =
sig
 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Ascribe signature **QUEUE** to structure **Queue1**

```
structure Queue1 : QUEUE =
struct
```

name

```
end
```

Ascribe means:
The structure provides all the items specified in the signature. (It may contain additional items, e.g., helper functions, which will not be visible outside the structure, e.g., to a client.)

Single list representation of alpha queue

```
signature QUEUE =
sig
 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
struct



end
```

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
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structure Queue1 : QUEUE =
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```
end
```

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
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structure Queue1 : QUEUE =
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```
end
```

Single list representation of alpha queue

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structure Queue1 : QUEUE =
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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Single list representation of alpha queue

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list



end
```

```
signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Single list representation of alpha queue

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structure Queue1 : QUEUE =
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  type 'a queue = 'a list

end
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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
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Single list representation of alpha queue

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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

representation type

Single list representation of alpha queue

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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list

end
```

representation type

Single list representation of alpha queue

transparent ascription

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list

end
```

```
signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

representation type

Single list representation of alpha queue

transparent ascription

```
structure Queue1 : QUEUE =
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  type 'a queue = 'a list

end
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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
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 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

representation type

Transparent ascription means:
The representation type of the abstract type alpha queue will be visible outside the structure, e.g., to the client.

Single list representation of alpha queue

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list




end
```

```
signature QUEUE =
sig
 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Single list representation of alpha queue

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list
  val empty = nil



end
```

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signature QUEUE =
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
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Single list representation of alpha queue

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list
  val empty = nil
  fun enq (q,x) = q @ [x]



end
```

```
signature QUEUE =
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 type 'a queue          (* abstract type *)
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 val enq : 'a queue * 'a -> 'a queue
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 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

Single list representation of alpha queue

```
structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list
  val empty = nil
  fun enq (q,x) = q @ [x]
  val null = List.null


end
```

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signature QUEUE =
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 type 'a queue          (* abstract type *)
 val empty : 'a queue
 val enq : 'a queue * 'a -> 'a queue
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Single list representation of alpha queue

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structure Queue1 : QUEUE =
struct
  type 'a queue = 'a list
  val empty = nil
  fun enq (q,x) = q @ [x]
  val null = List.null
  exception Empty
  fun deq (x::q) = (x,q)
    | deq (nil) = raise Empty
end
```

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signature QUEUE =
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 type 'a queue          (* abstract type *)
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Let's interact with `Queue1`:

Single list representation of alpha queue

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Let's interact with `Queue1`:

```
val q = Queue1.enq(Queue1.enq(Queue1.empty, 1), 2)
```

Single list representation of alpha queue

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Let's interact with `Queue1`:

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Single list representation of alpha queue

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Let's interact with `Queue1`:

```
val q = Queue1.enq(Queue1.enq(Queue1.empty, 1), 2)
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What is the type of `q`?

```
int Queue1.queue
```

Single list representation of alpha queue

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signature QUEUE =
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Let's interact with `Queue1`:

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val q = Queue1.enq(Queue1.enq(Queue1.empty, 1), 2)
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What is the type of `q`?

`int Queue1.queue` , i.e.,

Single list representation of alpha queue

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Single list representation of alpha queue

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Single list representation of alpha queue

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`int Queue1.queue` , i.e., `int list`

Why? Because:

First, the signature specifies that queues have type `'a queue`, with `'a` representing the element type. That is `int` here.

Second, we have implemented queues using a structure called `Queue1`. The type is defined inside the structure, so the type has the qualified name `'a Queue1.queue`, here with `'a` instantiated with `int`. The representation type `int list` is visible due to transparent ascription.

Single list representation of alpha queue

Let's interact with `Queue1`:

```
val q = Queue1.enq(Queue1.enq(Queue1.empty, 1), 2)
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What is the type of `q`?

```
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Also, `q` will be bound to:

```
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```

We can see the representation type list because of transparent ascription (and because list is defined in the Basis Library and thus in the top-level scope).

Single list representation of alpha queue

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Single list representation of alpha queue

Single list representation of alpha queue

How long does enqueuing take?

Single list representation of alpha queue

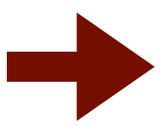
How long does enqueuing take?

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fun enq (q,x) = q @ [x]
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Single list representation of alpha queue

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O(n), with n the number of elements in **q**.

Single list representation of alpha queue

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- Can we improve that?

Single list representation of alpha queue

How long does enqueuing take?

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- **O**(n), with n the number of elements in **q**.
- Can we improve that?
  - Yes, let's choose a different representation type!

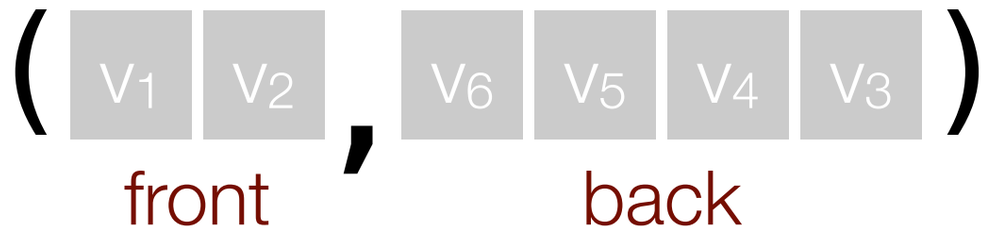
Pair of list representation of alpha queue

Pair of list representation of alpha queue

Implementation II: **represent** alpha queue as a pair of list.

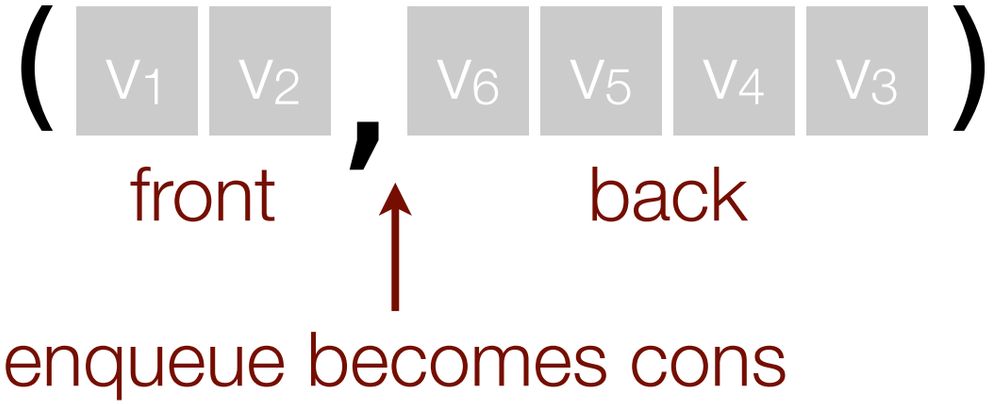
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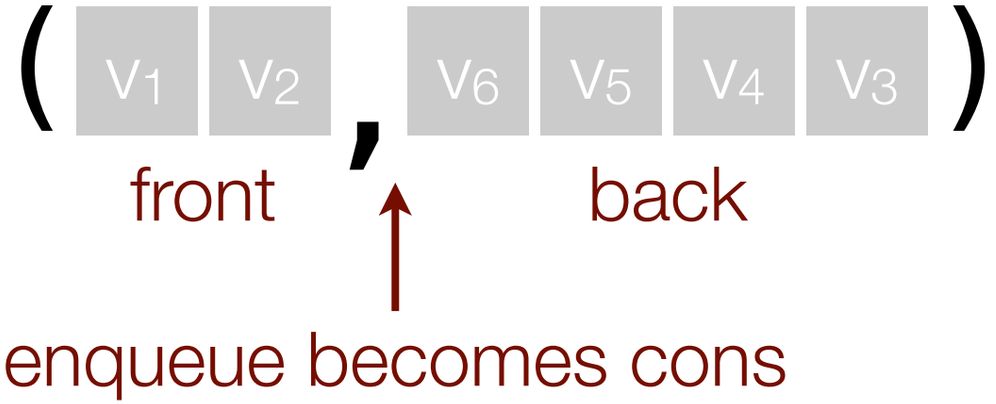
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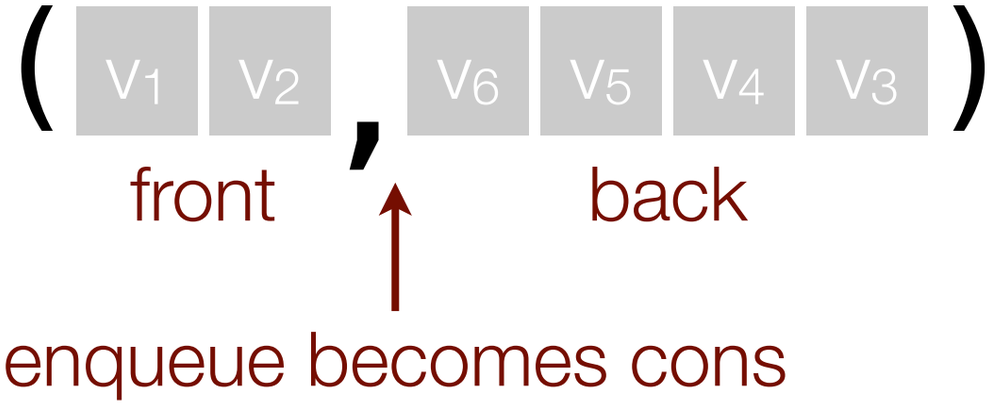
Implementation II: **represent** alpha queue as a pair of list.



Abstraction function:

Pair of list representation of alpha queue

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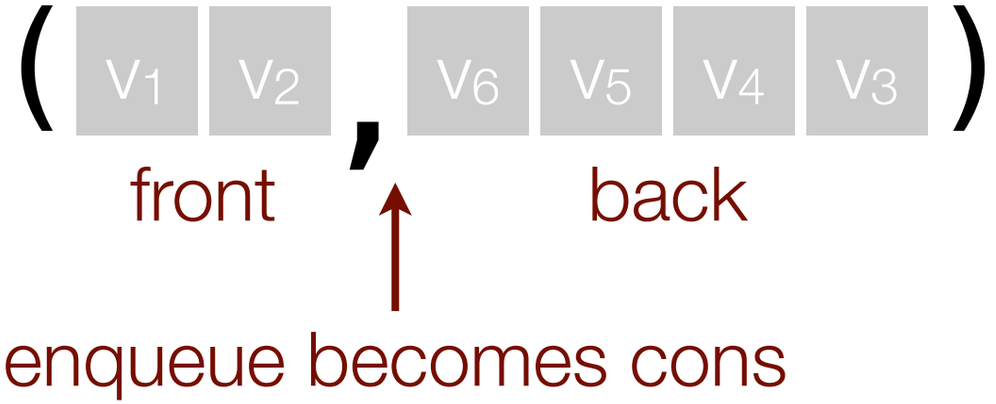


Abstraction function:

```
front @ (rev back)
```

Pair of list representation of alpha queue

Implementation II: **represent** alpha queue as a pair of list.



Abstraction function:

```
front @ (rev back)
```

Represents alpha queue elements in arrival order.

Pair of list representation of alpha queue

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```
signature QUEUE =
sig
 type 'a queue          (* abstract type *)
 val empty : 'a queue
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 val null : 'a queue -> bool
 exception Empty
 val deq : 'a queue -> 'a * 'a queue
end
```

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opaque ascription

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Opaque ascription means that the representation type of the abstract type alpha queue will be hidden outside the structure, e.g., from the client. ML will only print a dash.

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→ Ensures information hiding and integrity of data structure.

→ Not guaranteed with transparent ascription!

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Pair of list representation of alpha queue

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  val empty = (nil, nil)

  fun enq ((front, back), x) = (front, x::back)

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Satisfies requirement that `f @ (rev(x::b))` constitutes queue elements in arrival order.

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```

reverse `back`, if front is empty!

Pair of list representation of alpha queue

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Let's interact with `Queue2`:

Pair of list representation of alpha queue

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[1/a, 1/c, 2/d]
```

Pair of list representation of alpha queue

Pair of list representation of alpha queue

How long does enqueuing take?

Pair of list representation of alpha queue

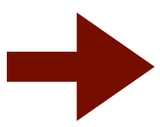
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O(1)!

Pair of list representation of alpha queue

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fun enq ((front, back), x) = (front, x::back)
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- **O**(1)!
- Dequeuing can now take varying amounts of time.

Pair of list representation of alpha queue

How long does enqueuing take?

```
fun enq ((front, back), x) = (front, x::back)
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- **O**(1)!
- Dequeuing can now take varying amounts of time.
- However, enqueuing and dequeuing n items will take **O**(n) time total, so the **amortized** cost of each is **O**(1).

Comparing the two implementations

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Operation:	Queue1:	Queue2:

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Operation:	Queue1:	Queue2:
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`enq 1`	`[1]`	`([], [1])`

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Operation:	Queue1:	Queue2:
`empty`	`[]`	`([], [])`
`enq 1`	`[1]`	`([], [1])`
`enq 2`	`[1,2]`	`([], [2,1])`

Comparing the two implementations

Operation:	Queue1:	Queue2:
`empty`	`[]`	`([], [])`
`enq 1`	`[1]`	`([], [1])`
`enq 2`	`[1,2]`	`([], [2,1])`
`deq`	`[2]`	

Comparing the two implementations

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`empty`	`[]`	`([], [])`
`enq 1`	`[1]`	`([], [1])`
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		briefly this:
`deq`	`[2]`	`([1,2], [])`
		then this:
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`enq 2`	`[1,2]`	`([], [2,1])`
		briefly this:
`deq`	`[2]`	`([1,2], [])`
		then this:
		`([2], [])`
`enq 3`	`[2,3]`	`([2], [3])`

Dictionary module

Another example: dictionary

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A dictionary is a collection of pairs of the form

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where all keys must be unique within a dictionary.

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signature DICT =
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  type key = string                    (* concrete type *)
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  type 'a dict                         (* abstract type *)



end
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Let's use strings as keys for now

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Let's use strings as keys for now

Permit value type to be polymorphic

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end
```

Replace entry, if key already exists

Search tree representation of dictionary

Search tree representation of dictionary

Implementation: **represent** dictionary as a binary search tree, where

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`(key, value)`

Search tree representation of dictionary

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→ Representation **invariant**:

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- Representation **invariant**:
  - Tree must be sorted.
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Search tree representation of dictionary

Implementation: **represent** dictionary as a binary search tree, where

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  - Tree must be sorted.
  - Keys must be unique.
- All functions declared by structure

Search tree representation of dictionary

Implementation: **represent** dictionary as a binary search tree, where

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(key, value)
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- Representation **invariant**:
  - Tree must be sorted.
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Search tree representation of dictionary

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  - Tree must be sorted.
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  - may **assume** that received tree is sorted,
  - and must **assert** that returned tree is sorted.

Search tree representation of dictionary

Implementation: **represent** dictionary as a binary search tree, where

```
(key, value)
```

are stored in nodes.

- Representation **invariant**:
  - Tree must be sorted.
  - Keys must be unique.
- All functions declared by structure
  - may **assume** that received tree is sorted,
  - and must **assert** that returned tree is sorted.
  - (Similarly for key uniqueness.)

Search tree representation of dictionary

Search tree representation of dictionary

```
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  type key = string          (* concrete type *)
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  val insert : 'a dict * 'a entry -> 'a dict
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```

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end
```

```
structure BST : DICT =
struct
  type key = string
  type 'a entry = key * 'a
  datatype 'a tree =
    Empty | Node of 'a tree * 'a entry * 'a tree
  type 'a dict = 'a tree



end
```

Search tree representation of dictionary

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end
```

Search tree representation of dictionary

Explore

```
structure BST : DICT =
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```

```
signature DICT =
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  type key = string          (* concrete type *)
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  val empty : 'a dict
  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

→ Transparent ascription can be useful for debugging purposes.

Search tree representation of dictionary

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  type key = string          (* concrete type *)
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struct
  type key = string
  type 'a entry = key * 'a
  datatype 'a tree =
    Empty | Node of 'a tree * 'a entry * 'a tree
  type 'a dict = 'a tree



end
```

no other choice

Search tree representation of dictionary

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signature DICT =
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  type key = string          (* concrete type *)
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  val empty : 'a dict
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→ Because datatype is not declared in signature, constructors (and thus pattern matching) are not available outside structure.

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→ Because datatype is not declared in signature, constructors (and thus pattern matching) are not available outside structure.

→ But bindings externally visible due to transparent ascription.

Search tree representation of dictionary

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signature DICT =
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Search tree representation of dictionary

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structure BST : DICT =
struct
  type key = string
  type 'a entry = key * 'a
  datatype 'a tree =
    Empty | Node of 'a tree * 'a entry * 'a tree
  type 'a dict = 'a tree
  val empty = Empty
  fun lookup ...
  fun insert ...
end
```

Search tree representation of dictionary

Search tree representation of dictionary

```
(* ins : 'a dict * 'a entry -> 'a dict *)
fun insert (Empty, e) = Node(Empty, e, Empty)
  | insert (Node(lt, e' as (k',_), rt),
            e as (k,_)) =
```

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Layered pattern matching

Search tree representation of dictionary

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  | insert (Node(lt, e' as (k',_), rt),
                e as (k,_)) =
    (case String.compare(k,k') of
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fun insert (Empty, e) = Node(Empty, e, Empty)
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       EQUAL => Node(lt, e, rt)
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```

Replace existing entry with new one

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     | LESS => Node(insert(lt,e), e', rt)
     | GREATER => Node(lt, e', insert(rt,e)))
```

Search tree representation of dictionary

Search tree representation of dictionary

```
(* lookup : 'a dict -> key -> 'a option *)
fun lookup tree key =
 let
     fun lk (Empty) = NONE
     | lk (Node(left, (k,v), right)) =
       (case String.compare(key,k) of
         EQUAL => SOME(v)
         | LESS => lk left
         | GREATER => lk right)
in
   lk tree
end
```

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bound to argument to call of **lookup**

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end
```

Search tree representation of dictionary

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(* lookup : 'a dict -> key -> 'a option *)
fun lookup tree key =
 let
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end
```

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end
```

bound to argument to call of **`lookup`**

Search tree representation of dictionary

Search tree representation of dictionary

Let's interact with **BST**:

Search tree representation of dictionary

Let's interact with **BST**:

```
val d = BST.insert(BST.insert(BST.insert(
          BST.empty,(“a",1)),("b",2)),("c",3))
```

Search tree representation of dictionary

Let's interact with **BST**:

```
val d = BST.insert(BST.insert(BST.insert(
          BST.empty,(“a",1)),("b",2)),("c",3))
```

What is the type of **d**?

Search tree representation of dictionary

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int BST.dict
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The binding for **d** will be revealed because of transparent ascription. However, because the tree datatype is not declared in the signature, a client cannot pattern match on its constructors.

Search tree representation of dictionary

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Now consider:

Search tree representation of dictionary

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The binding for **d** will be revealed because of transparent ascription. However, because the tree datatype is not declared in the signature, a client cannot pattern match on its constructors.

Now consider: `val look = BST.lookup d`

Search tree representation of dictionary

Let's interact with **BST**:

```
val d = BST.insert(BST.insert(BST.insert(
          BST.empty,("a",1)),("b",2)),("c",3))
```

What is the type of **d**?

```
int BST.dict
```

The binding for **d** will be revealed because of transparent ascription. However, because the tree datatype is not declared in the signature, a client cannot pattern match on its constructors.

Now consider: `val look = BST.lookup d`

What is the type of **look**?

Search tree representation of dictionary

Let's interact with **BST**:

```
val d = BST.insert(BST.insert(BST.insert(
          BST.empty,("a",1)),("b",2)),("c",3))
```

What is the type of **d**?

```
int BST.dict
```

The binding for **d** will be revealed because of transparent ascription. However, because the tree datatype is not declared in the signature, a client cannot pattern match on its constructors.

Now consider: `val look = BST.lookup d`

What is the type of **look**?

```
BST.key -> int option
```

Search tree representation of dictionary

Let's interact with **BST**:

```
val d = BST.insert(BST.insert(BST.insert(
          BST.empty,("a",1)),("b",2)),("c",3))
```

What is the type of **d**?

```
int BST.dict
```

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val x = look "e"
val y = look "a"
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Bindings:

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Now consider:
```
val x = look "e"
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Bindings: `[NONE/x, (SOME 1)/y]`

That's all for today. See you on Thursday!