Higher-Order Functions II: Staging

15-150

Lecture 11: October 2, 2025

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Let's revisit foldl and foldr on lists

Higher-order function: fold

Combining elements in a list, given a binary operation and base value:

```
(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
```

combining function:

‘a: type of list elements

‘b: type of base value and of combined value

initial value

list to be combined

combined value

Higher-order function: fold

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```

Two implementations:

$$\texttt{foldl f z } \overrightarrow{[x_1,\ldots,x_n]} \cong f(x_n,\ldots f(x_3,f(x_2,f(x_1,z))))$$

$$\texttt{foldr f z } \underleftarrow{[x_1,\ldots,x_n]} \cong f(x_1,\ldots f(x_{n-2},f(x_{n-1},f(x_n,z))))$$

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2 (4-(3-(2-(1-0))))
```

Higher-order function: fold

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$\texttt{foldl f z } \overrightarrow{[x_1,\ldots,x_n]} \cong \texttt{f}(x_n,\ldots\texttt{f}(x_3,\texttt{f}(x_2,\texttt{f}(x_1,\texttt{z}))))$

$\texttt{foldr f z } \overleftarrow{[x_1,\ldots,x_n]} \cong \texttt{f}(x_1,\ldots\texttt{f}(x_{n-2},\texttt{f}(x_{n-1},\texttt{f}(x_n,\texttt{z}))))$

Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
foldr (op -) 0 [1,2,3,4] ==> ~2  (1-(2-(3-(4-0))))
```

Higher-order function: fold

Let's implement foldl and foldr:

```
(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
fun foldl f z [] =
  | foldl f z (x::xs) =
```

Higher-order function: fold

Let’s implement foldl and foldr:

```
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Higher-order function: fold

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fun foldl f z [] = z
  | foldl f z (x::xs) = foldl f (f(x,z)) xs

fun foldr f z [] = z
  | foldr f z (x::xs) = f(x, foldr f z xs)
```

Homework:

```
foldl (op ::) [] [1,2,3,4] ==> ?
foldr (op ::) [] [1,2,3,4] ==> ?
```

Map and fold over trees and other datatypes

Can we generalize map and fold?

So fare we have considered map and fold exclusively for lists.

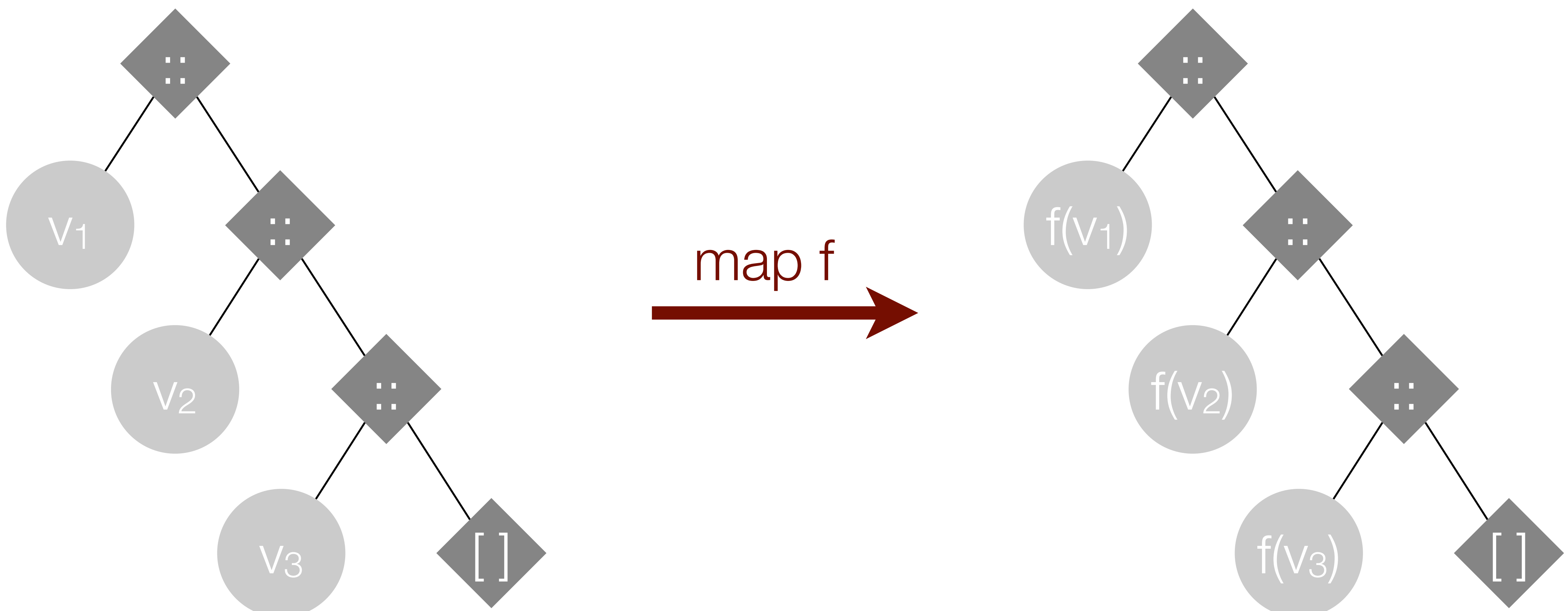
- map: transform elements in a list, given a transformation function
- fold: combines elements in a list, given a binary operation and base value

Can we generalize map and fold to, for example, binary trees?

- Yes! Let’s work it out.
- It may be helpful to visualize map and fold for lists diagrammatically first, to capture the underlying pattern.

The “pattern” underlying map

```
(* map: ('a -> 'b) -> 'a list -> 'b list *)
```



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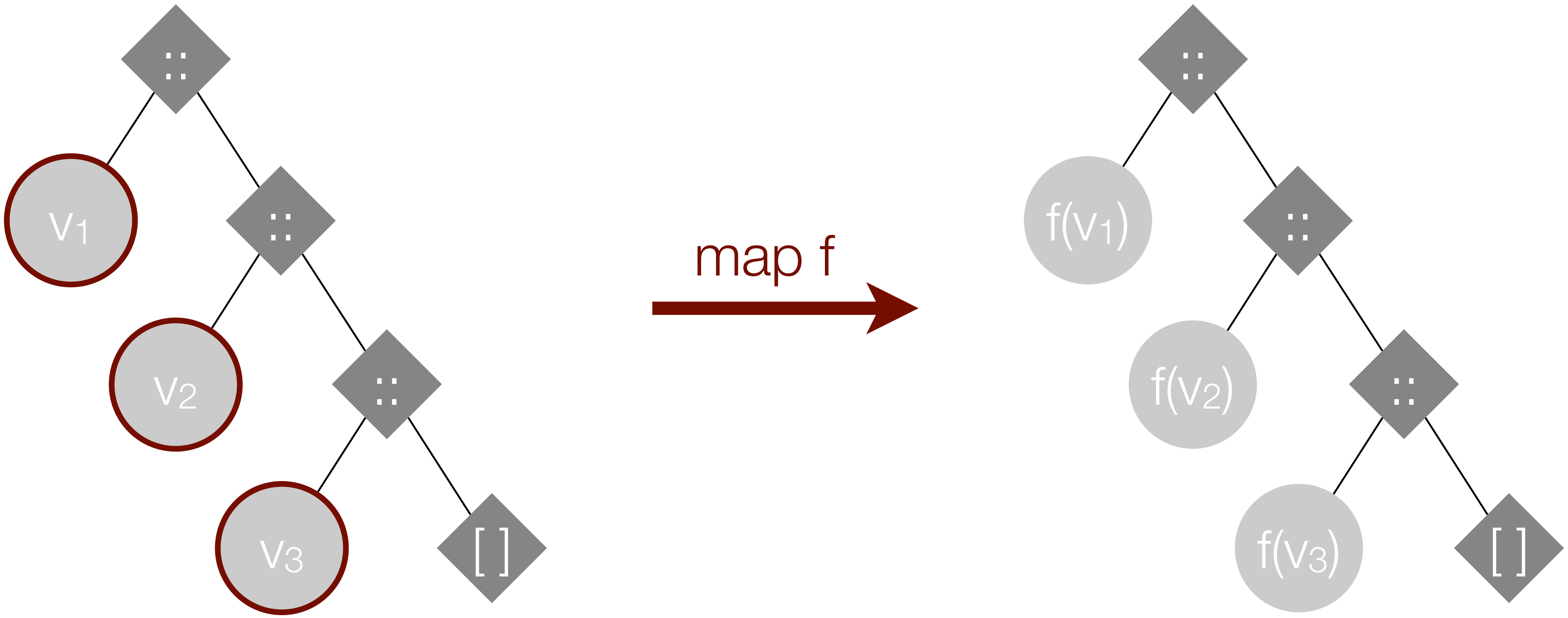

Replace every element value v_i with its transformed value $f(v_i)$.

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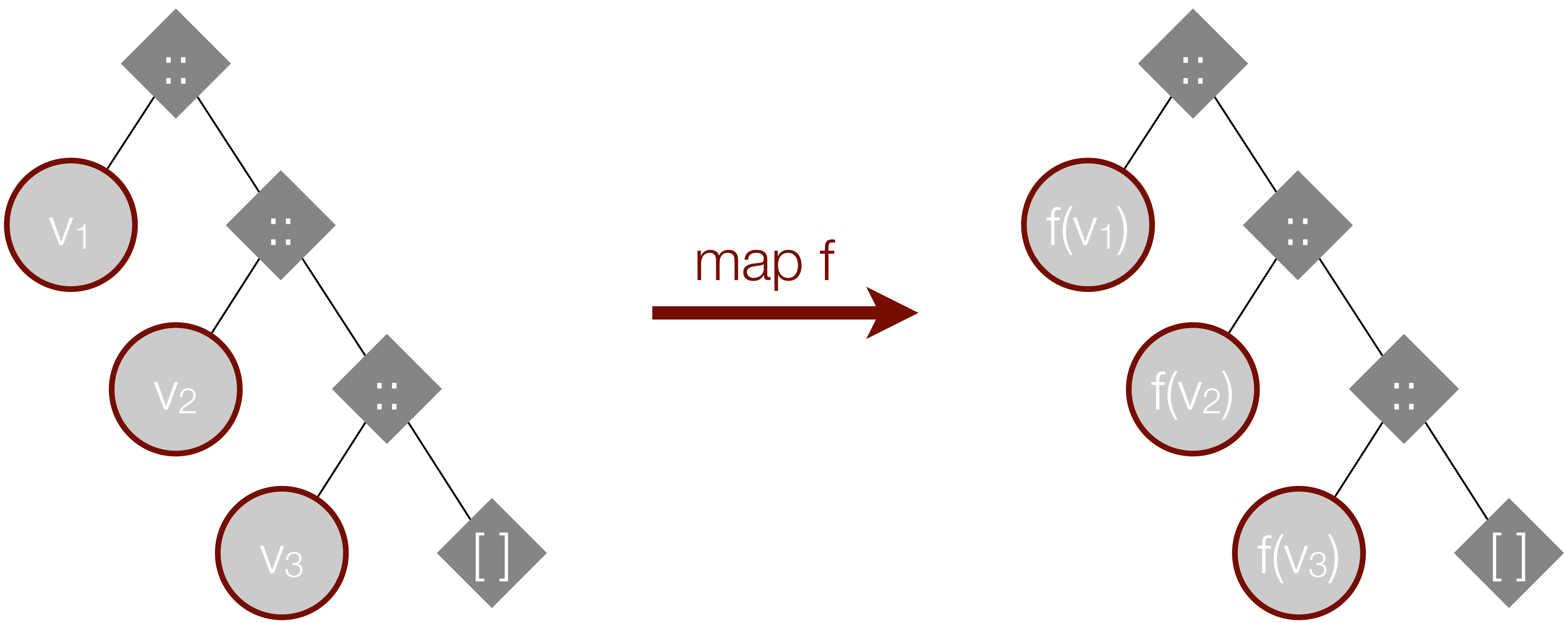

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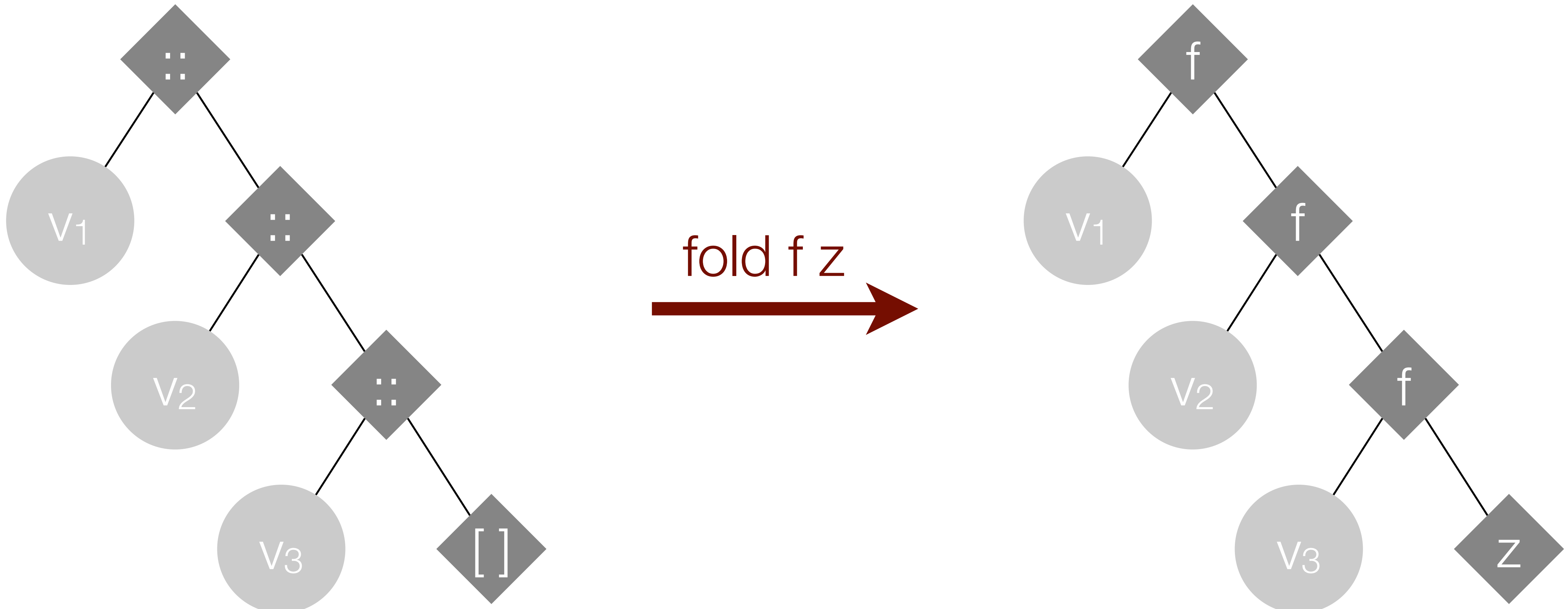

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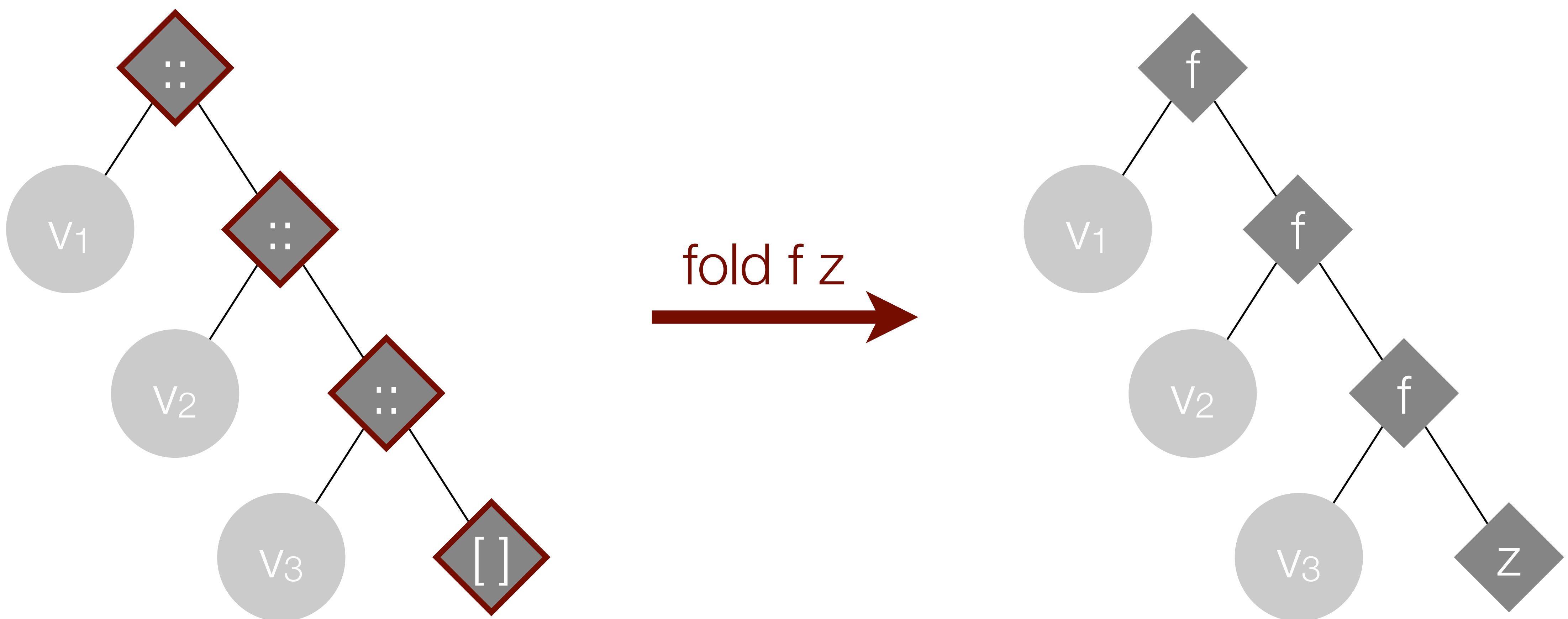

Replace every constructor with a function or value.

The “pattern” underlying fold

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(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
```
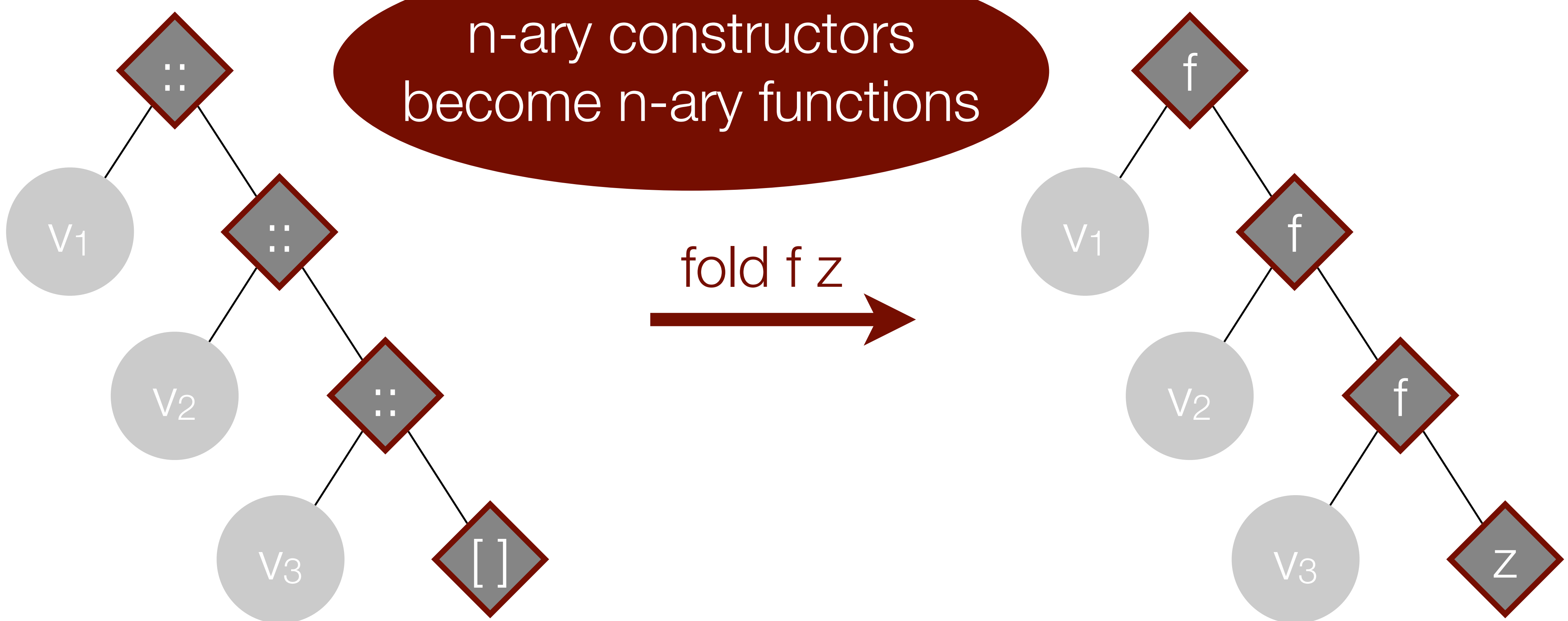

n-ary constructors
become n-ary functions



Replace every constructor with a function or value.

Map and fold for binary trees

```
datatype 'a tree = Empty | Node of 'a tree * 'a * 'a tree
```

Map and fold for binary trees

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```

same number
of arguments as
constructor

```
(* tfold : ('b * 'a * 'b -> 'b) -> 'b -> 'a tree -> 'b *)
```

result of fold
of left subtree

result of fold
of right subtree

base value for
empty

Map and fold for binary trees

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Examples for tmap and tfold

```
val stringify = tmap Int.toString

val treesum = tfold (fn (a,x,b) => a+x+b) 0
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Map and fold for leafy binary trees

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Map and fold for non-recursive datatypes

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datatype 'a option = NONE | SOME of 'a
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val osum = opfold (fn x => x) 0
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What are the types of `otringify` and `osum`?

```
(* ostringify : int option -> string option *)

(* osum : int option -> int *)
```

Staging

Another use of HOF: Staging

Staging is a coding technique
that has a function perform useful work
prior to receiving all its arguments.

- Concern: efficiency (“cost”) of evaluation
- Employs partial application
  - to factor out expensive part
  - to specialize inexpensive part for specific argument.
- Improves efficiency when specialized function used many times.

Staging

Consider the following function:

```
fun f (x:int, y:int) : int =
    let
       val z : int = horriblecomputation(x)
    in
       z + y
    end
```

Suppose the horrible computation takes 10 months.
(And suppose that addition takes a picosecond.)

Then each of these expressions takes at least 10 months to evaluate:

```
f (5,2)
f (5,3)
```

without mutation

If only we could recall `horriblecomputation(5)`!

Staging

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```
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    let
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    in
       z + y
    end
```

What is the type of `f`?

```
(* f : int * int -> int *)
```

➡ Maybe currying can help?

➡ Let’s define a curried version of f!

Staging

Curried version of f:

```
fun g (x:int) (y:int) : int =
    let
       val z : int = horriblecomputation(x)
    in
       z + y
    end
```

Now the type of **g** is `(* g : int -> int -> int *),`

so we can define `val g5 : int -> int = g(5)`

and then evaluate `g5 (2)  (* instead of f (5,2) *)`

```
g5 (3)  (* instead of f (5,3) *)
```

How long do the 3 lines above take?

Staging

How long do the 3 lines above take?

Remember, the declaration of **g** created the following binding:

```
[(fn x => fn y => let val z = hc(x) in z+y end)/g]
```

In declaring `val g5 = g(5)`, one evaluates

```
[(fn x => fn y => let val z = hc(x) in z+y end)/g] g(5)
==> (fn x => fn y => let val z = hc(x) in z+y end) (5)
==> [5/x] fn y => let val z = hc(x) in z+y end
```

This is a lambda, and thus s a value!

No application, and thus no evaluation of body!

Staging

How long do the 3 lines above take?

Remember, the declaration of `g` created the following binding:

```
[(fn x => fn y => let val z = hc(x) in z+y end)/g]
```

In declaring `val g5 = g(5)`, one evaluates

```
[(fn x => fn y => let val z = hc(x) in z+y end)/g] g(5)
==> (fn x => fn y => let val z = hc(x) in z+y end) (5)
==> [5/x] fn y => let val z = hc(x) in z+y end
```

This is the closure returned by `g(5)`.

The horrible computation has not yet happened :-(

Staging

We now have the following binding:

```
[ env
  [5/x]
  fn y => let val z = hc(x) in z+y end  /g5]
```

Evaluating `g5(2)`

```
==> [5/x, 2/y] let val z = hc(x) in z+y end
==> [5/x, 2/y, n/z] z+y          (for some integer n)
==> n
```

10 months!

Similarly, `g5(3)` will take 10 months.

Defining `g` in place of `f` has not yet helped!

Staging

Recall the lambda expression for `g`:

```
fn x => fn y => let val z = hc(x) in z+y end
```

Let's move this computation here.

Horrible computation hidden underneath inner lambda.

- Move is valid because the computation does not depend on `y`.
- Such rearrangement of code — putting it in the "right spot" — we refer to as staging.

Staging

Let's stage properly:

```
fun h (x:int) : int -> int =
    let
       val z : int = horriblecomputation(x)
    in
       (fn y : int => z + y)
    end
```

Inner lambda free of `hc(x)`!

Now the type of h is `(* h : int -> int -> int *)`,

so we can define `val h5 : int -> int = h(5)`

and then evaluate `h5 (2)`

`h5 (3)`

How long do the 3 lines above take?

Staging

How long do the 3 lines above take?

Remember, the declaration of h created the following binding:

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h]
```

In declaring `val h5 = h(5)`, one evaluates

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h] h(5)
==> (fn x => let val z = hc(x) in fn y => z+y end) (5)
==> [5/x] let val z = hc(x) in fn y => z+y end
==> [5/x, n/z] fn y => z+y
```

(for some integer n)

10 months!

Staging

How long do the 3 lines above take?

Remember, the declaration of h created the following binding:

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h]
```

In declaring `val h5 = h(5)`, one evaluates

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h] h(5)
==> (fn x => let val z = hc(x) in fn y => z+y end) (5)
==> [5/x] let val z = hc(x) in fn y => z+y end
==> [5/x, n/z] fn y => z+y
```

(for some integer n)

10 months!

This is a lambda, and thus s a value!

Staging

How long do the 3 lines above take?

Remember, the declaration of h created the following binding:

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h]
```

In declaring `val h5 = h(5)`, one evaluates

```
[(fn x => let val z = hc(x) in fn y => z+y end)/h] h(5)
==> (fn x => let val z = hc(x) in fn y => z+y end) (5)
==> [5/x] let val z = hc(x) in fn y => z+y end
==> [5/x, n/z] fn y => z+y
```

(for some integer n)

10 months!

This is the closure returned by `h(5)`.

Staging

We now have the following binding:

$\left[\boxed{\begin{array}{l}\text{env}\\ \texttt{[5/x, n/z]}\\ \texttt{fn y => z+y}\end{array}} \Big/ \texttt{h5}\right]$

Evaluating `h5(2)`

```
==> [5/x, n/z, 2/y] z+y
==> n'
```
(for some integer `n'`)

quick!

Similarly, `h5(3)` will be very quick.

Factoring `hc(x)` out of the inner lambda has improved efficiency!

Staging

Summary:

`f (5,2)`	> 10 months
`f (5,3)`	> 10 months
`val g5 = g(5)`	fast
`g5 (2)`	> 10 months
`g5 (3)`	> 10 months
`val h5 = h(5)`	> 10 months
`h5 (2)`	fast
`h5 (3)`	fast

That's all for today. Have a good weekend!