Higher-Order Functions II: Staging

15-150

Lecture 11: October 2, 2025

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Let's revisit foldl and foldr on lists

Higher-order function: fold

Combining elements in a list, given a binary operation and base value:

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(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
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- `'b`: type of base value and of combined value

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Two implementations:

$\texttt{foldl f z } [x_1,\ldots,x_n] \cong \texttt{f}(x_n,\ldots\texttt{f}(x_3,\texttt{f}(x_2,\texttt{f}(x_1,\texttt{z}))))$

$\texttt{foldr f z } [x_1,\ldots,x_n] \cong \texttt{f}(x_1,\ldots\texttt{f}(x_{n-2},\texttt{f}(x_{n-1},\texttt{f}(x_n,\texttt{z}))))$

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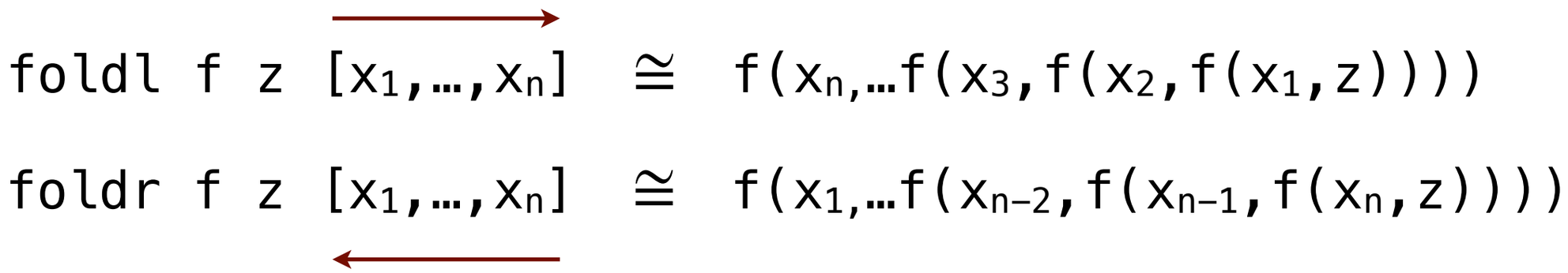

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$\overrightarrow{\texttt{foldl}}$ `f z` $[x_1,\ldots,x_n] \cong f(x_n,\ldots f(x_3,f(x_2,f(x_1,z))))$

`foldr f z` $[x_1,\ldots,x_n] \cong f(x_1,\ldots f(x_{n-2},f(x_{n-1},f(x_n,z))))$ $\overleftarrow{\texttt{foldr}}$

Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
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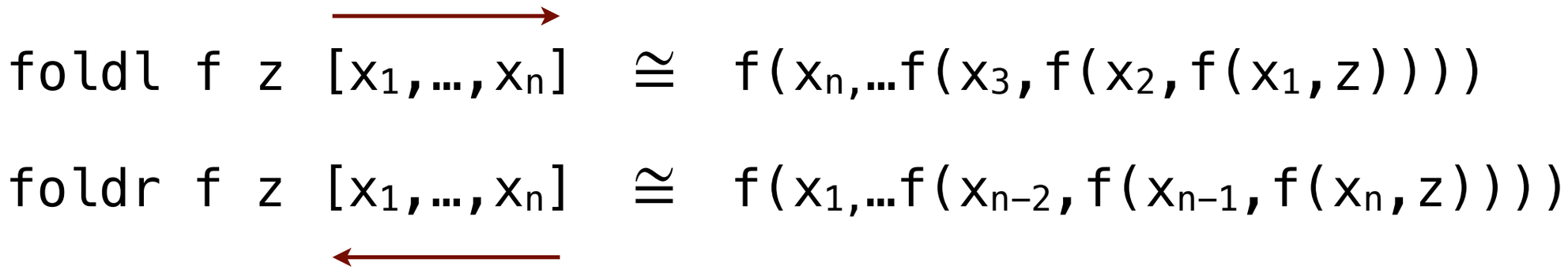

Examples:

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foldl (op -) 0 [1,2,3,4] ==> 2 (4-(3-(2-(1-0))))
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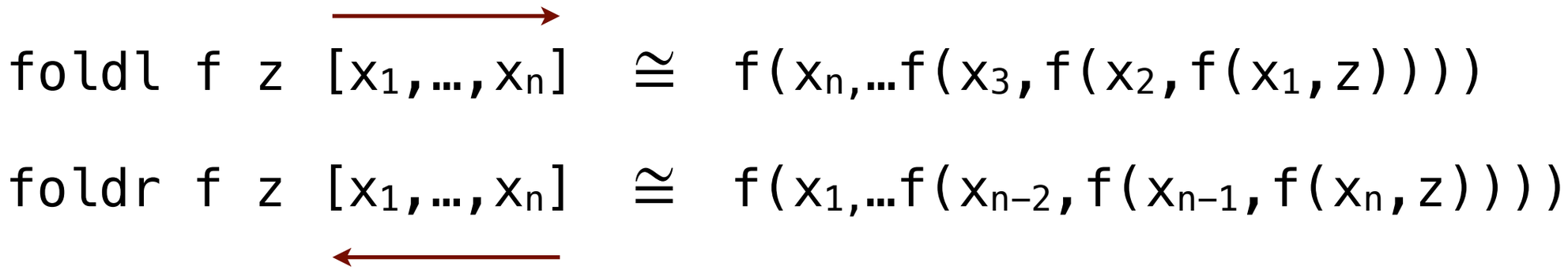

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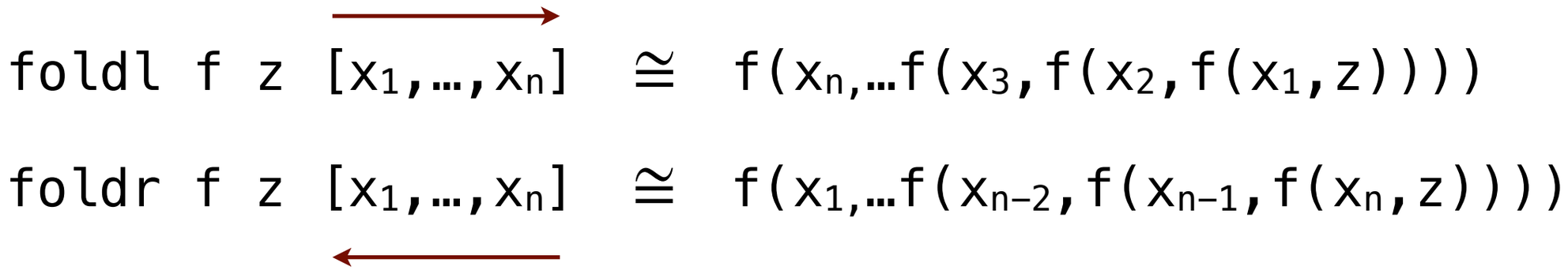

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Homework:

```
foldl (op ::) [] [1,2,3,4] ==> ?
foldr (op ::) [] [1,2,3,4] ==> ?
```

Map and fold over trees and other datatypes

Can we generalize map and fold?

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So fare we have considered map and fold exclusively for lists.

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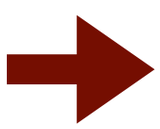

map: transform elements in a list, given a transformation function

Can we generalize map and fold?

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- map: transform elements in a list, given a transformation function
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Can we generalize map and fold to, for example, binary trees?

- Yes! Let's work it out.
- It may be helpful to visualize map and fold for lists diagrammatically first, to capture the underlying pattern.

The “pattern” underlying map

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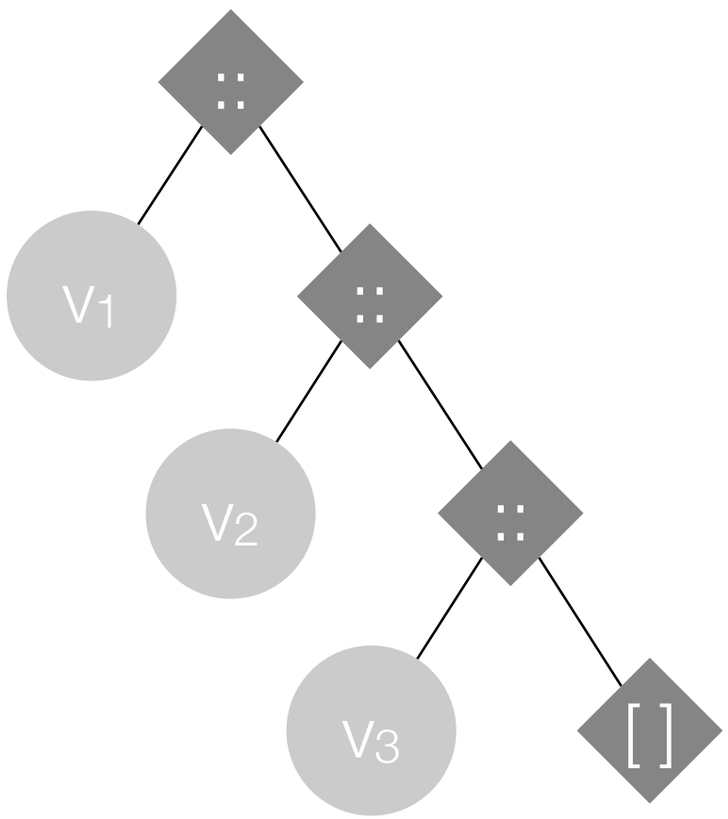

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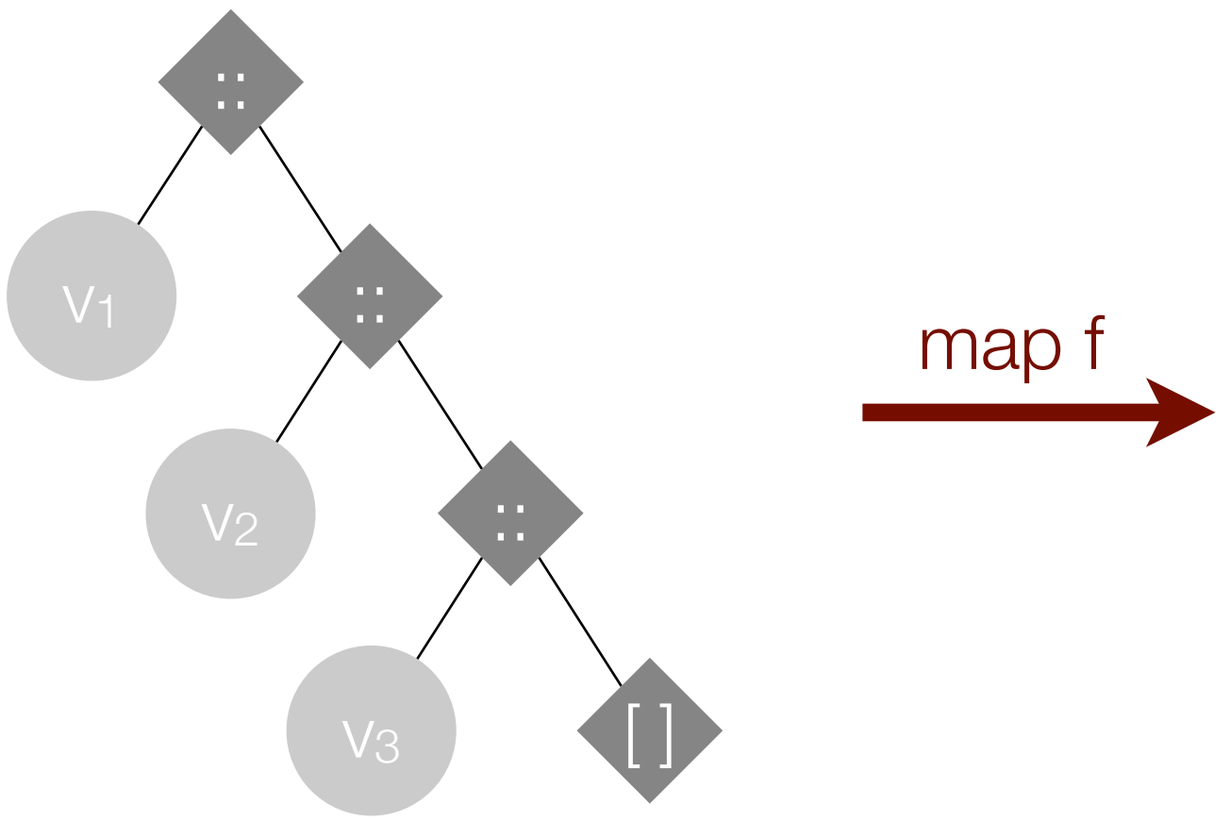

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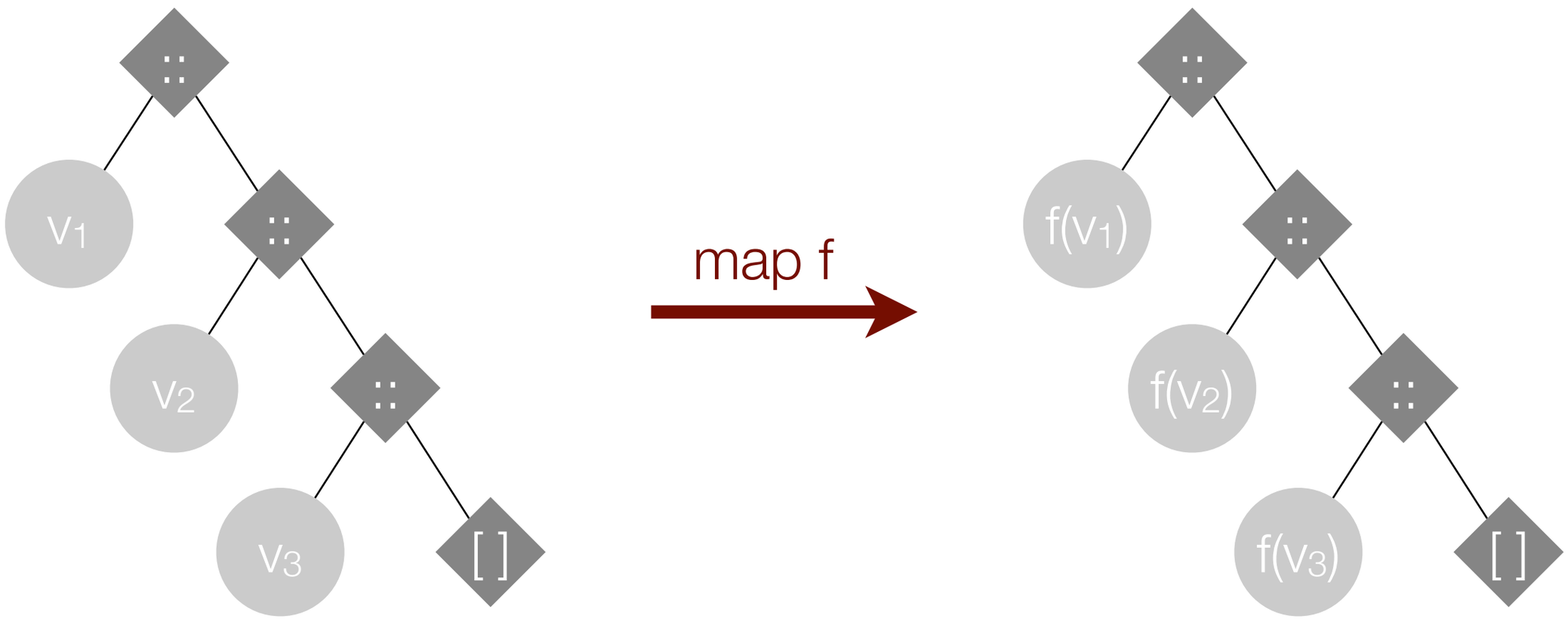

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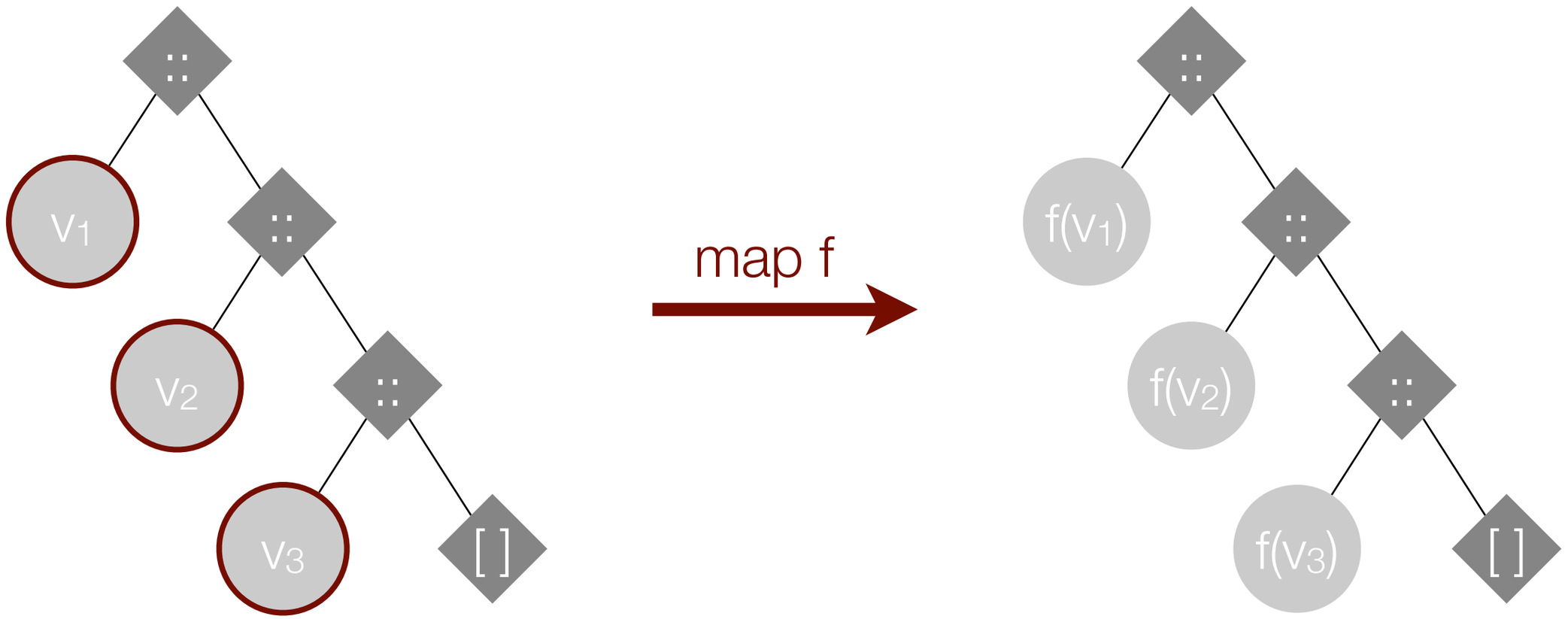

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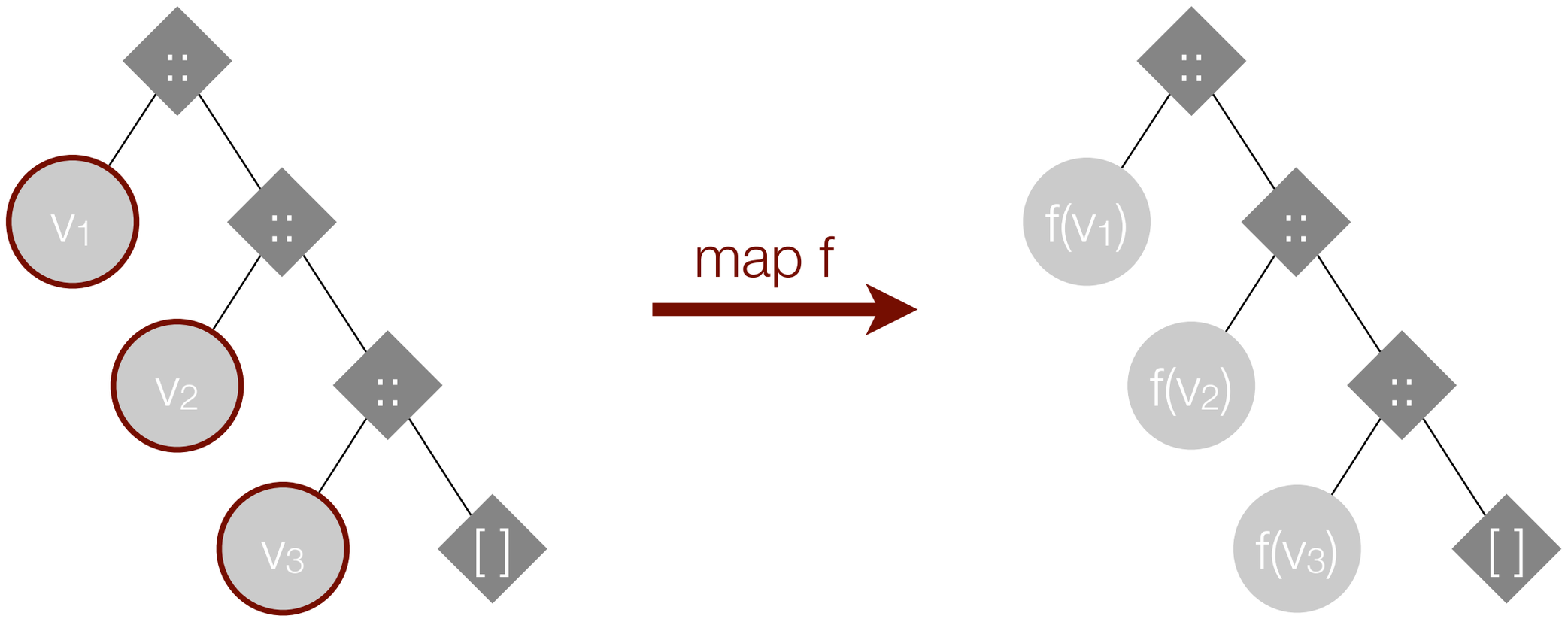

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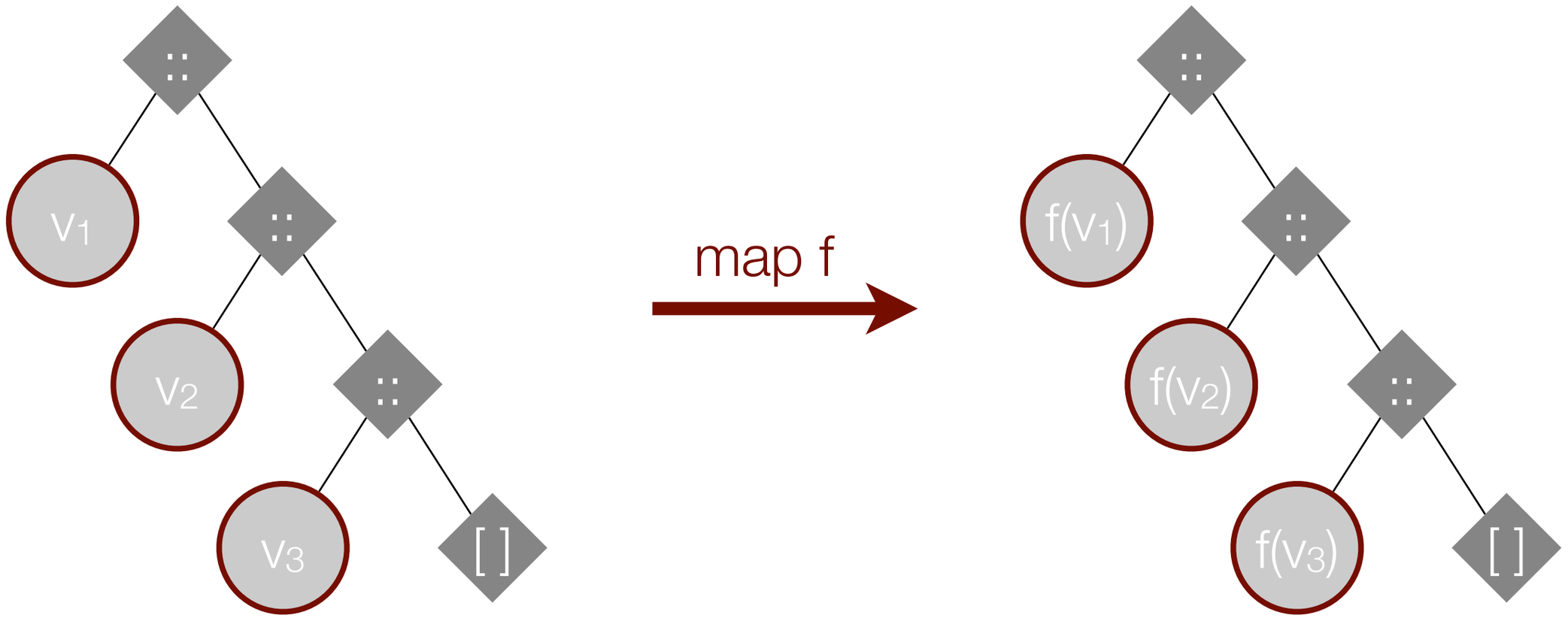

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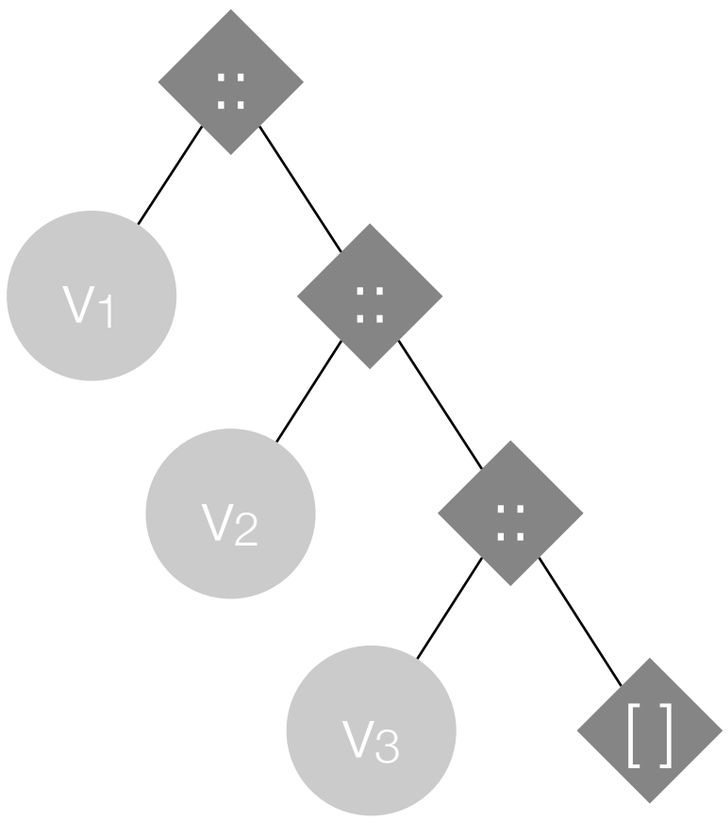

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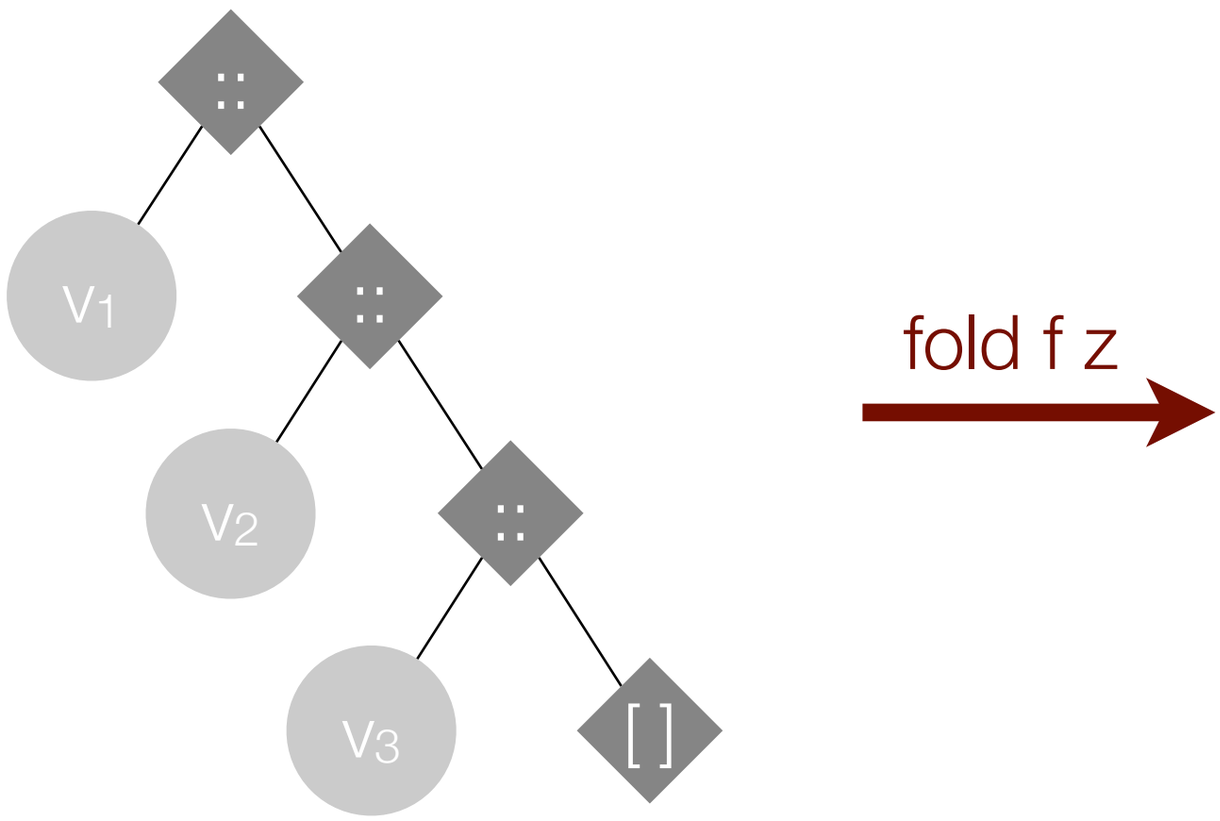

fold f z

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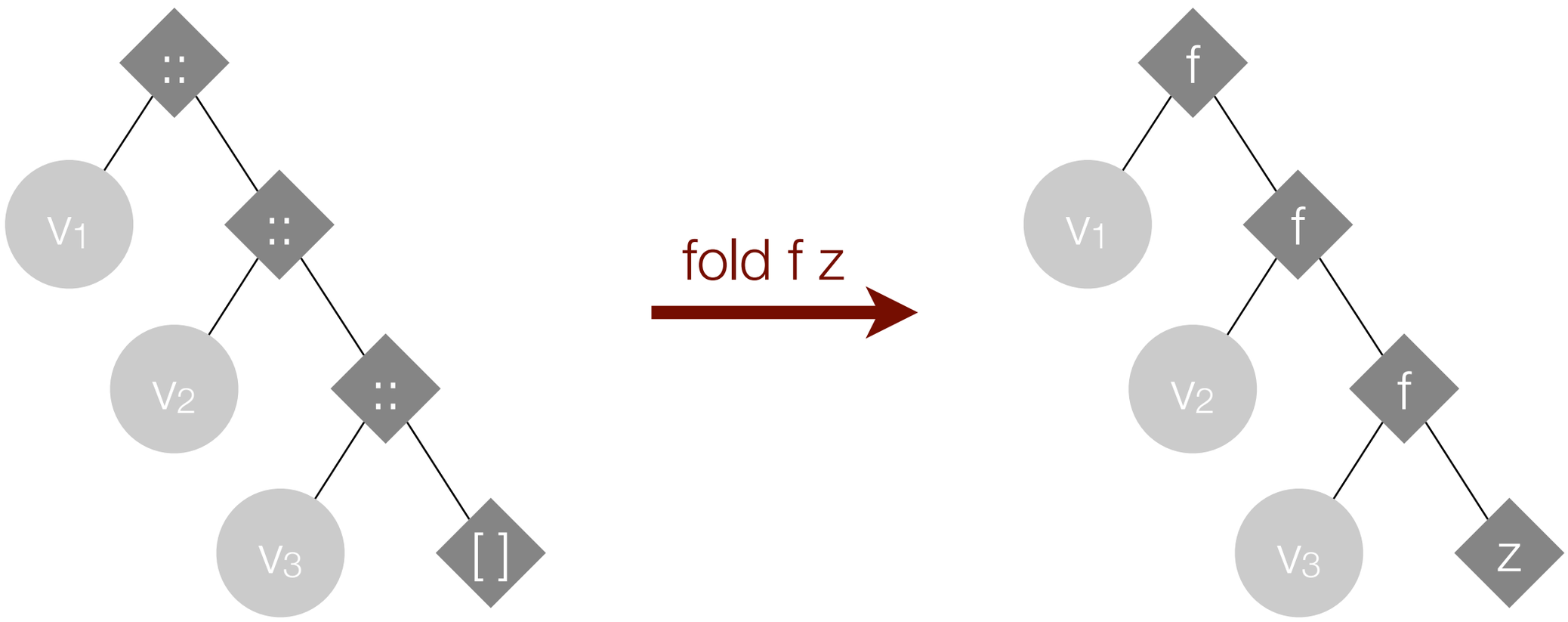

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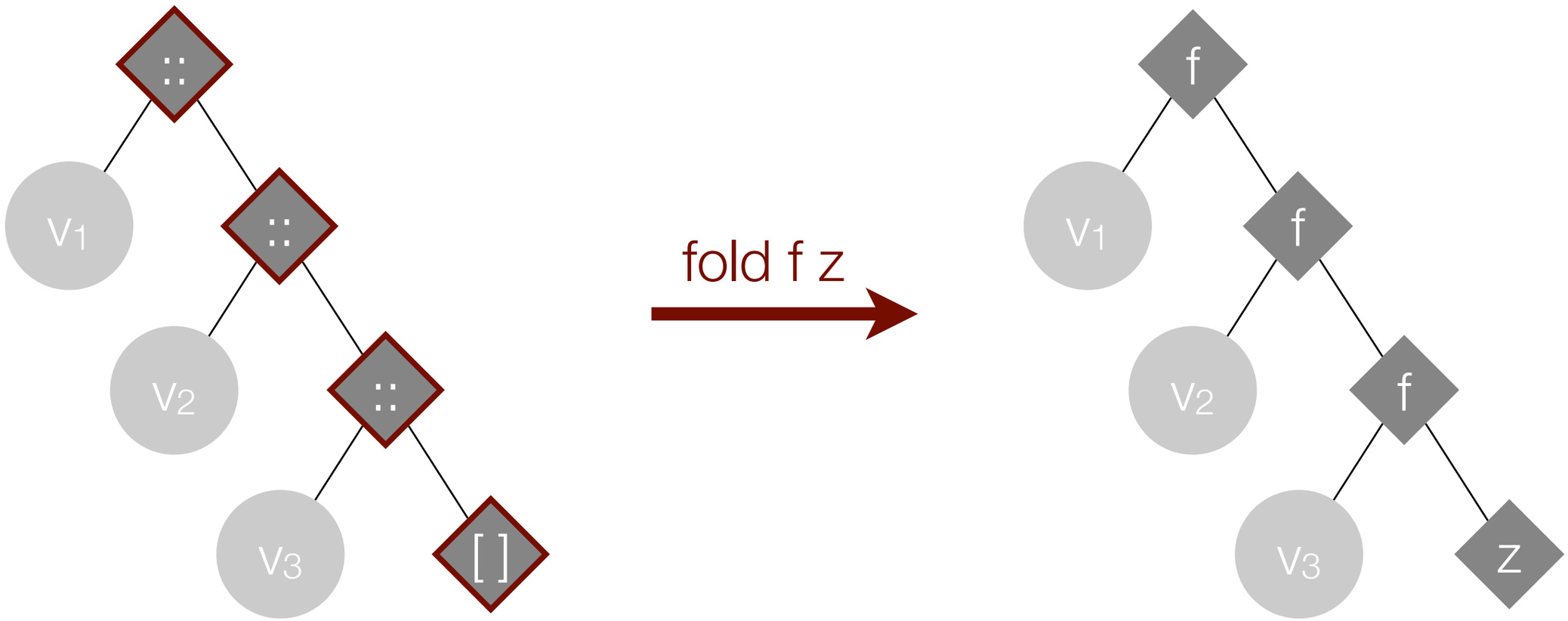

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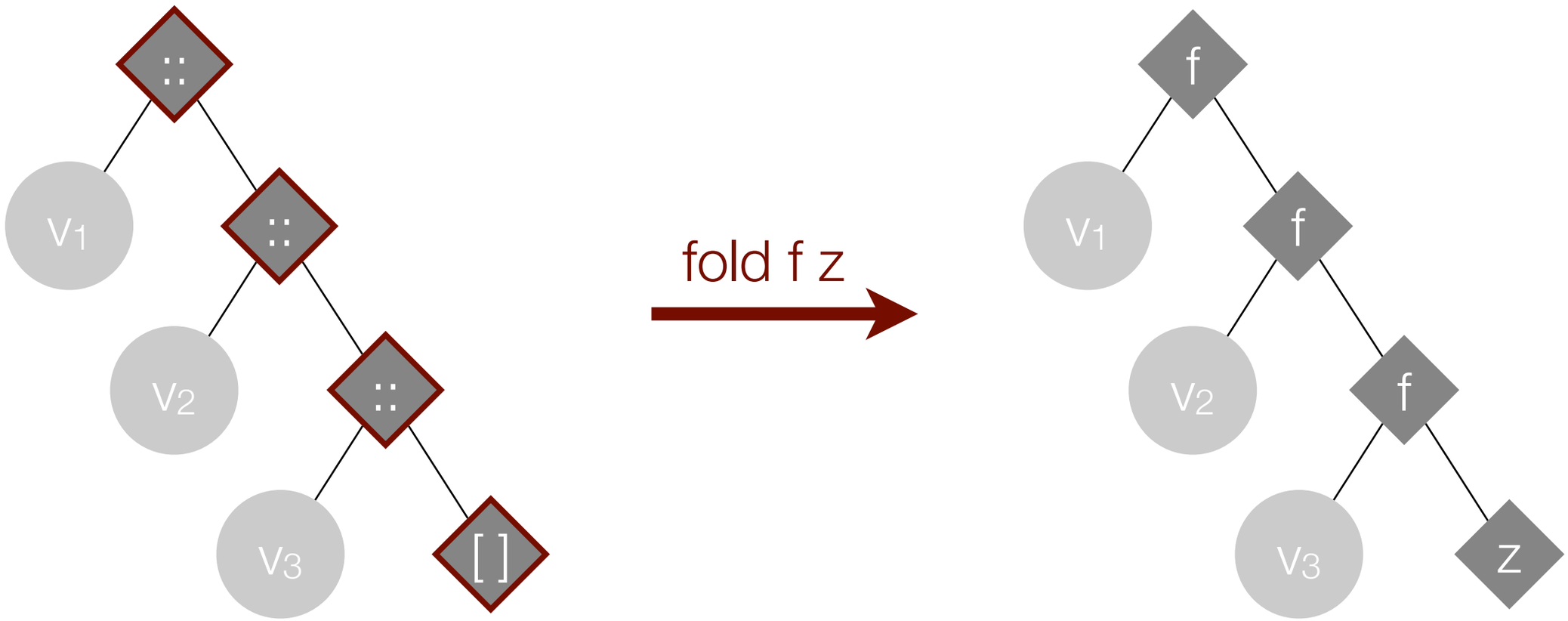

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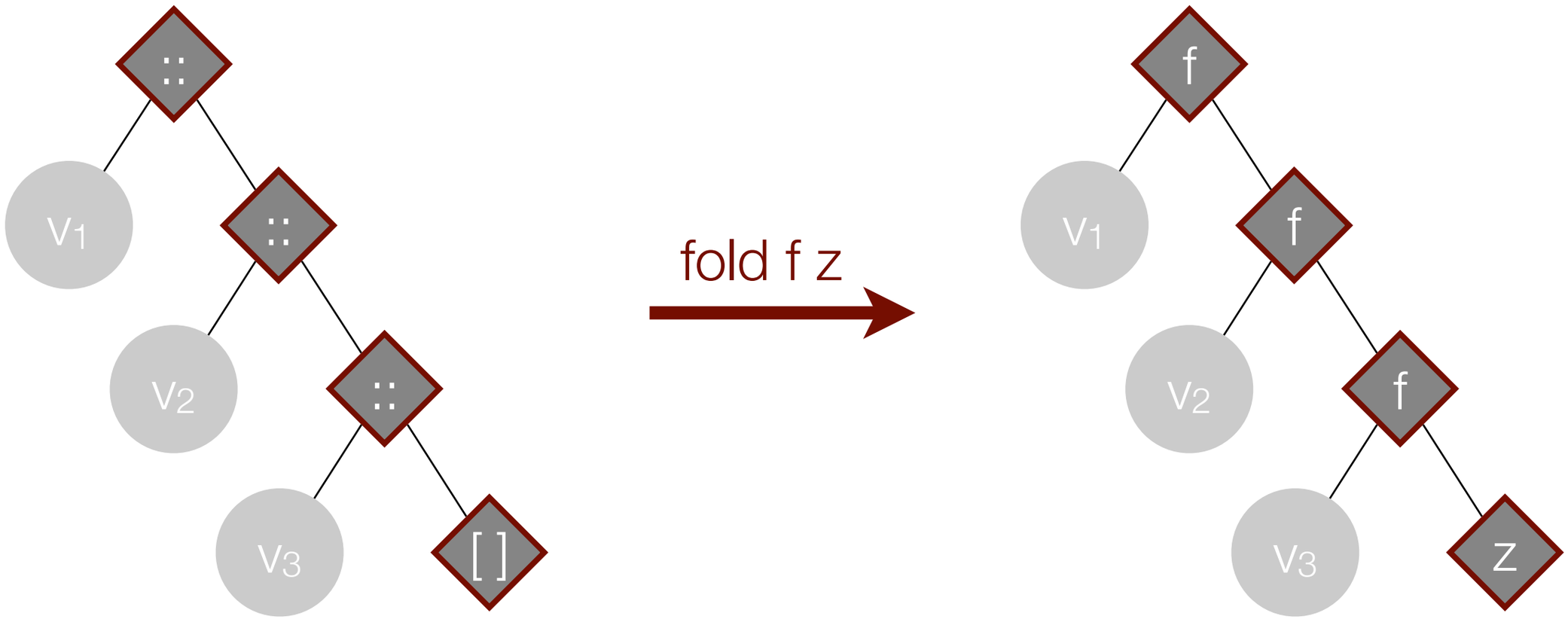

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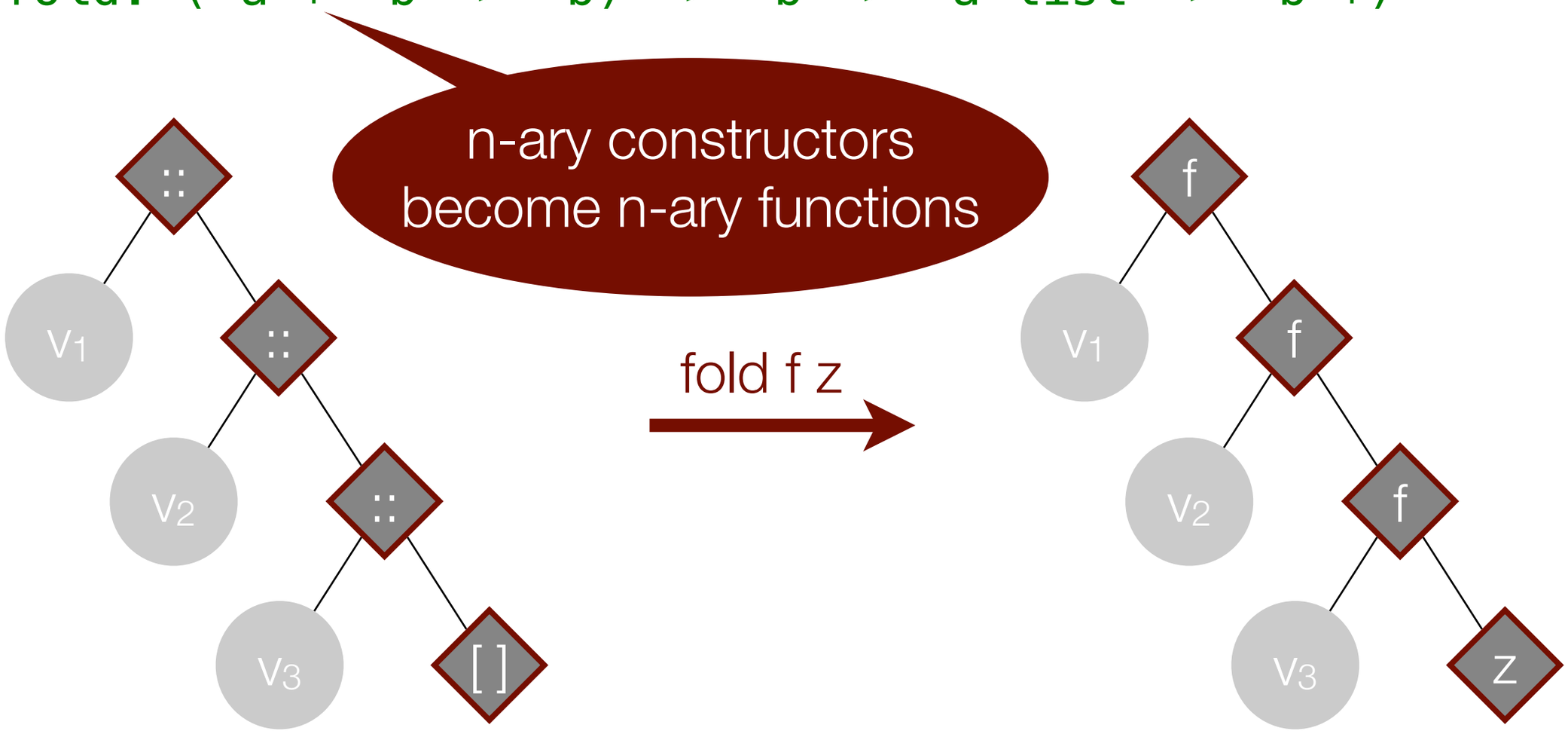

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Map and fold for binary trees

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What are the types of `stringify` and `treesum`?

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Staging

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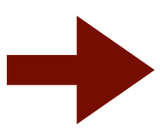

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- Improves efficiency when specialized function used many times.

Staging

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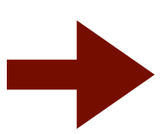

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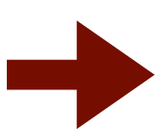

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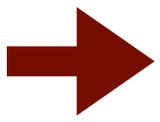

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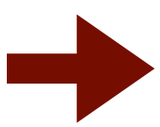

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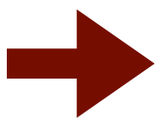

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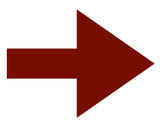

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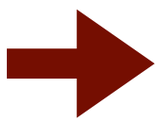

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No application, and thus no evaluation of body!

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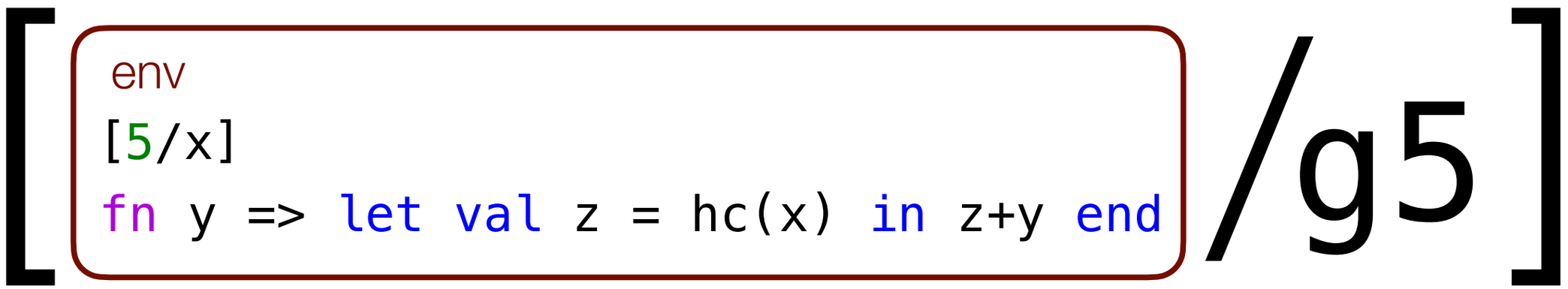

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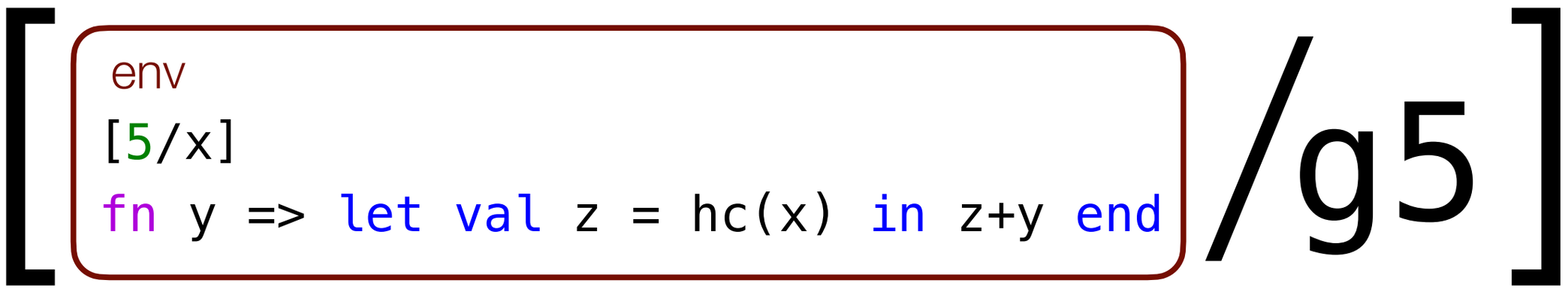

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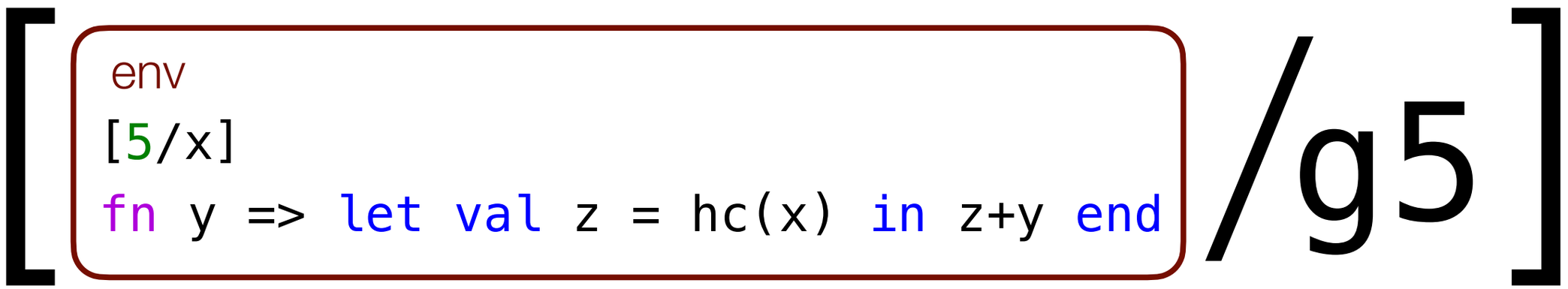

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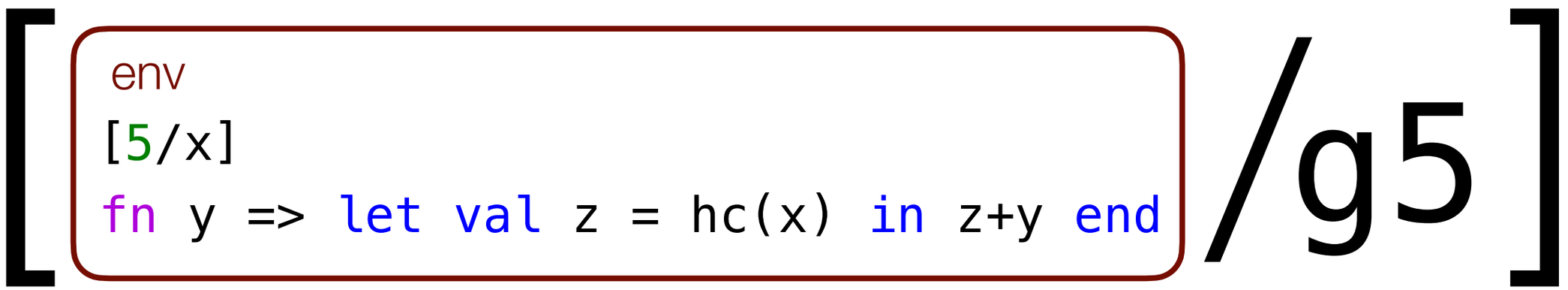

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10 months!

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Similarly, `g5(3)` will take 10 months.

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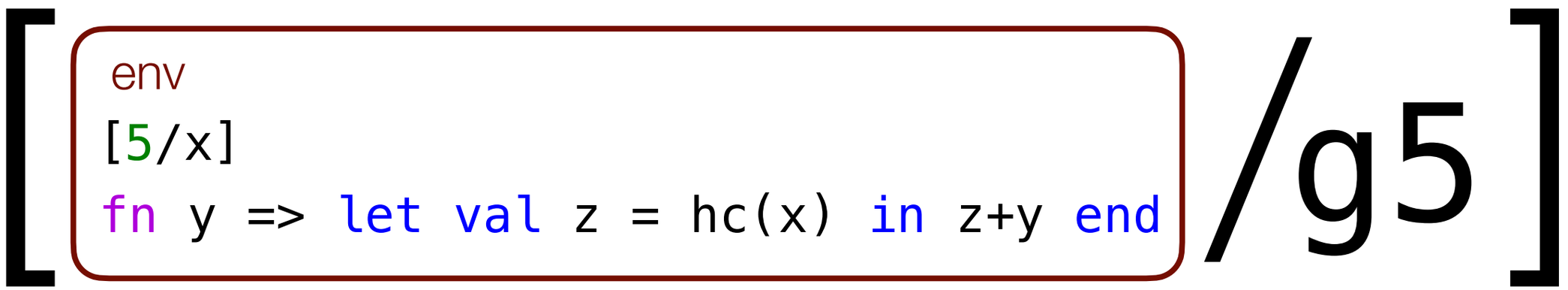

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Similarly, `g5(3)` will take 10 months.

→ Defining `g` in place of `f` has not yet helped!

Staging

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Let's move this computation here.

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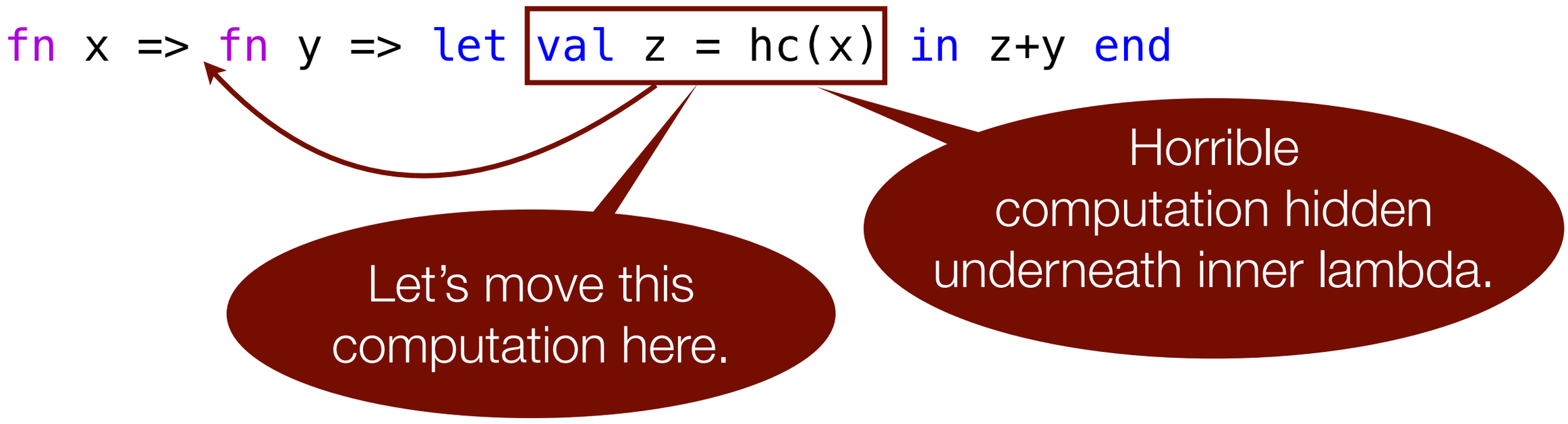

→ Move is valid because the computation does not depend on `y`.

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Let's move this computation here.

Horrible computation hidden underneath inner lambda.

→ Move is valid because the computation does not depend on **y**.

→ Such rearrangement of code — putting it in the "right spot" — we refer to as staging.

Staging

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fun h (x:int) : int -> int =
    let
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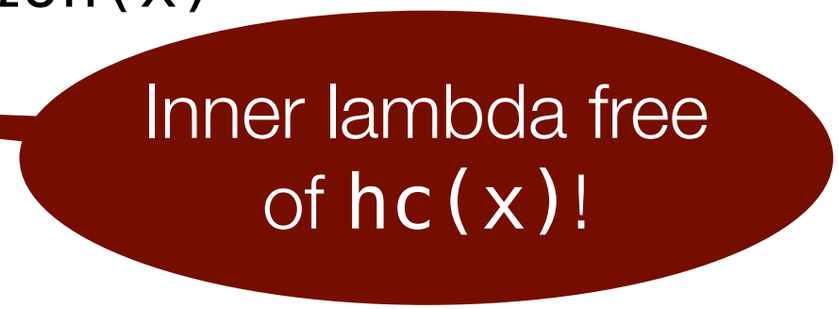

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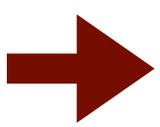

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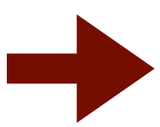

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This is the closure returned by **h(5)**.

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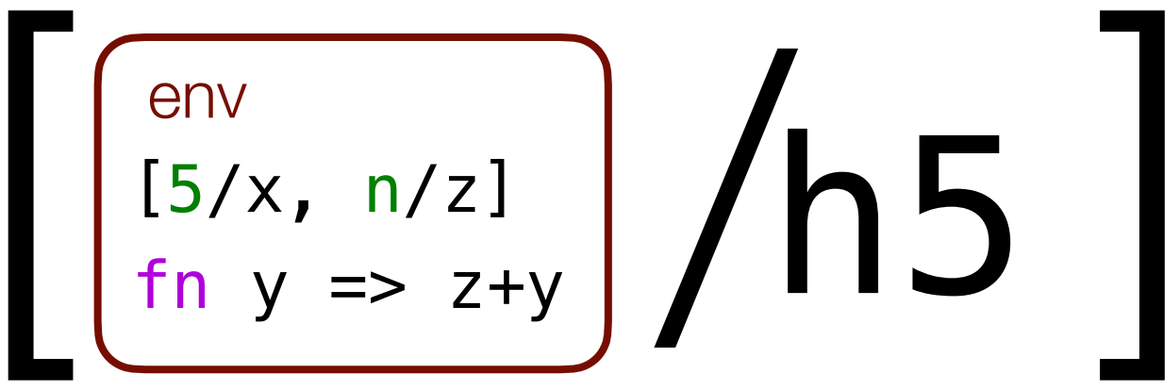

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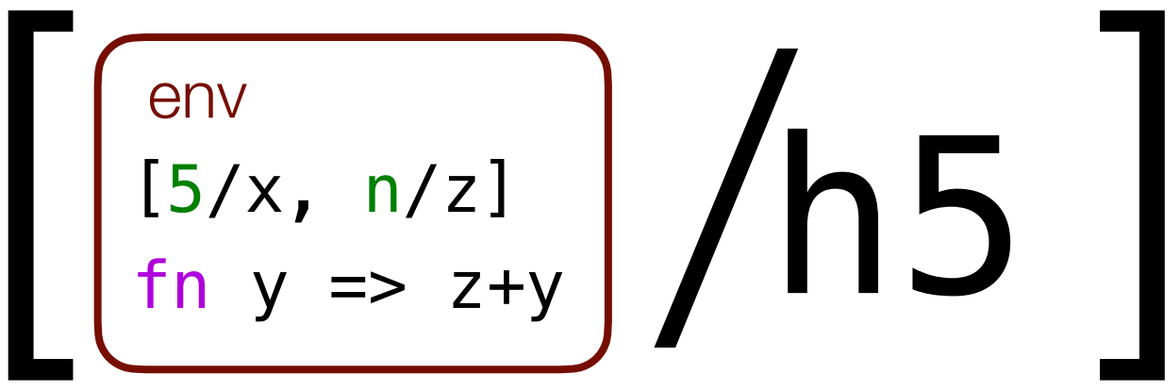

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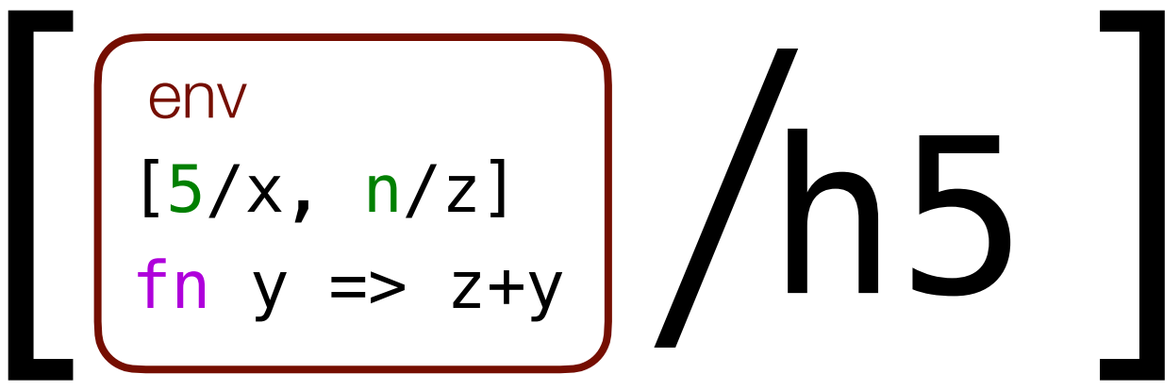

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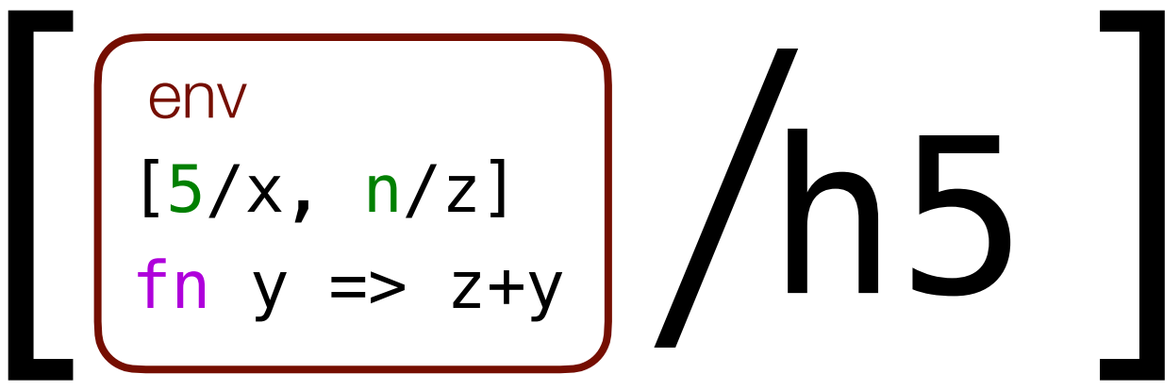

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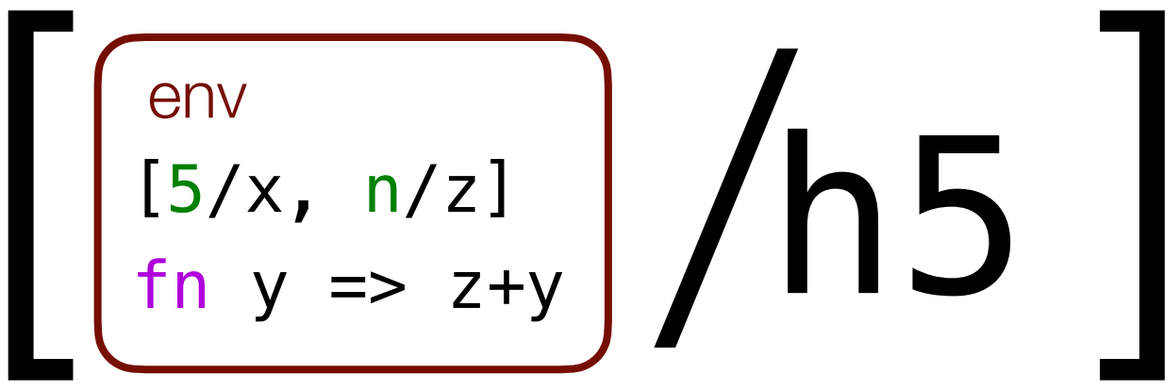

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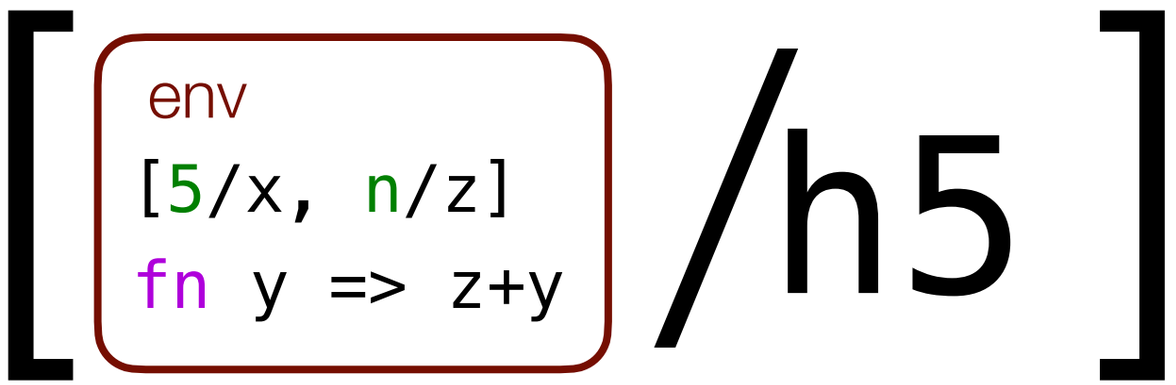

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Staging

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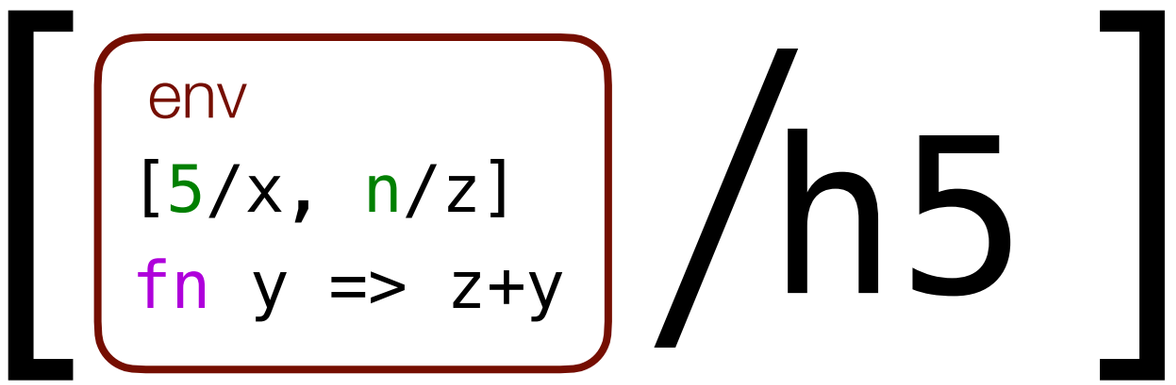

Evaluating `h5(2)`

```
==> [5/x, n/z, 2/y] z+y
==> n'    (for some integer n')
```

quick!

Similarly, `h5(3)` will be very quick.

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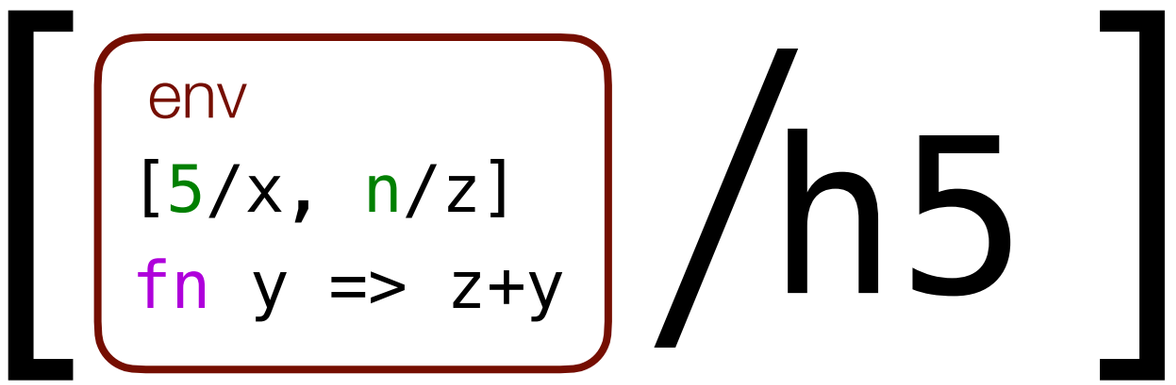

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==> [5/x, n/z, 2/y] z+y
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```

quick!

Similarly, `h5(3)` will be very quick.

→ Factoring `hc(x)` out of the inner lambda has improved efficiency!

Staging

Staging

Summary:

Staging

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`f (5,2)`	> 10 months
`f (5,3)`	> 10 months

Staging

Summary:

`f (5,2)`	> 10 months
`f (5,3)`	> 10 months
`val g5 = g(5)`	fast
`g5 (2)`	> 10 months
`g5 (3)`	> 10 months

Staging

Summary:

`f (5,2)`	> 10 months
`f (5,3)`	> 10 months
`val g5 = g(5)`	fast
`g5 (2)`	> 10 months
`g5 (3)`	> 10 months
`val h5 = h(5)`	> 10 months
`h5 (2)`	fast
`h5 (3)`	fast

That's all for today. Have a good weekend!