Higher-Order Functions I

15-150

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Recap

Recap

- Polymorphic types (type families) and instantiation.

Recap

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  - parameterized data types

Recap

Polymorphic types (type families) and instantiation.

parameterized data types

```
datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

```
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- Typing rules and most general type.

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

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datatype 'a tree =
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```

- Typing rules and most general type.
- Type guarantee:

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

```
datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

- Typing rules and most general type.
- Type guarantee:

$$\text{If } e : t \text{ and } e \hookrightarrow v \text{ then } v : t.$$

Today’s topic

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Function currying.

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- Higher-order functions:

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- Higher-order functions:
  - Functions whose argument and/or return types are functions.

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- Function currying.
- Higher-order functions:
  - Functions whose argument and/or return types are functions.
- More polymorphism.

Currying

Currying

Consider:

```
fun add (x,y) = x + y
```

Currying

Consider:

```
fun add (x,y) = x + y
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add:                    *)
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int          *)
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Recall, the function definition introduces a binding in the environment:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Recall, the function definition introduces a binding in the environment:

```
[ env
  fn (x,y) => x + y  /add]
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

```
fun plus x = fn y => x + y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
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Let's consider another function:

```
fun plus x = fn y => x + y
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

```
fun plus x = fn y => x + y
                           (* plus:                   *)
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let’s consider another function:

```
fun plus x = fn y => x + y
                          (* plus: int ->           *)
```

What’s its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let’s consider another function:

```
fun plus x = fn y => x + y
                              (* plus: int -> int ->     *)
```

What’s its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

```
fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

$$t_1 \texttt{ -> } t_2 \texttt{ -> } t_3$$

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

```
f x y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

`t1 -> t2 -> t3` means `t1 -> (t2 -> t3)`

Correspondingly, function application left-associates!

`f x y` means `(f x) y`

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Binding for plus:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Binding for plus:

$$\left[\boxed{\begin{array}{l}\text{env}\\ \texttt{fn x => fn y => x + y}\end{array}}\,\texttt{/plus}\right]$$

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

Function `plus` is the curried form of `add`.

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                             (* plus: int -> int -> int *)
```

- Function `plus` is the curried form of `add`.
- Currying: changing “`*`” to “`->`”.

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

- Function `plus` is the curried form of `add`.
- Currying: changing "*" to "->".
  - Named after Haskell Curry.

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                               (* plus: int -> int -> int *)
```

- Function `plus` is the curried form of `add`.
- Currying: changing “`*`” to “`->`”.
  - Named after Haskell Curry.
  - Useful for staging (next lecture!).

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                          (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                         (* plus: int -> int -> int *)
```

Let's evaluate:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Let's evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                                (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                          (* plus: int -> int -> int *)
```

Let's evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Let's evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
==> […] 7
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Let’s define yet another function:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                             (* plus: int -> int -> int *)
```

Let's define yet another function:

```
val incr3 = plus 3
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

Let's define yet another function:

```
val incr3 = plus 3
```

What's its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
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Let’s define yet another function:

```
val incr3 = plus 3      (* incr3:             *)
```

What’s its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Let’s define yet another function:

```
val incr3 = plus 3      (* incr3: int ->     *)
```

What’s its type?

Currying

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fun add (x,y) = x + y   (* add: int * int -> int *)

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Let’s define yet another function:

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val incr3 = plus 3      (* incr3: int -> int *)
```

What’s its type?

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Let’s define yet another function:

```
val incr3 = plus 3         (* incr3: int -> int *)
```

What’s its type?

partial application

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3         (* incr3: int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Binding for incr3:

Currying

Consider:

```
fun add (x,y) = x + y    (* add: int * int -> int *)

fun plus x = fn y => x + y
                         (* plus: int -> int -> int *)

val incr3 = plus 3       (* incr3: int -> int *)
```

Binding for incr3:

```
[ env
  [3/x]
  fn y => x + y  / incr3 ]
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3         (* incr3: int -> int *)
```

Let’s evaluate:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let’s evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                         (* plus: int -> int -> int *)

val incr3 = plus 3       (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

lambda

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

It's a value!

lambda

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let’s evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                         (* plus: int -> int -> int *)

val incr3 = plus 3       (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

closure

Currying

Consider:

```
fun plus x = fn y => x + y
                                   (* plus: int -> int -> int *)

val incr3 = plus 3                 (* incr3: int -> int *)
```

Currying

Consider:

```
fun plus x = fn y => x + y
                                (* plus: int -> int -> int *)

val incr3 = plus 3              (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                                (* plus: int -> int -> int *)

val incr3 = plus 3              (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                                  (* plus: int -> int -> int *)

val incr3 = plus 3                (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
```

Currying

Consider:

```
fun plus x = fn y => x + y
                             (* plus: int -> int -> int *)

val incr3 = plus 3           (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
```

Currying

Consider:

```
fun plus x = fn y => x + y
                                  (* plus: int -> int -> int *)

val incr3 = plus 3          (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                                  (* plus: int -> int -> int *)

val incr3 = plus 3           (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
==> […] 7
```

Currying: syntactic sugar

Currying: syntactic sugar

A maybe more convenient notation

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

space!

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

which is itself syntactic sugar for

```
val rec plus = fn x => fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

which is itself syntactic sugar for

```
val rec plus = fn x => fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

which is itself syntactic sugar for

```
val rec plus = fn x => fn y => x + y
```

`fun` definitions are recursive!

Currying: some more examples

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f:                               *)
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

fun g x y z = (x + y) div z
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

fun g x y z = (x + y) div z
(* g:                          *)
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

fun g x y z = (x + y) div z
(* g: int -> int -> int -> int *)
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

fun g x y z = (x + y) div z
(* g: int -> int -> int -> int *)

fun h (x, y) z = (x + y) div z
```

Currying: some more examples

```
fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

fun g x y z = (x + y) div z
(* g: int -> int -> int -> int *)

fun h (x, y) z = (x + y) div z
(* h:                          *)
```

Currying: some more examples

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fun f (x, y, z) = (x + y) div z
(* f: int * int * int -> int *)

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(* g: int -> int -> int -> int *)

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```

Higher-order functions

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                              (* int -> bool *)
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type?

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Filter in action:

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val keepevens = filter (fn n => n mod 2 = 0)

val [2,~6, 10] = keepevens [1,2,~5,~6,11,10,13]
```

Higher-order function: composition

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Function composition, abstractly:

Higher-order function: composition

Function composition, abstractly: $(f \circ g)\,(x) = f\,(g\,(x))$

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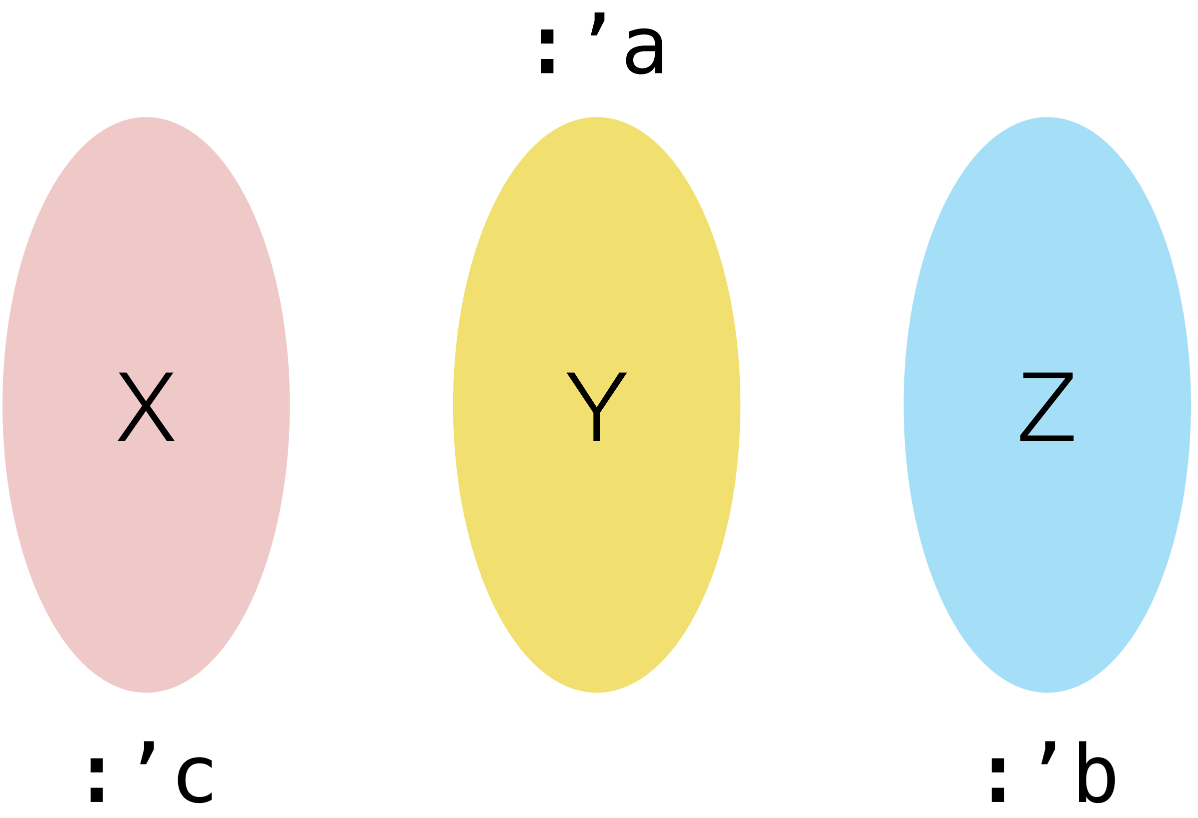

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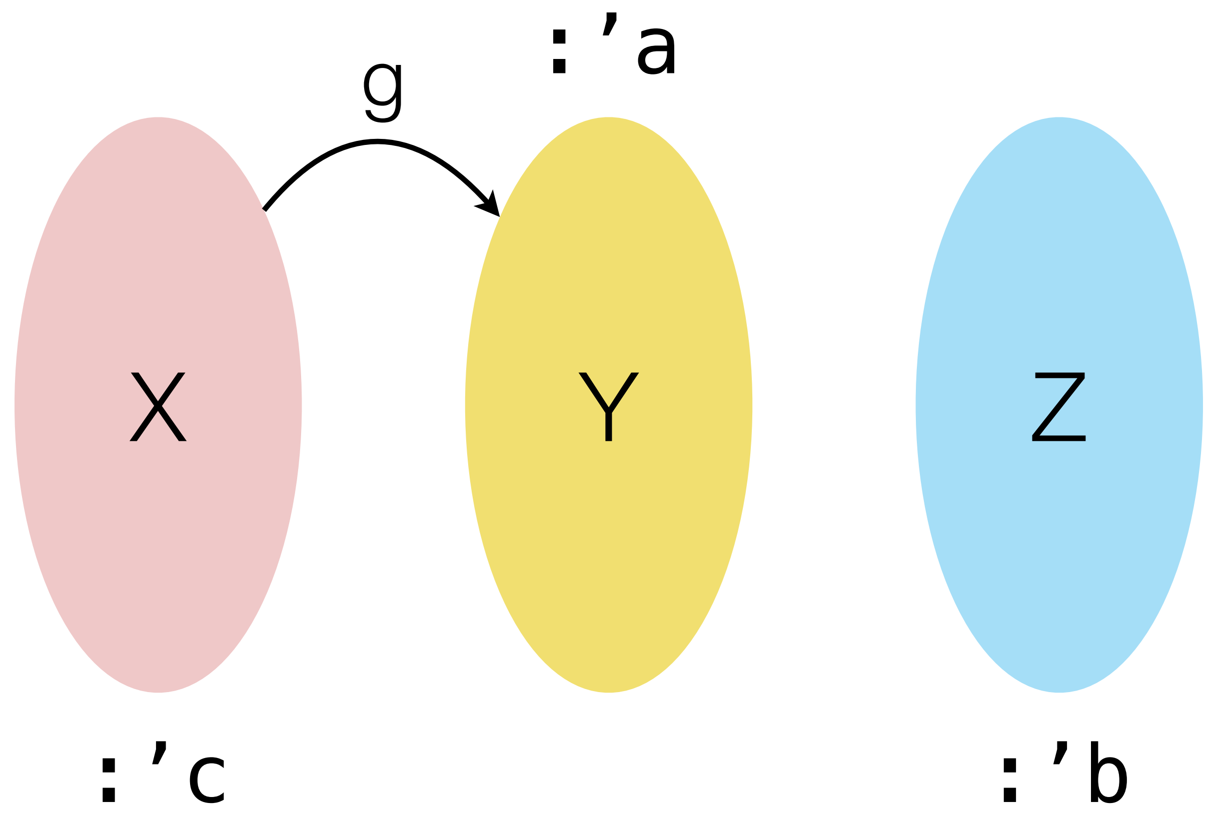

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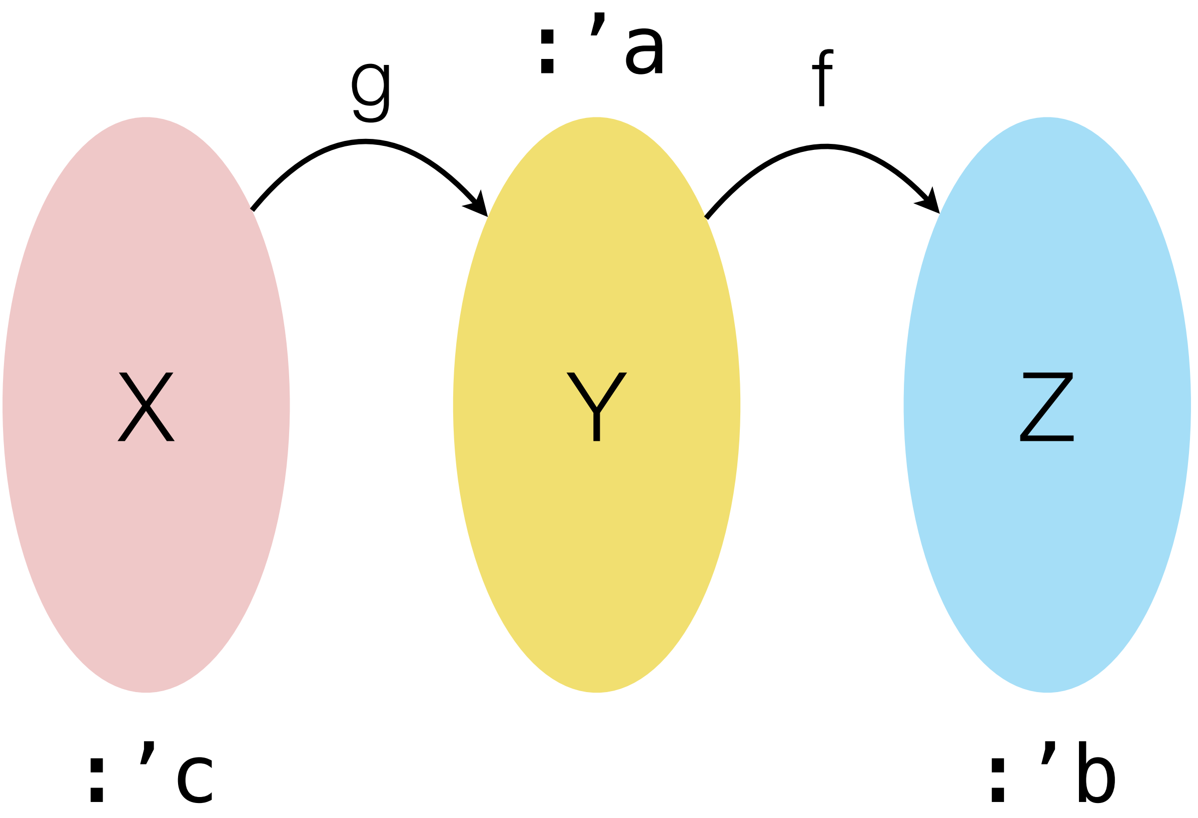
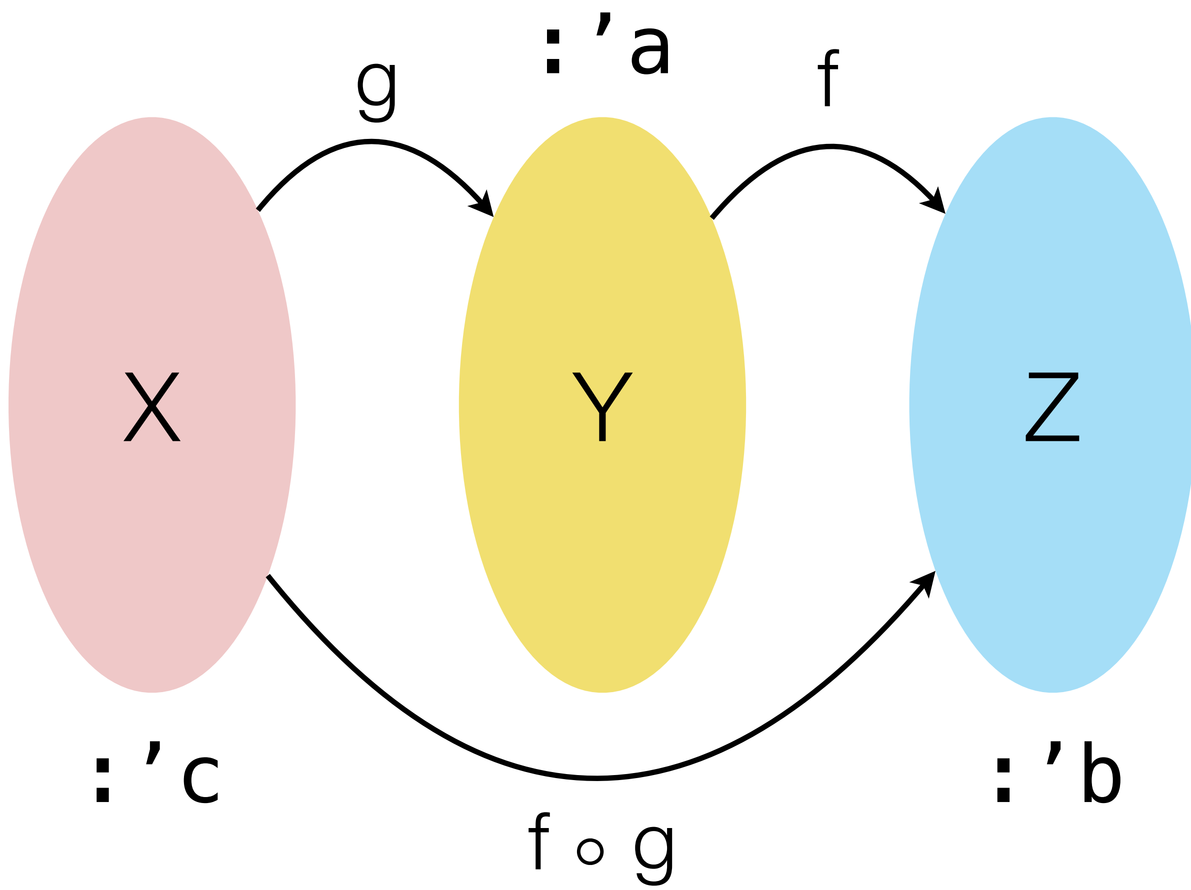


Higher-order function: composition

Function composition, abstractly: $(f \circ g)(x) = f(g(x))$



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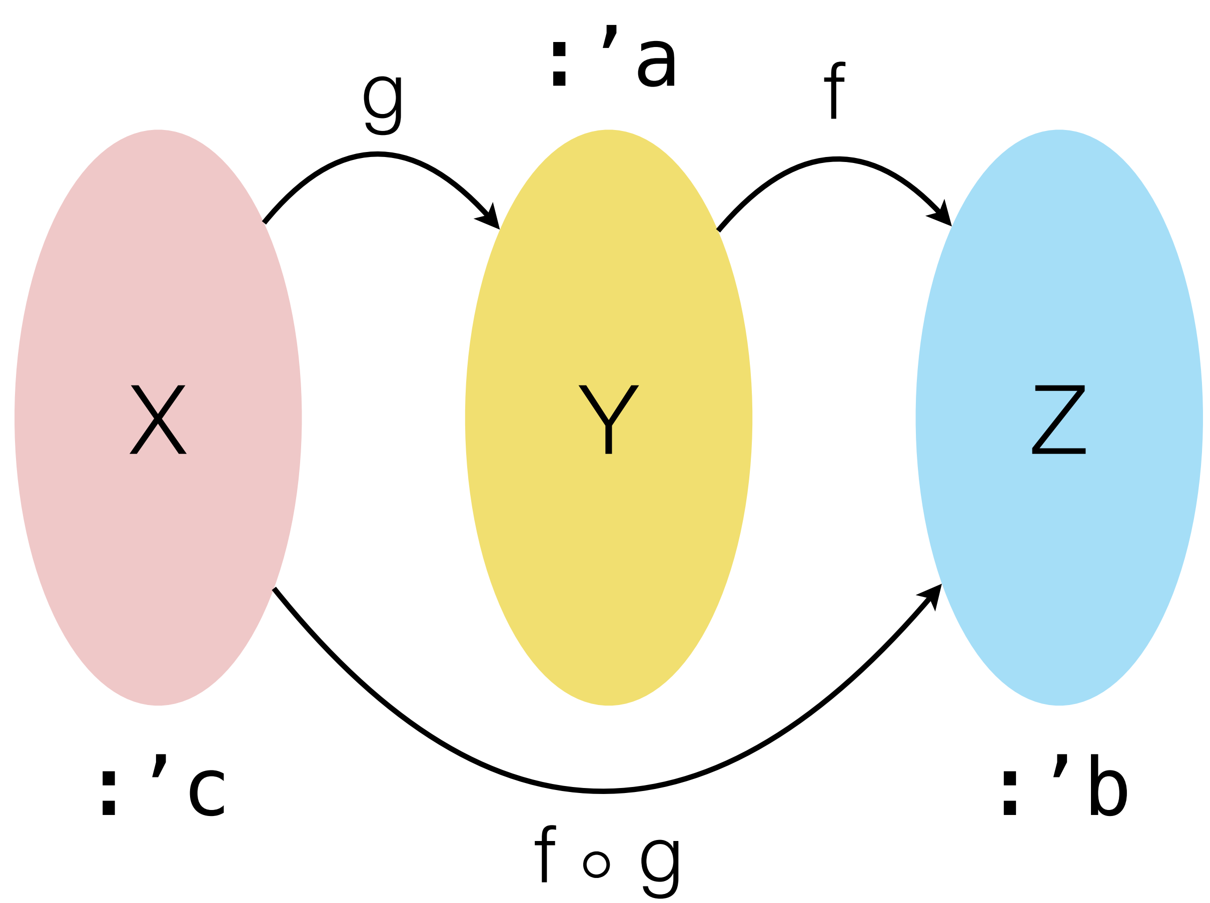

$\circ$ is a higher-order function, i.e., a combinator, expecting two functions as arguments and returning a function.

Higher-order function: composition

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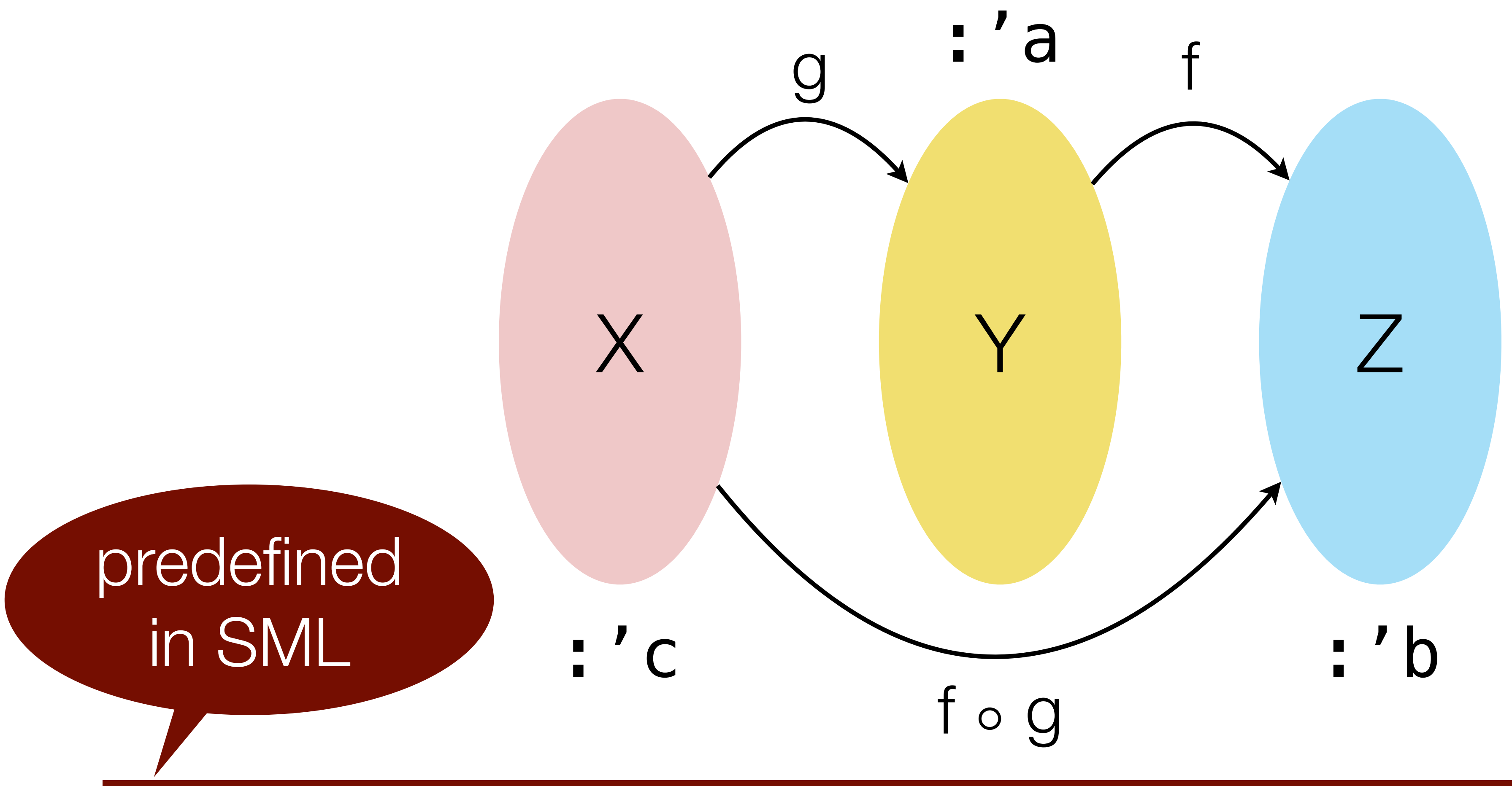

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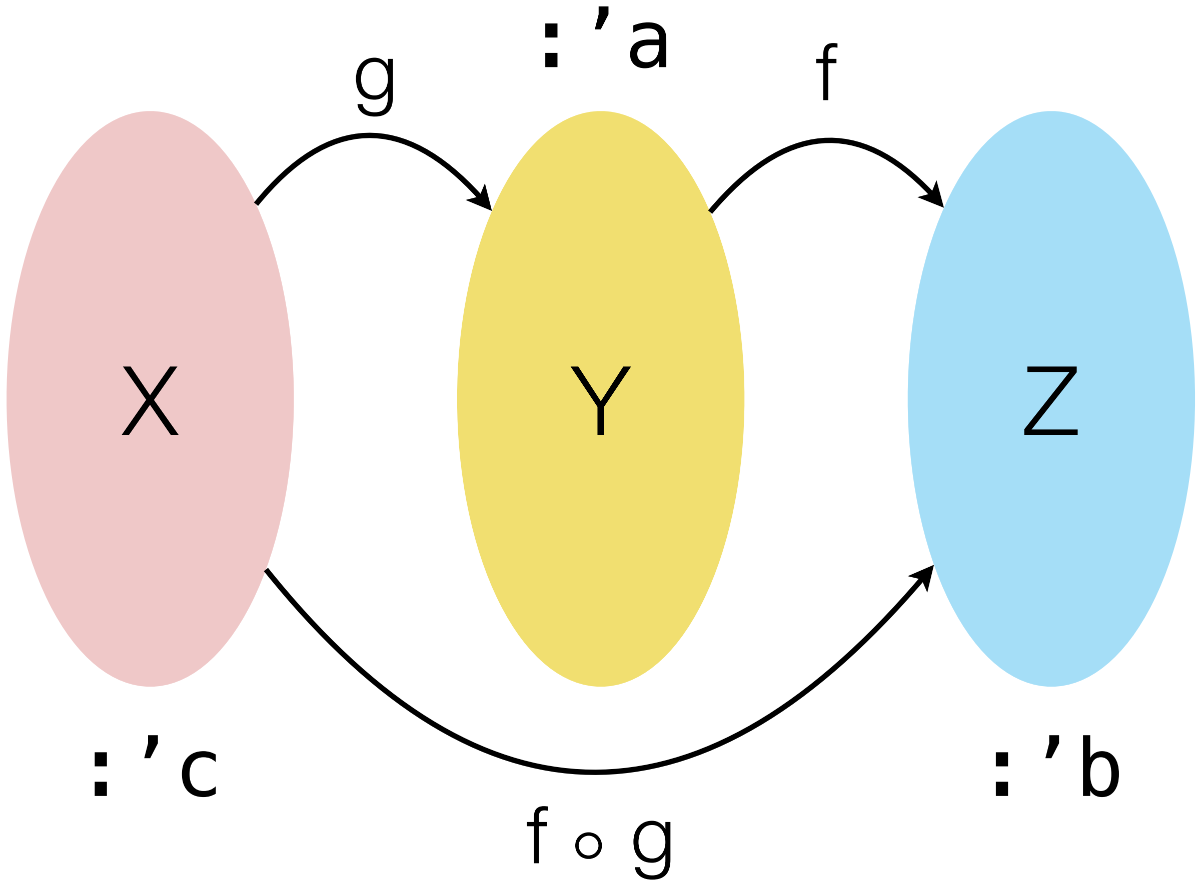
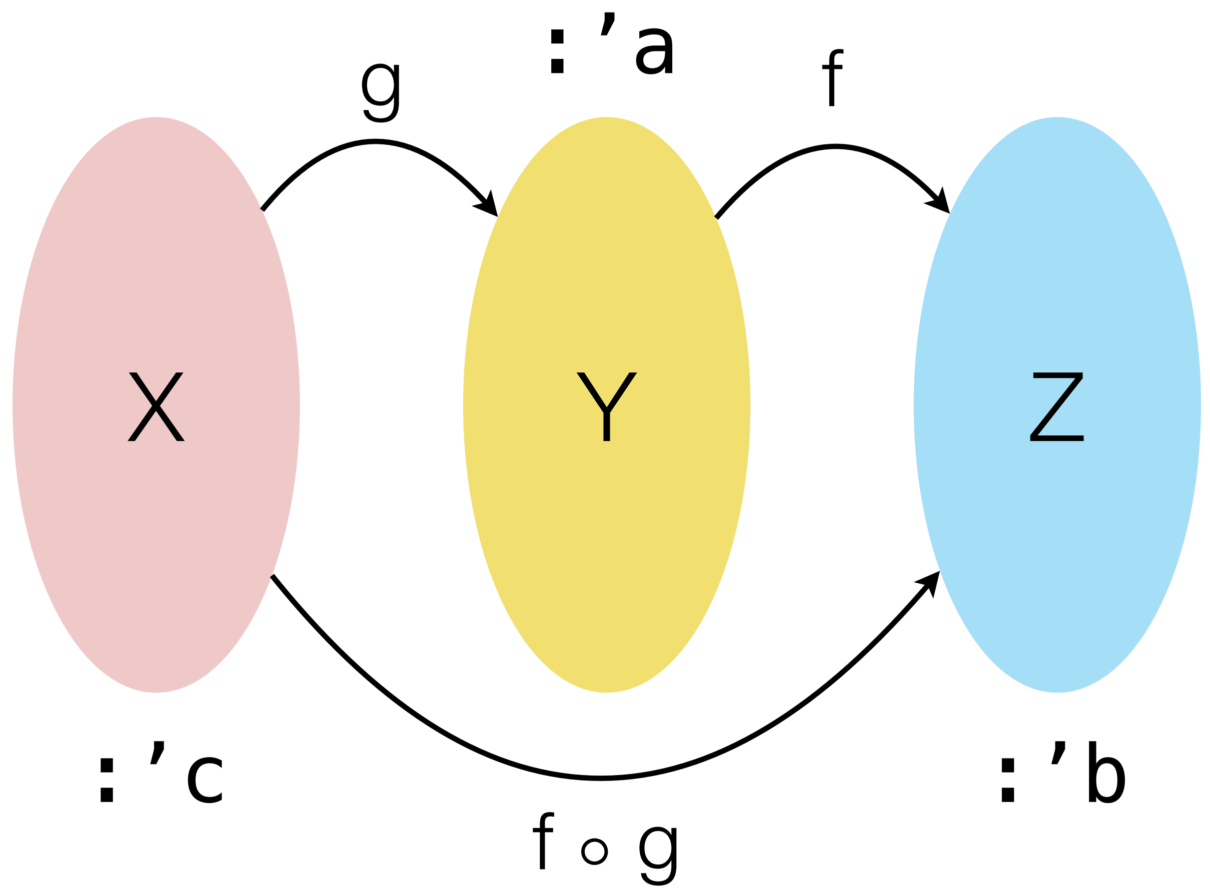
Higher-order function: composition



Higher-order function: composition

```
infix o
fun f o g = fn x => f(g(x))
```



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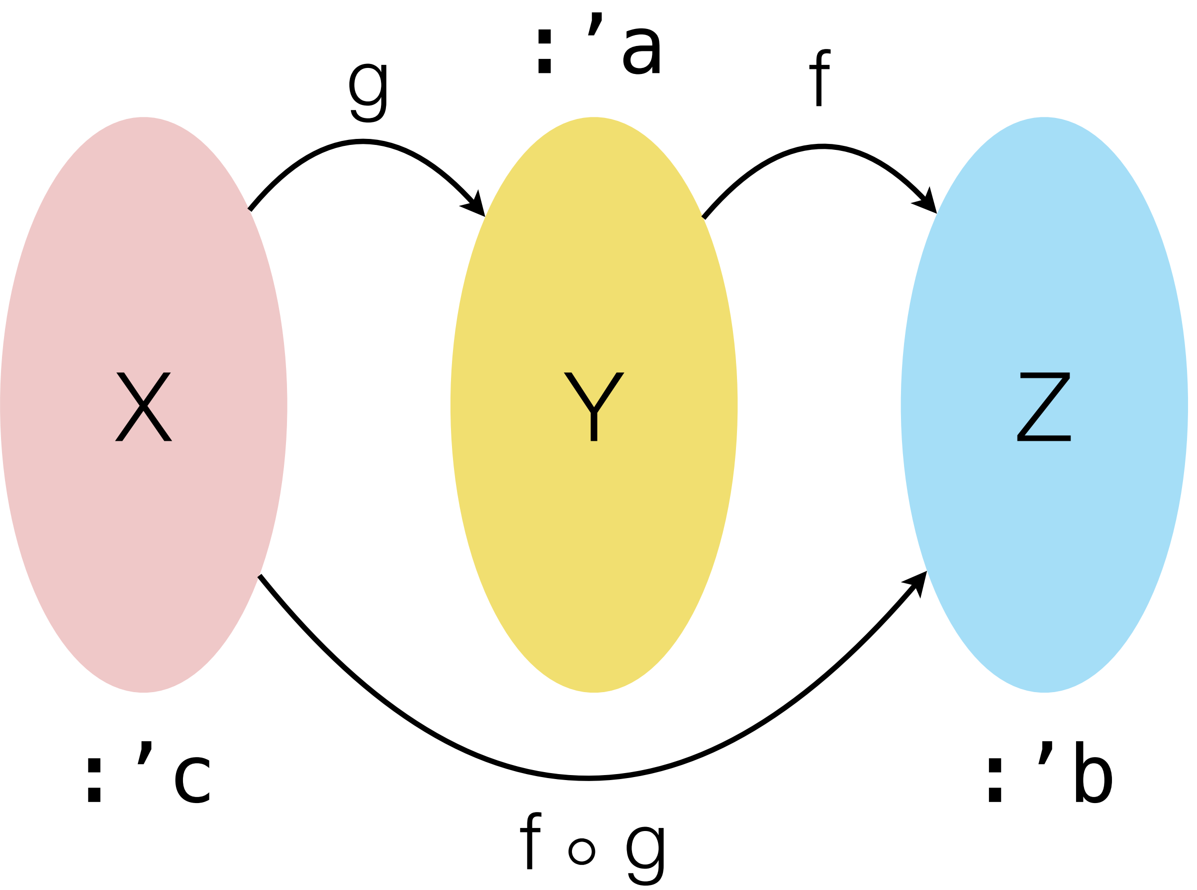
What is its type?



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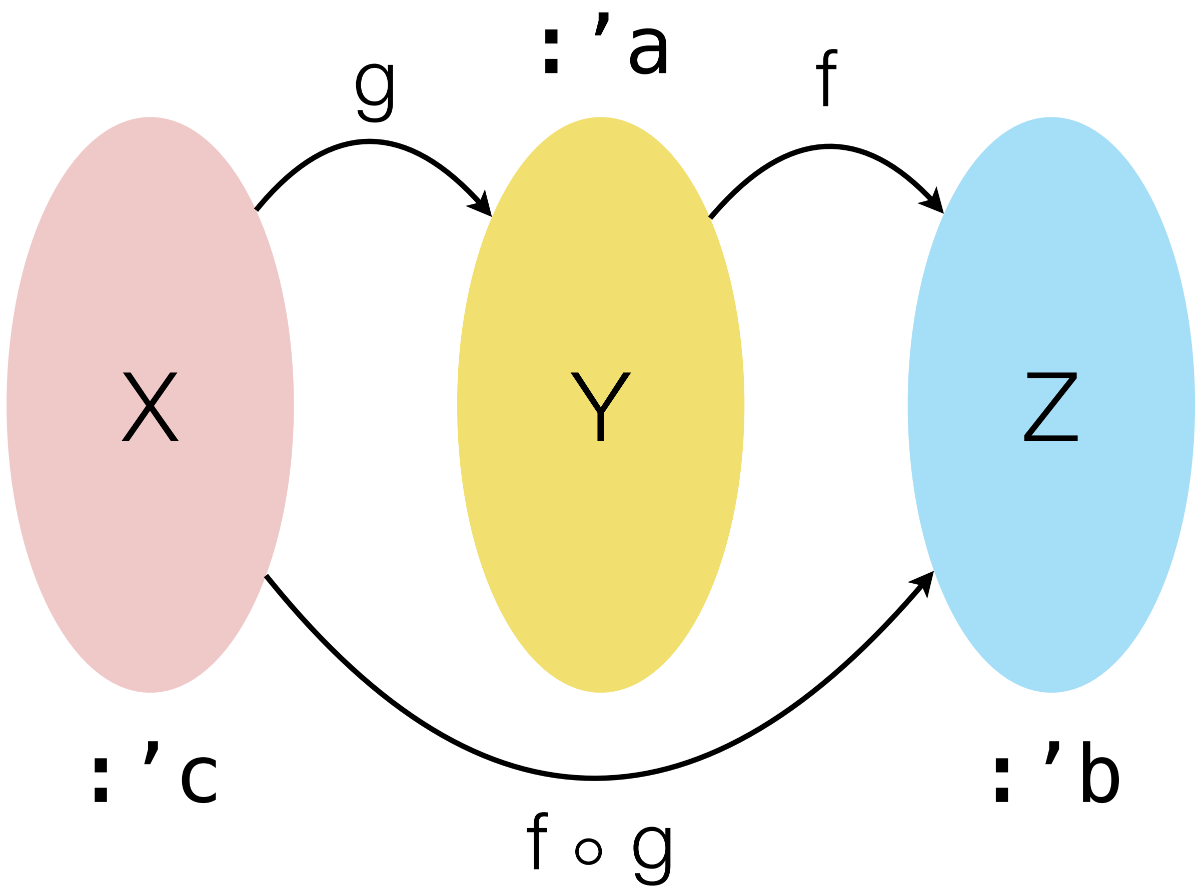


```
(* (op o):                                        *)
```

Higher-order function: composition

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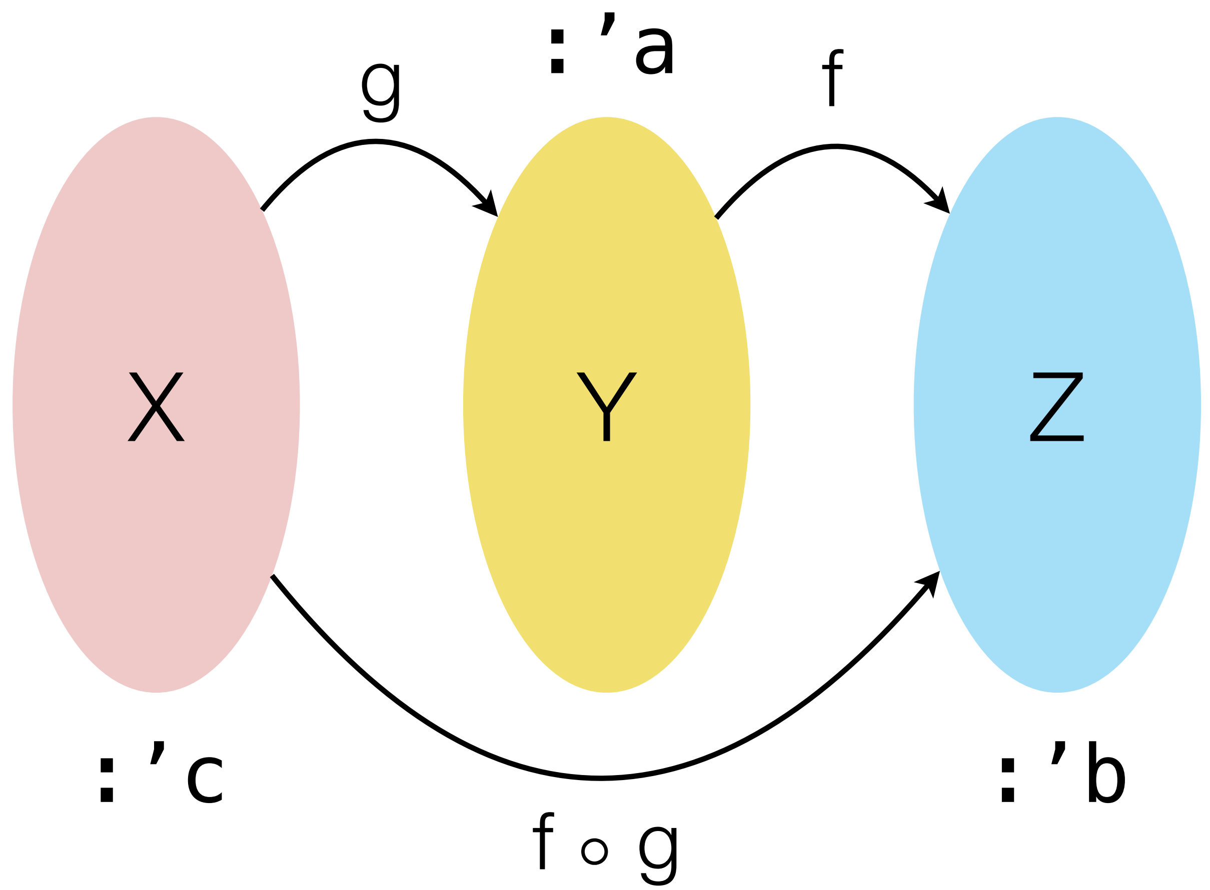


```
(* (op o): ('a -> 'b) *                                    *)
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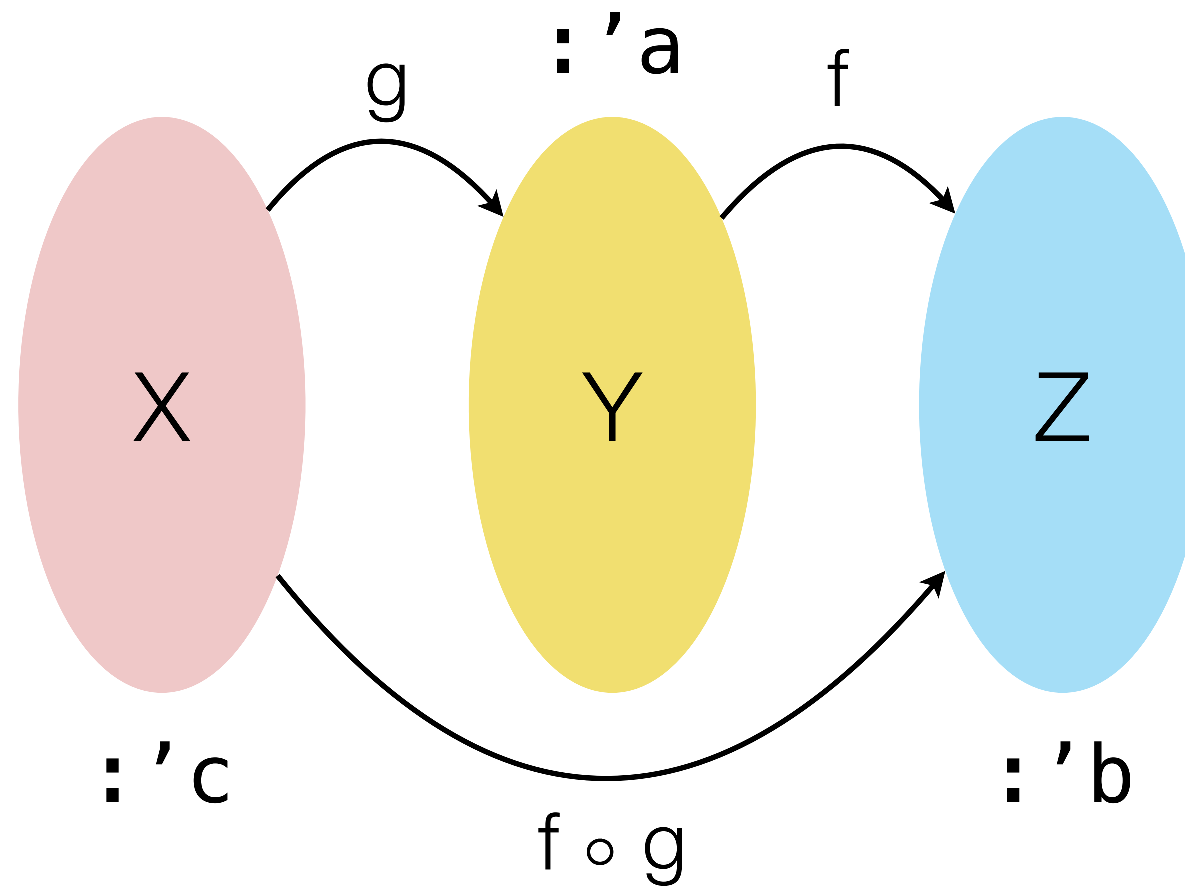


```
(* (op o): ('a -> 'b) * ('c -> 'a)                *)
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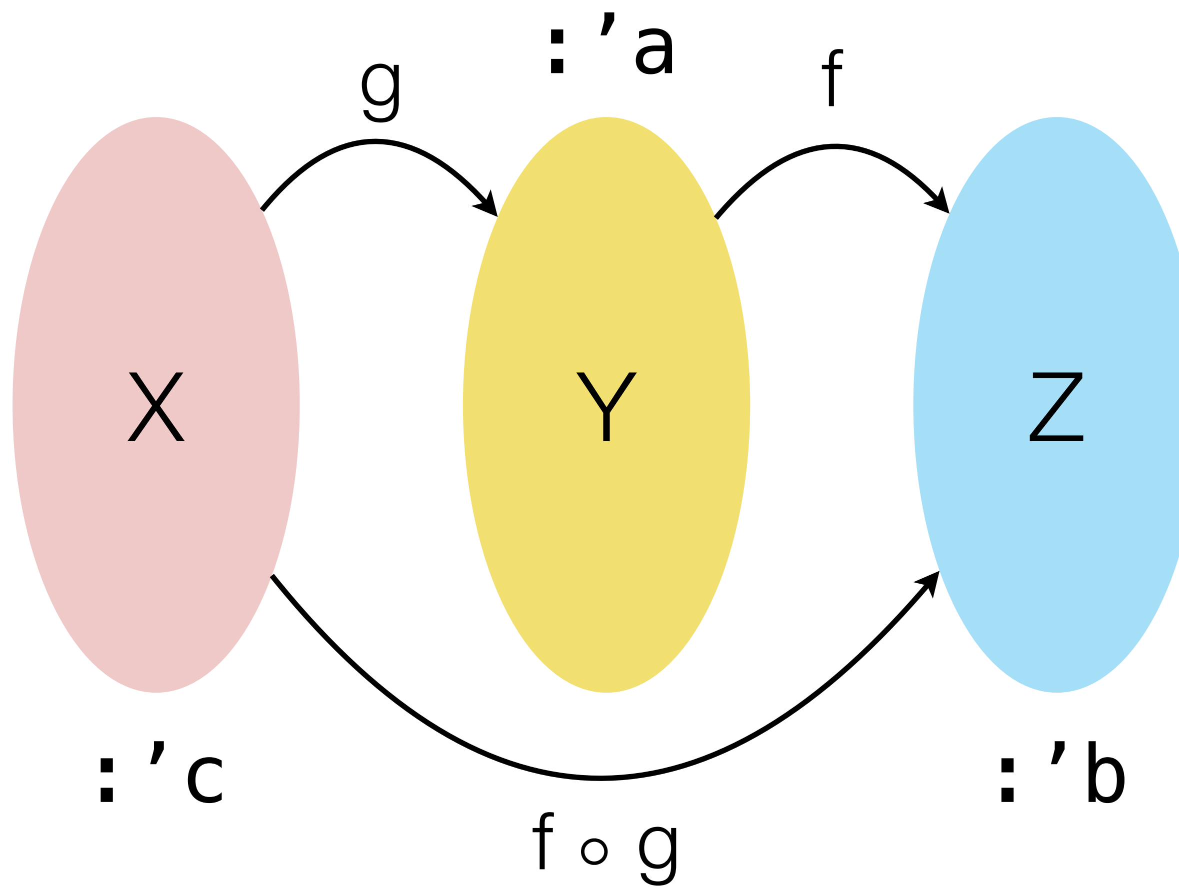


```
(* (op o): ('a -> 'b) * ('c -> 'a) -> 'c -> 'b *)
```

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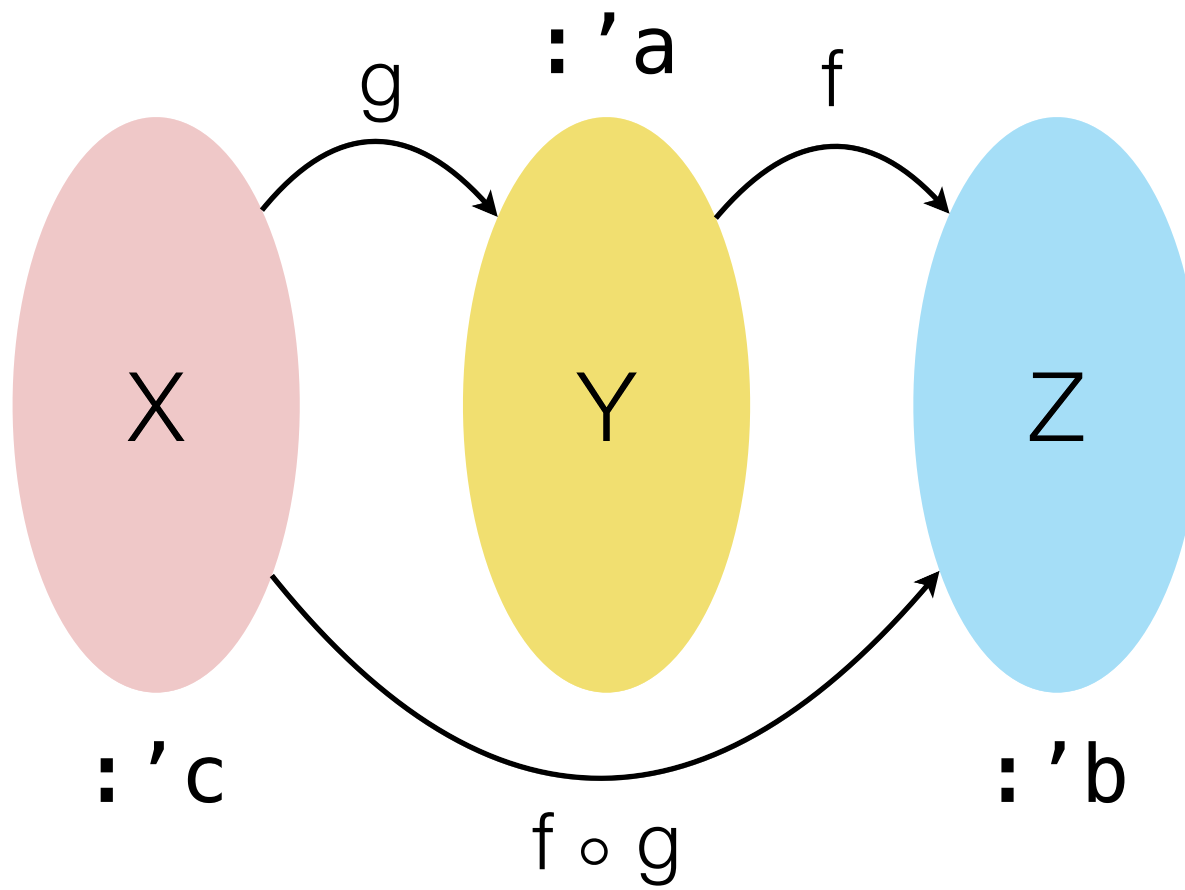

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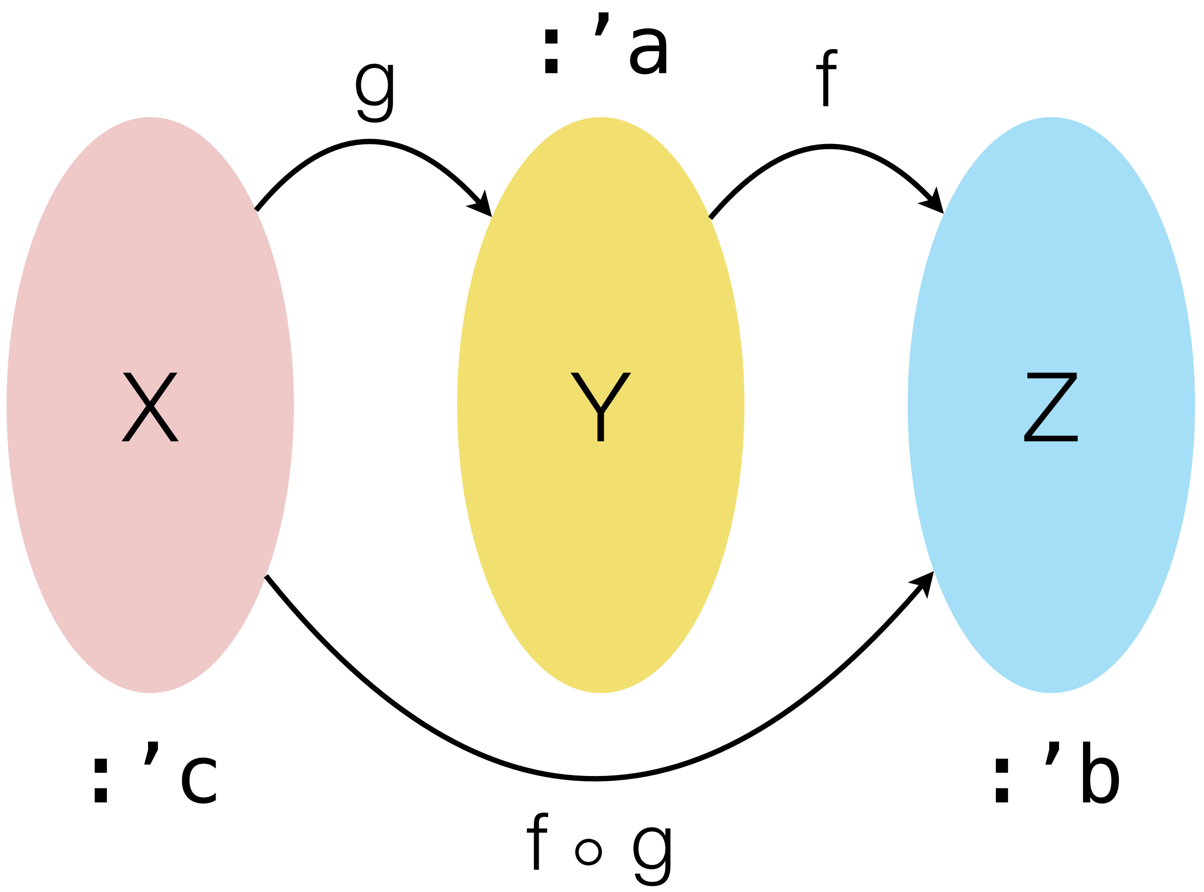
Examples:



Higher-order function: composition

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Examples:

```
fun incr x = x + 1
fun double x = 2 * x
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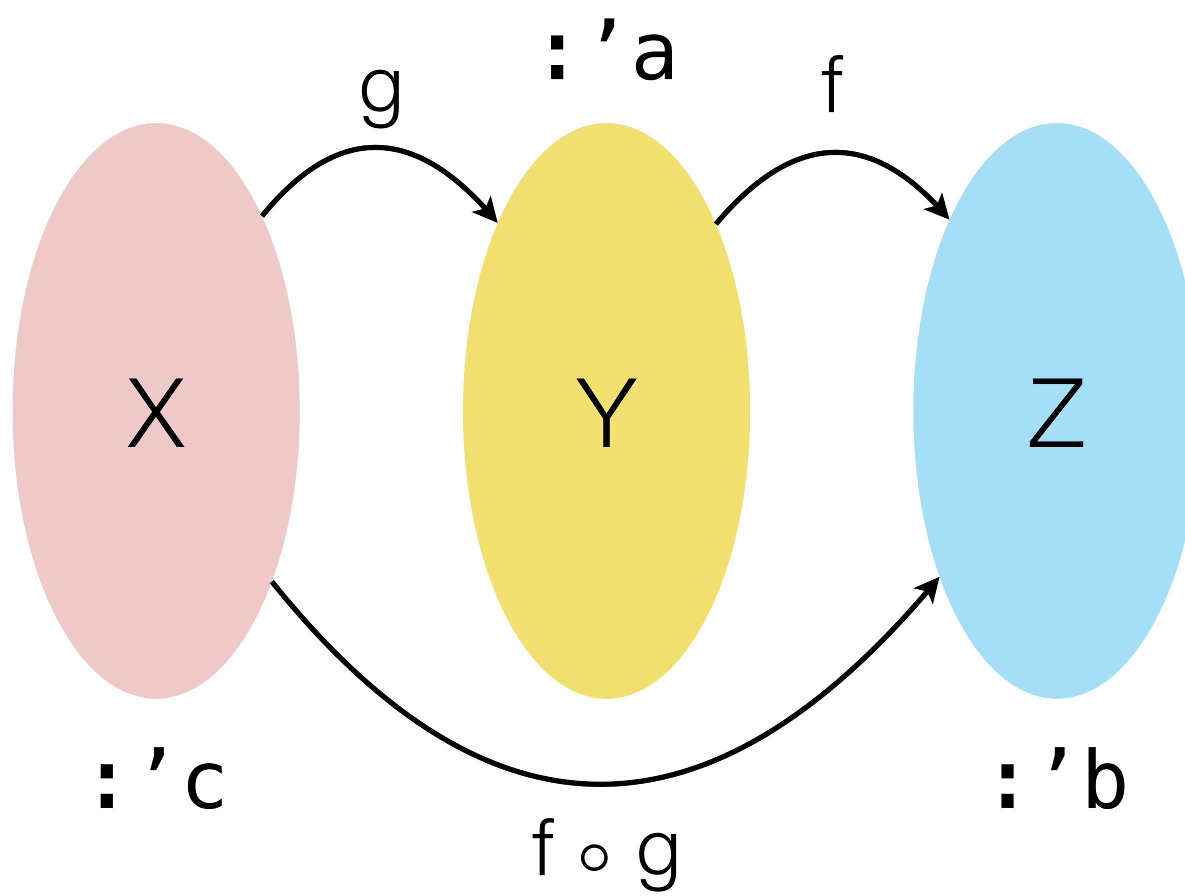

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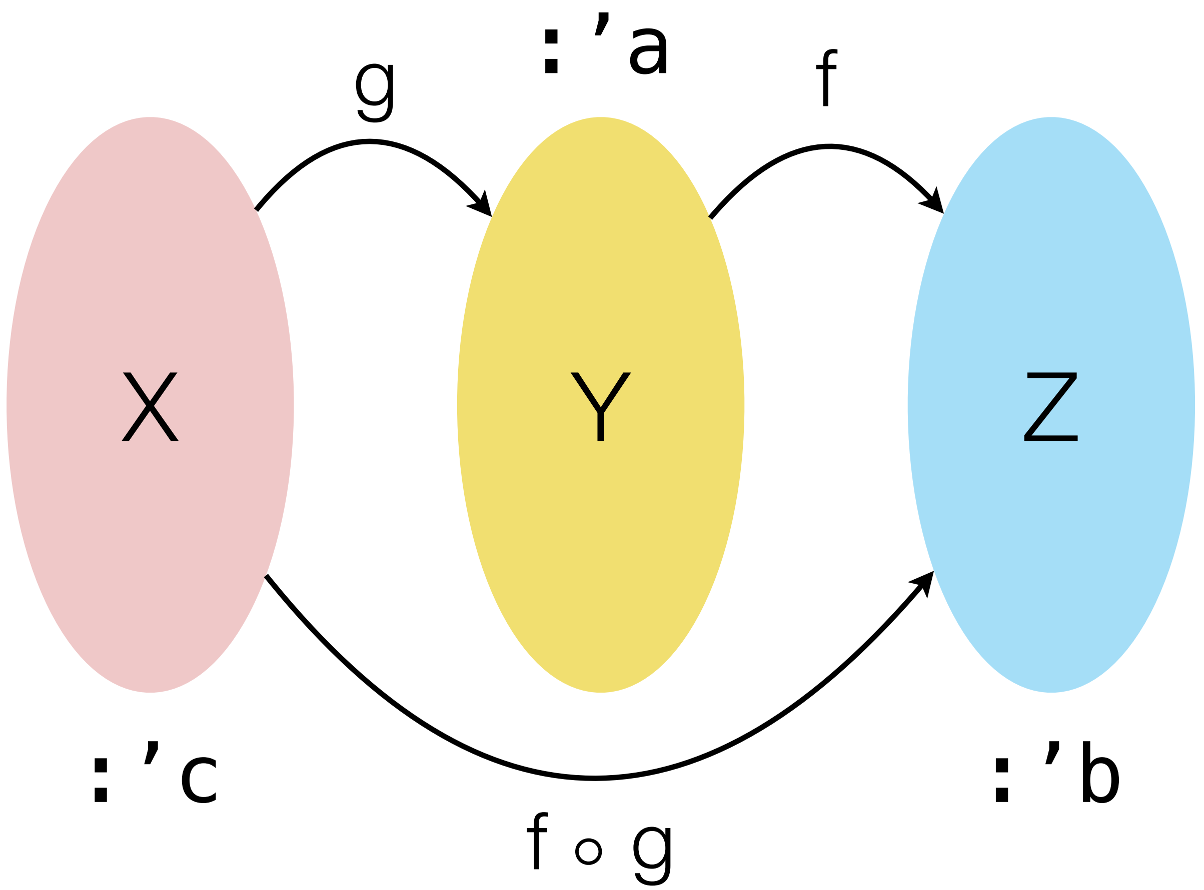

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Higher-order function: composition

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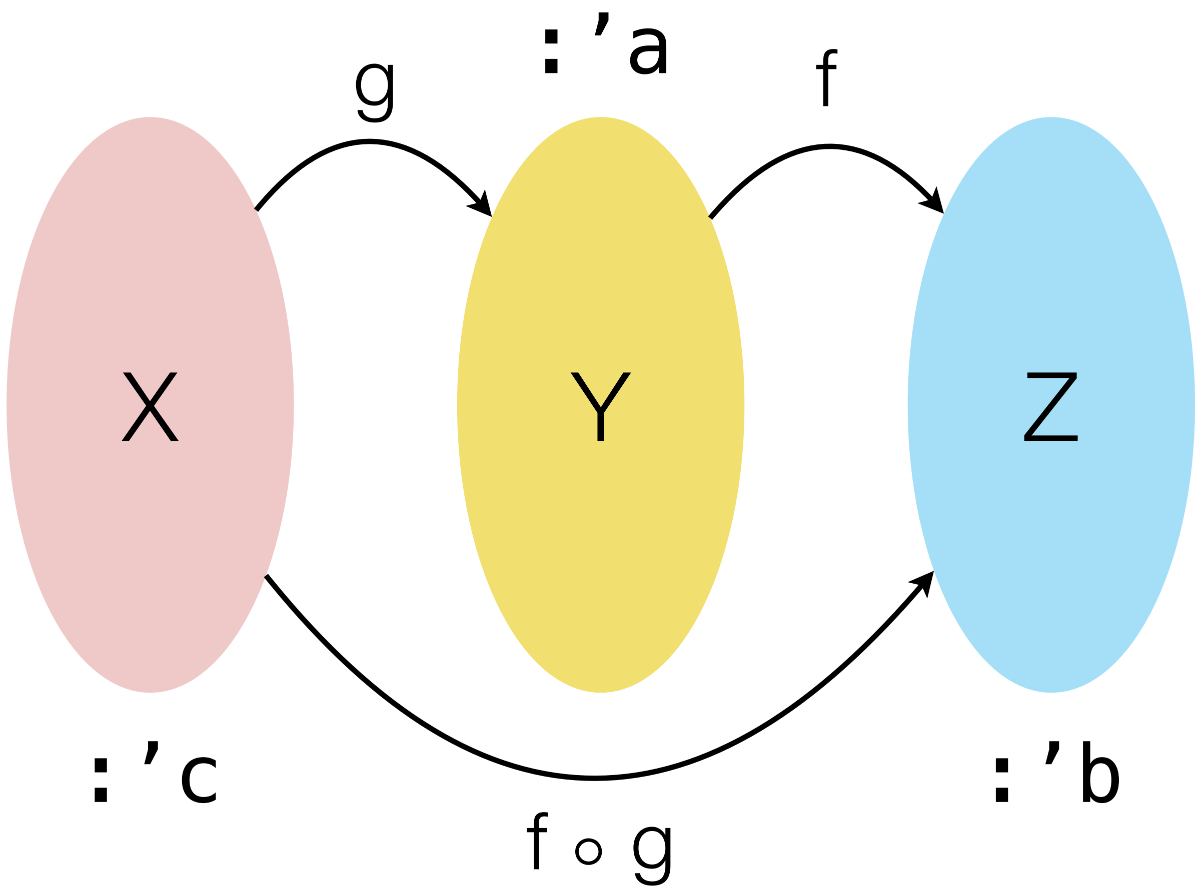

Examples:

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fun incr x = x + 1
fun double x = 2 * x
```

Then we have:

```
double o incr  ≅  fn x => 2x + 2
incr o double  ≅  fn x => 2x + 1
```

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Transforming elements in a list, given a transformation function:

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(* map: ('a -> 'b) -> 'a list -> 'b list
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Example:

```
map double [1,2,3] ==> [2,4,6]
```

Higher-order function: fold

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Combining elements in a list, given a binary operation and base value:

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combining function:

`'a`: type of list elements

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initial value

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$$\texttt{(* fold: } \underbrace{\boxed{\texttt{('a * 'b -> 'b)}}}_{\text{combining function}} \texttt{ -> } \underbrace{\boxed{\texttt{'b ->}}}_{\text{initial value}} \boxed{\texttt{'a list}} \texttt{ -> 'b *)}$$

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Two implementations:

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Two implementations:

$$\texttt{foldl f z } [x_1,\ldots,x_n] \;\cong\; f(x_n,\ldots f(x_3,f(x_2,f(x_1,z))))$$

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```

Two implementations:

$\texttt{foldl f z [}x_1,\ldots,x_n\texttt{]} \;\cong\; \texttt{f(}x_n,\ldots\texttt{f(}x_3\texttt{,f(}x_2\texttt{,f(}x_1\texttt{,z))))}$

$\texttt{foldr f z [}x_1,\ldots,x_n\texttt{]} \;\cong\; \texttt{f(}x_1,\ldots\texttt{f(}x_{n-2}\texttt{,f(}x_{n-1}\texttt{,f(}x_n\texttt{,z))))}$

Higher-order function: fold

Combining elements in a list, given a binary operation and base value:

```
(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
```

Two implementations:

$\texttt{foldl f z } \overrightarrow{[x_1,\ldots,x_n]} \cong \texttt{f}(x_n,\ldots\texttt{f}(x_3,\texttt{f}(x_2,\texttt{f}(x_1,z))))$

$\texttt{foldr f z } [x_1,\ldots,x_n] \cong \texttt{f}(x_1,\ldots\texttt{f}(x_{n-2},\texttt{f}(x_{n-1},\texttt{f}(x_n,z))))$

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Examples:

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==>
```

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
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Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2 (4-(3-(2-(1-0))))
```

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foldr (op -) 0 [1,2,3,4] ==>
```

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
foldr (op -) 0 [1,2,3,4] ==> ~2
```

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
foldr (op -) 0 [1,2,3,4] ==> ~2  (1-(2-(3-(4-0))))
```

Higher-order function: fold

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Let’s implement foldl and foldr:

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(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
fun foldl f z [] =
  | foldl f z (x::xs) =
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  | foldr f z (x::xs) = f(x, foldr f z xs)
```

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Let's implement foldl and foldr:

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fun foldr f z [] = z
  | foldr f z (x::xs) = f(x, foldr f z xs)
```

Homework:

```
foldl (op ::) [] [1,2,3,4] ==> ?
foldr (op ::) [] [1,2,3,4] ==> ?
```

That's all for today. See you on Thursday!